PRELIMINARY-TEST ESTIMATION OF A MIS-SPECIFIED LINEAR MODEL WITH SPHERICALLY SYMMETRIC DISTURBANCES

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ABSTRACT

This thesis considers some finite sample properties of preliminary test (pre-test) estimators of the unknown parameters of a (possibly) mis-specified linear regression model. We investigate two types of mis-specification which may or may not occur simultaneously. The first relates to the distribution of the regression disturbances, which is assumed to be normal, when in fact, the error distribution belongs to a broader family of spherically symmetric distributions. The second mis-specification is that the model's design matrix may exclude relevant regressors.

We analyse some finite sample properties of three pre-test estimators. The first is an estimator of the prediction vector after a pre-test for exact linear restrictions on the location vector. Secondly, we consider an estimator of the error variance after the same pre-test. Finally, we analyse an estimator of the error variance after a pre-test for homogeneity of the variances in the two-sample linear regression model. In each case we extend the existing literature by generalising the model's error distribution and allowing for model mis-specification through the omission of regressors.

To provide a setting for this research, we survey the relevant pre-testing literature in Chapter Two. This discussion assumes that the errors are normally distributed. There is a body of research, however, which proposes that some economic data series may be generated by processes whose underlying distributions have thicker tails than that which would result from a normality assumption. We briefly examine this literature in Chapter Three.

One alternative family of distributions, which has received

considerable attention, is the spherically symmetric family of distributions. Well known members of this family include the normal and the multivariate Student-t distributions. So, we include in Chapter Three a rationale for investigating spherically symmetric regression disturbances as an alternative to the usual normality assumption. We also discuss several studies which consider the linear regression model under a spherically symmetric disturbance assumption.

Having provided a setting and rationale for our research in Chapters Two and Three, Chapters Four, Five and Six present the finite sample properties of the aforementioned pre-test estimators. In each of these chapters we derive the exact bias and the exact risk functions (under quadratic loss) of the estimators under the mis-specified regression model. We also give the non-null distributions of the commonly used test-statistics for the investigated pre-tests, and we generalise many of the results reported in the existing literature. In particular, we derive the critical values of the test which result in a minimum of the bias and of the risk of the pre-test estimators of the error variance.

To illustrate the results we assume multivariate Student-t regression disturbances, rather than the general spherically symmetric family, and numerically evaluate the derived expressions for various cases. Our results suggest, when estimating the prediction vector, that the mis-specification of the distribution of the regression disturbances has little impact on the qualitative properties of the predictor pre-test estimator, though there are quantitative effects.

However, when estimating the error variance, after either a pre-test for linear restrictions or for homogeneity of the error variances, we find that mis-specifying the error distribution can have a substantial qualitative, and quantitative, impact on the bias and the risk functions of the estimators. Imposing the linear restrictions, even if they are valid, or always pooling the samples, even if the error variances are identical, may often be inappropriate strategies.

The final chapter, Chapter Seven, contains some concluding remarks. In particular, we consider some possible future research topics.

CHAPTER ONE

INTRODUCTION

1.1 Introductory Comments

In the experimental sciences data can be obtained with relative accuracy and certainty, experiments can be repeated, and models of processes can be specified with known error. In economics, however, we cannot identically repeat an environment, the accuracy of data is uncertain, and we are unsure of the "true" driving forces behind economic processes. Our economic information is ambiguous, incomplete, and usually inaccurate. Econometricians, therefore, inevitably work with false models, even though the statistical tools we frequently use assume that the model specification is correct.

For instance, our economic theory may suggest competing sets of variables to explain an economic process. We may be uncertain whether the classical assumption of normal regression disturbances is applicable, or whether our errors are homoscedastic or uncorrelated, or whether the data contain measurement errors. If such uncertainties exist our usual procedure in practice is to specify an initial model and then allow the data to inform us of any possible specification problems. So, we may undertake preliminary tests of the validity of the model assumptions that we are uncertain about. Then, on the basis of the outcome of such tests, we estimate the parameters of the model. For example, it is common practice to undertake preliminary tests of the significance of the regressors, to test for serial correlation, for homoscedasticity, and for normality, of the regression disturbances prior to specifying the "final" model.

Accordingly, the estimators of the parameters we use are conditional on the preliminary tests we have undertaken. Such conditional or pre-test estimators are not equivalent to the estimators we would have used had we not pre-tested and merely ignored, or simply imposed, any prior beliefs without testing.

There has been substantial investigation in the literature of some of the finite sample properties of many classical pre-test estimators. Traditionally, pre-test estimators have been examined within the context of the standard linear statistical model assuming normal, independent, and identically distributed regression disturbances and a correctly specified design matrix. It seems rather trite to propose that researchers pre-test because of uncertainty and then proceed to investigate the properties of pre-test estimators assuming a correctly specified model. There have been exceptions to this, as some recent research considers the effects of model mis-specification on the properties of pre-test estimators. Some of these investigations consider the effects of omitting relevant regressors or including irrelevant variables, or both, within the framework of the standard linear regression model, while the remainder investigate, via Monte Carlo experiments, the effects of non-normal regression disturbances.

In this thesis we contribute to this pool of knowledge of the consequences of pre-testing with mis-specified models. We consider two common pre-test problems. First, estimating the paramaters of the model after a pre-test for exact linear restrictions and secondly, estimating the error variance after a pre-test for homogeneity. We investigate the consequences on some finite sample properties of estimators of these parameters if the model is mis-specified in two possible ways, which may or may not occur simultaneously. The specification errors relate to the error term distribution assumption and to the variables included in the design matrix. We assume that the researcher specifies the classical linear regression model when in fact first, the disturbances follow the laws of the spherically symmetric family of distributions, and secondly, the design matrix omits pertinent variables.

1.2 An Outline of the Thesis

The layout of this thesis is as follows. In the next section we establish the broad framework within which our pre-test estimators are investigated. We give particular attention to the criterion we use to compare the estimators, which is risk under quadratic loss.

Chapters Two and Three set the scene and provide the motivation for the research presented in this thesis. Chapter Two reviews the literature which considers the pre-test problems that we investigate. The two pre-tests of interest are first, a pre-test for exact linear restrictions on the location vector and secondly, a pre-test for homogeneity of the error variances in the two-sample linear regression model. We follow the literature by assuming throughout this discussion that the regression disturbances are normally distributed.

For each of the pre-tests, we survey the studies which have considered the estimation of the coefficient vector (or of the prediction vector) and of the error variance. Though we do not extend the existing literature on the pre-test estimator of the coefficient vector after a pre-test for homogeneity in this thesis we include a discussion of this research so as to highlight that the approach used to investigate the finite sample properties of this estimator is somewhat different from that of the other three pre-test estimators that we consider. We also include in this chapter a brief review of the optimal size literature.

Our discussion in Chapter Two assumes that the regression errors are normal. We question the validity of this assumption in Chapter Three. We examine the suitability of this assumption and then follow this with a review of the literature which questions its applicability for certain economic data series. This research suggests that many economic data series have more kurtosis (and hence fatter tails) than the normal distribution. One alternative is to replace the normality assumption with the wider one of spherical symmetry. Well known members of this family include the normal and the multivariate Student-t distributions. We provide some justification for this extension and we give some necessary definitions. The remainder of Chapter Three is devoted to a brief survey of the literature which considers the linear regression model with non-normal errors.

The next three chapters investigate some finite sample properties of the pre-test estimators under consideration. Chapter Four considers the estimators of the conditional forecast of y, or the so-called estimators of the prediction vector, in the linear regression model after a pre-test for exact linear restrictions. Chapter Five investigates the estimators of the error variance under the same framework as that used in Chapter Four, while Chapter Six considers the problem of the estimation of the error variance in the two-sample linear regression model when it is suspected that that the sample regressions have a common coefficient vector but possibly different error variances. In each chapter we assume that the researcher proposes a mis-specified model.

The first specification error relates to the assumption about the disturbance distribution. We assume he specifies normal errors when in fact the errors are spherically symmetric. The second specification error is that there may be omitted relevant regressors. We derive both the exact bias and the exact risk of the pre-test estimators, and their component estimators, when the model is mis-specified in these ways.

We also derive the non-null distribution of the relevant test statistics, and we generalise many of the results reported in the existing literature. In particular, we derive the critical values of the pre-test which result in a minimum of the pre-test bias and of the pre-test risk of the error variance. To illustrate the results we numerically evaluate the derived expressions assuming multivariate Student-t errors.

Some final remarks are given in Chapter Seven.

1.3 Some Definitions and Performance Measures

Throughout this thesis the model under consideration is the classical linear regression model

$$y = X\beta + e \ , \tag{1.3.1}$$

where y is a (T×1) random vector, X is a (T×k) non-stochastic matrix of rank k (< T), β is a (k×1) vector of unknown parameters and e is a (T×1) random error vector. e is assumed to have a zero mean vector $\left(E(e)=0\right)$ and a finite variance-covariance matrix $\left(E(ee')=\Sigma\right)$, where E(.) is the usual expectation operator.[1] We assume that the regression disturbances are spherically symmetric.[2] Deferring a discussion of this family of

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1 Note that the assumption of finite first and second moments for e precludes e from being distributed according to those members of the spherically symmetric family with undefined integer moments, such as the Cauchy distribution. This assumption is necessary for 'risk' to be a meaningful performance measure.

In Chapters Four and Five we assume that the spherical symmetry holds for all T observations, while in Chapter Six we assume that the regression disturbances for each sample are spherically symmetric but that their scale parameters, and hence their error variances, may be different.

distributions until Chapter Two, briefly, this means that e has a joint probability density function, which is assumed to exist, of the form

$$f(e) = \vartheta(e'e) \tag{1.3.2}$$

with respect to the Lebesgue measure on R^T, where ϑ: $[0,\infty) \rightarrow [0,\infty)$. The location vector β and variance-covariance matrix Σ are unknown and the objective is to estimate them using (say) estimators $\hat{\beta}$ and $\hat{\Sigma}$ respectively, under some loss function which measures the consequence of using the estimators rather than their true values. For example if θ is the unknown true value of a parameter then the problem is to select an appropriate estimator of θ, say $\hat{\theta}(y)$, which minimizes some loss function $L\left(\theta,\hat{\theta}(y)\right)$. This approach merely frames the estimation problem as a statistical decision problem on which there exists a wealth of literature. A detailed discussion of this literature is far beyond the scope of this thesis: some references include Raiffa and Schlaifer (1961), Ferguson (1967), De Groot (1970), Judge and Bock (1978), Berger (1985), and Judge *et al.* (1985a).

As $L\left(\theta,\hat{\theta}(y)\right)$ is random, one common criterion is to seek the estimator which minimizes the risk function

$$\rho\left(\theta,\hat{\theta}(y)\right) = E\left[L\left(\theta,\hat{\theta}(y)\right)\right]$$

for all values of θ. Unfortunately, as it is always possible to reduce the risk at any given point θ_0 to zero, no uniformly best estimator exists (see, for instance, Lehmann (1983) for further discussion) and so, some limit must be placed on the estimator set. From a decision theoretic viewpoint, we may consider that an estimator is desirable if it:

(i) Is admissible. An estimator $\hat{\theta}_0$ is said to dominate an estimator $\hat{\theta}$ if, for all possible θ, $\rho(\theta,\hat{\theta}_0) \leq \rho(\theta,\hat{\theta})$. If, in addition, the strict inequality holds for some θ, then $\hat{\theta}_0$ strictly dominates $\hat{\theta}$. An estimator is called admissible if it is not strictly dominated by any other estimator.

(ii) Minimizes average risk

$$E\left(\rho(\theta,\hat{\theta}_0)\right) = \int_{\theta}\rho(\theta,\hat{\theta}_0)w(\theta)d(\theta)$$

for some weight function w. If w(θ) is a density function which represents our uncertainty about the true value of θ, then the resulting Bayes' estimator minimizes average risk (see, for example, Ferguson (1967), Zellner (1971), Judge and Bock (1978), Berger (1985) and Judge *et al.* (1985a)). A Bayes' estimator with a proper prior distribution is admissible.

(iii) Minimizes the maximum of the risk function. Such an estimator is called a minimax estimator. (Some reference sources include Ferguson (1967), Judge and Bock (1978), Barnett (1982) and Lehmann (1983).) Note that minimax estimators are also Bayes' estimators.

So, our criterion for evaluating estimator performance is risk for which we require an explicit loss function. The choice of loss function is at the investigator's discretion[3] but it should clearly not be proportional to the error committed for then a positive error would be negated by a negative one. Consequently, a loss function that is frequently advocated in the literature is squared error loss. This results in a risk function

$$\begin{aligned}\rho\left(\theta,\hat{\theta}(y)\right) &= E\left[\left(\hat{\theta}(y)-\theta\right)'\left(\hat{\theta}(y)-\theta\right)\right] \\ &= E\left[\mathrm{tr}\left(\hat{\theta}(y)-\theta\right)\left(\hat{\theta}(y)-\theta\right)'\right] \\ &= \mathrm{tr}\left\{\mathrm{cov}\left(\hat{\theta}(y)\right) + \left[\mathrm{bias}\left(\hat{\theta}(y)\right)\right]\left[\mathrm{bias}\left(\hat{\theta}(y)\right)'\right]\right\}\end{aligned} \tag{1.3.3}$$

where tr{.} is the usual trace operator. Hence, risk under squared error loss is the trace of the matrix mean squared error (MMSE) and hence the

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[3] The loss function is widely discussed in the literature. See, for instance, De Groot (1970), Judge and Bock (1978), Barnett (1982), Martz and Waller (1982) and Zellner (1986). We limit our attention to squared error loss.

unweighted sum of the MSE's for each element of $\hat{\theta}(y)$:[4]

$$\rho\left(\theta,\hat{\theta}(y)\right) = \mathrm{tr}\left(\mathrm{MMSE}\right) = \sum_i \mathrm{MSE}\left(\hat{\theta}(y)_i\right),$$

where $\mathrm{MSE}\left(\theta(y)_i\right) = E\left(\hat{\theta}(y)_i - \theta_i\right)^2 = \mathrm{var}\left(\theta(y)_i\right) + \left(\mathrm{bias}(\hat{\theta}(y)_i\right)^2.$

This bias/variance tradeoff gives the squared error loss function intuitive appeal. Also, its simplicity and its mathematical convenience further enhances its attractiveness. See, for instance, Toro-Vizcarrondo and Wallace (1968), Wallace (1972), Judge and Bock (1978) and Judge *et al.* (1985a) for further discussion.

Using risk under squared error loss as the performance measure to compare alternative estimators contrasts with the classical approach of comparing estimators on the basis of such separate properties as unbiasedness, invariance, sufficiency, minimum variance unbiasedness and so on: criteria which are not necessarily satisfactory under the overall criterion of minimizing risk. For instance, limiting our attention to unbiased estimators may result in unnecessarily high variances compared with some biased estimators: a tradeoff which, from equation (1.3.3), has risk consequences (at least under squared error loss).

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4 We could consider the weighted square error loss function,

$$L\left(\theta,\hat{\theta}(y)\right) = \left[\left(\hat{\theta}(y)-\theta\right)' W\left(\hat{\theta}(y)-\theta\right)\right]$$

where W is a known symmetric positive definite weight matrix. However, it is usually possible to reparameterize the model in such a way as to reduce the criterion to squared error loss. Note that in our case W=I, and the squared error loss for each parameter is equally weighted.

CHAPTER TWO

PRELIMINARY-TEST ESTIMATION IN THE LINEAR REGRESSION MODEL WITH NORMAL DISTURBANCES : A REVIEW

2.1 Introduction

In this chapter we survey the literature that directly pertains to the subject matter of this dissertation.[1] We assume in this discussion, as has the literature, that the errors are normally distributed. The next section, Section 2.2, considers the papers which have contributed to the analysis of the pre-test estimators we will investigate. As noted in the last chapter we will analyse the finite sample properties of three pre-test estimators. The first is an estimator of the prediction vector after a pre-test for exact linear restrictions on the location vector. Secondly, after the same pre-test we consider an estimator of the error variance. Finally, we analyse an estimator of the error variance after a pre-test for homogeneity of the variances in the two-sample linear regression model. In each case we extend the existing literature by generalising the model's error distribution and allowing for model mis-specification through the omission of relevant regressors.

These particular estimators are examined first, because they are commonly encountered in applied econometric research. Secondly, we consider them because they have received significant attention in the pre-test literature. Thirdly, we analyse these particular estimators because they

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[1] An excellent bibliography of papers that have been written in the pre-test area is contained in Bancroft and Han (1977). To our knowledge, this bibliography has not been updated.

constitute a cohesive group: we can investigate their sampling properties using the same analytic and computational techniques. It is for this third reason that we do not consider the estimation of the location vector after a pre-test for homogeneity. Despite that, in this literature review, we include a brief survey of the studies relating to this pre-test estimator, so as to highlight the different nature of the risk function of this estimator, as opposed to the others that we consider and extend.

Unless there exists a strictly dominating estimator this research does not analytically address the problem of choosing the optimal pre-test size for a particular pre-test problem. Nevertheless, as the sampling properties of pre-test estimators crucially depend on the chosen test size, we include a brief review of the literature that has considered this question for the pre-tests under investigation.

There are, of course, many other pre-test problems investigated in the literature. In Section 2.5, where we conclude this chapter, mention is made of some other pre-test situations in econometrics. We also concentrate on what might be called "traditional" or "conventional" pre-test estimators: those pre-test estimators whose component estimators are the traditional ones used for the problem under investigation (traditional either in the literature or in applied research). In particular, we will not consider Bayesian pre-test estimators or pre-test estimators whose components are Stein-like estimators. The analysis of Stein (or James-Stein) estimators is integrally related to the analysis of pre-test estimators, as both are alternatives to the classical estimators, and both are in the class of "shrinkage" estimators. They are both typically biased, and so we usually compare their performance to that of the classical estimators using the same criterion.[2]

These estimators, and various extensions to them, have received considerable attention in the literature: a coverage of which is far beyond the scope of this thesis. Judge and Bock (1978), Vinod and Ullah (1981), Judge and Bock (1983), and Judge *et al.* (1985a), for example, survey some of this extensive literature; they also investigate the sampling properties of many of these estimators.

2.2 The Linear Restrictions Pre-Test Estimators

In this section we assume that interest lies in the estimation of the parameters of model (1.3.1) when the errors are normally distributed with mean vector zero and variance-covariance matrix $\sigma^2 I_T$. That is,

$$y = X\beta + e \quad , \quad e \sim N(0,\sigma^2 I_T). \tag{2.2.1}$$

We also assume that the researcher, in addition to the sample information contained in (2.2.1) has some non-sample (prior) information about the unknown parameters, β, which he can specify as m independent exact linear restrictions

$$R\beta = r \ , \tag{2.2.2}$$

where R is an $(m\times k)$ known matrix of rank m $(\le k)$ and r is an $(m\times 1)$ vector of

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[2] Stein (1955) proved that the traditional least squares (maximum likelihood) estimator, b, of the location vector is inadmissible under squared error loss when the number of parameters in the linear model with orthonormal regressors exceeds two. James and Stein (1961), Baranchik (1964) and Stein (1966) demonstrate, under these conditions, that the estimators $b_s = (1-a/b'b)b$ (the Stein or James-Stein estimator) and $b_{+s} = I_{[a,\infty)}(b'b)(1-a/b'b)b$ (the positive-part Stein or James-Stein estimator), which shrink toward a fixed vector zero, dominate b when $0 \le a \le 2(k-2)$. $I_{[a,\infty)}(.)$ is an indicator function which is one if (.) lies with the subscripted range, zero otherwise. Furthermore, b_{+s} dominates b_s but is itself inadmissible.

known non-stochastic elements. Define $\delta = R\beta - r$, so that δ represents an $(m\times 1)$ specification error vector of the prior information. If the information is valid then $R\beta = r$ and $\delta = 0$. This situation is one that is commonly encountered in applied econometrics, except that usually the researcher is uncertain of the accuracy of the prior beliefs. Accordingly, the procedure usually followed in practice is to (pre-)test the validity of the restrictions and if the outcome of the pre-test suggests that they are correct then the model's parameters are estimated incorporating the restrictions. If the pre-test rejects the accuracy of the non-sample information then the parameters are estimated from the sample information alone. The properties of this pre-test estimator differ from those of either the estimator which ignores the non-sample information or the estimator which imposes the restrictions without first testing their validity.

Prior to considering the properties of such pre-test estimators of β and σ^2 we will briefly review the estimators which ignore the restrictions (the "unrestricted" estimators) and those which assume the restrictions are correct (the "restricted" estimators).

The unrestricted ordinary least squares (and maximum likelihood) estimator of β is well known to be given by $b = S^{-1}X'y$, where $S = (X'X)$. b is normally distributed with mean β and variance-covariance matrix $\sigma^2 S^{-1}$, and consequently its risk under squared error loss is

$$\rho(\beta,b) = E\left((b-\beta)'(b-\beta)\right) = \sigma^2 tr(S^{-1}).$$

From the Gauss-Markov theorem we know that b is the best linear unbiased estimator (BLUE).[3] It is minimax and, among the class of unbiased

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[3] e need not be normal for this result to hold, of course. With normal e, b is best unbiased.

estimators, minimises risk under quadratic loss.[4]

A best (minimum variance) quadratic unbiased estimator of σ^2 is the usual least squares estimator, given by $\tilde{\sigma}_L^2=(y-Xb)'(y-Xb)/v=\tilde{e}'\tilde{e}/v=e'Me/v$ where $v = (T-k)$, $\tilde{e} = (y-Xb)$ is the residual vector and $M = \left(I_T-XS^{-1}X'\right)$ is an idempotent matrix of rank v. Now $(e'Me/\sigma^2)\sim\chi_v^2$ and so, from the moments of a central Chi-square random variate, it follows directly that the risk of $\tilde{\sigma}_L^2$ is $\rho(\sigma^2,\tilde{\sigma}_L^2)=\mathrm{var}(\sigma^2,\tilde{\sigma}_L^2)=2\sigma^4/v$.

If we allow the estimator of σ^2 to be biased then in this family the estimator of σ^2 with smallest MSE is $\tilde{\sigma}_M^2=(y-Xb)'(y-Xb)/(v+2)$, and its risk is $\rho(\sigma^2,\tilde{\sigma}_M^2)=2\sigma^4/(v+2)$. The maximum likelihood estimator of σ^2 is $\tilde{\sigma}_{ML}^2=(y-Xb)'(y-Xb)/T$ and $\rho(\sigma^2,\tilde{\sigma}_{ML}^2)=(2v+k^2)\sigma^4/T^2$.

Imposing the restrictions, we estimate β by $b^*=b+S^{-1}R'[RS^{-1}R']^{-1}(r-Rb)$, which, with $D=S^{-1}R'[RS^{-1}R']^{-1}RS^{-1}$, has a risk under squared error loss,

$$\rho(\beta,b^*)=\sigma^2\mathrm{tr}\left(S^{-1}-D\right)+\mathrm{tr}\left(S^{-1}R'[RS^{-1}R']^{-1}\delta\delta'[RS^{-1}R']^{-1}RS^{-1}\right).$$

b^* is unbiased if and only if the restrictions are correct ($\delta=0$). Further, as D is at least positive semi-definite, $\mathrm{var}(b_i^*)\le\mathrm{var}(b_i)$, $i=1,2,\ldots,k$, and within the class of linear estimators of β, b^* is BLUE in this case.

Let the corresponding residual vector be $e^*=y-Xb^*$ so that the corresponding restricted estimators of σ^2 are $\sigma_L^{*2}=(y-Xb^*)'(y-Xb^*)/(v+m)$ $=e^{*\prime}e^*/(v+m)$, $\sigma_M^{*2}=e^{*\prime}e^*/(v+m+2)$, and $\sigma_{ML}^{*2}=e^{*\prime}e^*/T$. Now $\left(e^{*\prime}e^*/(v+m)\right)\sim\chi_{v+m;\lambda}^{2\prime}$ where the non-centrality parameter, λ, defined by $\lambda=\delta'[RS^{-1}R']^{-1}\delta/2\sigma^2$, is a measure of the validity of the linear restrictions as it is monotonically

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[4] If no assumption is made regarding the distribution of e then the minimax and minimum risk properties of b hold only among the class of linear estimators. See Judge and Bock (1978).

related to the sum of squared errors in the individual restrictions.[5] If the restrictions are true, $\delta=0$ and so $\lambda=0$. From the moments of a non-central Chi-square random variate it follows that

$$\rho(\sigma^2,\sigma^{*2}_{L}) = 2(2\lambda^2+4\lambda+v+m)\sigma^4/(v+m)^2 ,$$

$$\rho(\sigma^2,\sigma^{*2}_{M}) = 2(2\lambda^2+v+m+2)\sigma^4/(v+m+2)^2 ,$$

and $$\rho(\sigma^2,\sigma^{*2}_{ML})= \left(2(m+v+4\lambda)+(m-k+2\lambda)^2\right)\sigma^4/T^2 .$$

The restricted least squares estimator, σ^{*2}_{L}, is unbiased and, the estimator, σ^{*2}_{ML}, is the minimum MSE estimator, only when the restrictions are true.

A number of studies[6] have considered the conditions under which the risk of b dominates that of b* and vice versa. These conditions are (generally) data specific, as their risk functions depend explicitly on the design matrix.[7] This limits the generality of any comparisons based on risk under squared error loss and, to avoid this complication we will, as others have, concentrate on the conditional forecast of y rather than on β itself.[8] So, the risk of Xb, the unrestricted estimator of E(y), is

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[5] See Farebrother (1975) and Wallace (1977) for further discussions on the interpretation of λ.

[6] For instance, see Toro-Vizcarrondo and Wallace (1968), Wallace and Toro-Vizcarrondo (1969), Wallace (1972), Goodnight and Wallace (1972), Yancey *et al.* (1973), Bock *et al.* (1973). See also Judge and Bock (1978) for a summary and a discussion.

[7] In contrast, Toro-Vizcarrondo and Wallace (1968) show that the risk difference matrix $E\left((b-\beta)(b-\beta)'\right) - E\left((b^*-\beta)(b^*-\beta)'\right)$ is non-negative definite if $\lambda \le 1/2$.

[8] This is equivalent to assuming orthonormal regressors (i.e. $X'X = I_k$) in the β space. So, though similar conclusions are drawn from comparing the risk functions, the mapping from the conditional mean (or orthonormal regressors) case to that of considering the unweighted risk of estimators of β (i.e. nonorthonormal regressors) is not direct and is significantly more complicated. See, for instance, Wallace (1972), Brook (1972, 1976), Bock *et al.* (1973), Yancey *et al.* (1973), Judge and Bock (1978). Brook (1972, 1976), Bock *et al.* (1973), and Judge and Bock (1978) also consider the unweighted risk function of the pre-test estimator of β.

$$\rho\left(E(y),Xb\right) = E\left((Xb-X\beta)'(Xb-X\beta)\right) = E\left((b-\beta)'X'X(b-\beta)\right)$$
$$= \sigma^2 k, \qquad (2.2.3)$$

while that of the restricted estimator, Xb*, is

$$\rho\left(E(y),Xb^*\right) = \sigma^2(k-m+2\lambda). \qquad (2.2.4)$$

Comparing (2.2.3) and (2.2.4) we see that the risk of the restricted estimator is less than or equal to that of the unrestricted estimator if $\lambda \le m/2$. This arises because even when the restrictions are false ($\delta>0$ and b^* is biased) the $var(b^*_i)<var(b_i)$, for $i=1,2,\ldots,k$. It is only for $\lambda>m/2$ that the bias of Xb* outweighs the reduction in variance[9] so that the risk of Xb* exceeds that of Xb.

Similarly, there is a λ-range over which the risk of the restricted estimator of σ^2 is less than or equal to that of the unrestricted estimator. The values of λ, λ^*_j , ($j=L,M,ML$) for which the risks are equal depends on the estimation method[10] and are given by

$$\lambda^*_L = -1+1/\left[4v^2+2mv(v+m)\right]^{1/2}/(2v) > 0 ,$$

$$\lambda^*_M = \left[\left(m(v+m+2)\right)/\left(2(v+2)\right)\right]^{1/2} > (m/2)^{1/2} ,$$

and $$\lambda^*_{ML} = \left\{(k-m-2)+\left[(k-m-2)^2+m(2k-m-2)\right]^{1/2}\right\} > 0.$$

Note that $\lambda^*_j \ne m/2$ and so, if the researcher desired the minimum risk estimators of E(y) and σ^2, there will be some λ-range over which his strategy should be to use the restricted estimator of σ^2 but the unrestricted estimator of E(y). This occurs because of the way λ (and hence the restrictions specification error) impacts on the restricted estimator

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[9] That is, more correctly, the difference matrix between the variance-covariance matrices of Xb and Xb* is non-negative definite.

[10] Note that though the restricted risk function is a quadratic in λ, there is only one positive value of λ for which $\rho(\sigma^2,\tilde{\sigma}^2_j)=\rho(\sigma^2,\sigma^{*2}_j)$.

risk functions of E(y) and σ^2. This feature suggests considering a joint risk function for E(y) and σ^2, which has not been pursued in the literature nor will be in this thesis.

We consider now the situation of the researcher undertaking a pre-test of the validity of the restrictions. Traditionally, the hypothesis

$$H_0 : \delta=0 \quad \text{vs} \quad H_1 : \delta\neq0 \tag{2.2.5}$$

is tested using the Wald (and Lagrange Multiplier) statistic

$$u = (Rb-r)'[RS^{-1}R']^{-1}(Rb-r)v/m(y-Xb)'(y-Xb). \tag{2.2.6}$$

If H_0 is correct, the test statistic u has a central F distribution with m and v degrees of freedom, $F_{(m,v)}$. Alternatively, if one or more of the restrictions are invalid, u has a non-central F distribution with m and v degrees of freedom and non-centrality parameter λ, $F_{(m,v;\lambda)}$. As λ is typically unknown, it is usual to test H_0 under the null distribution, and so we reject the hypothesis if $u>F^{\alpha}_{(m,v)}=c$, where c, the test critical value, is determined for a given significance level of the test α, by $\int_0^c dF_{(m,v)}=$ $\Pr.\left(F_{(m,v)} \le c\right)=(1-\alpha)$. This is a UMPI size-$\alpha$ test of the validity of the restrictions of interest. If H_0 is rejected we use the unrestricted estimators of E(y) and σ^2. If $u\le c$, we assume the restrictions are correct and use the restricted estimators of E(y) and σ^2.[11] So, this estimation procedure is dependent on a preliminary test of significance and, the estimators of E(y) and σ^2 actually reported are the pre-test estimators

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[11] Pertaining to the estimation of E(y), Wallace (1972) suggests that instead of using this test, we should be testing whether $\lambda\le m/2$, against $\lambda>m/2$, as this is the value of λ for which we would switch from the restricted to the unrestricted estimator of E(y). Bock *et al.* (1973) argue that perhaps the test should be whether λ is small enough to ensure that the risk of the pretest estimator is smaller than that of Xb, i.e. whether $\lambda\le m/4$ against $\lambda>m/4$. However, it is clearly irrelevant which test procedure is used as the critical values for one can be obtained from one of the others by appropriately varying α (see Bock *et al.* (1973) for further discussion).

$$\hat{Xb} = \begin{cases} Xb & \text{if } u > c \\ Xb^* & \text{if } u \le c \end{cases} , \tag{2.2.7}$$

and

$$\hat{\sigma}^2_j = \begin{cases} \tilde{\sigma}^2_j & \text{if } u > c \\ \sigma^{*2}_j & \text{if } u \le c \end{cases} , \tag{2.2.8}$$

j=(L, M, ML). It is useful to rewrite (2.2.7) and (2.2.8) as

$$\hat{Xb} = I_{[0,c]}(u)Xb^* + I_{(c,\infty)}(u)Xb \tag{2.2.9}$$

and

$$\hat{\sigma}^2_j = I_{[0,c]}(u)\sigma^{*2}_j + I_{(c,\infty)}(u)\tilde{\sigma}^2_j \tag{2.2.10}$$

where $I_{(.,.)}(u)$ is an indicator function which takes the value unity if u falls within the subscripted range and zero otherwise. From equations (2.2.9) and (2.2.10), it is clear that the pre-test estimators are functions of the data, the hypothesis, and the significance level of the test. Representing a pre-test estimator in this way highlights the difficulty of deriving its sampling properties; a pre-test estimator is the sum of two parts, both of which are composed of products of non-independent random variables.

Bancroft (1944) was the first to consider the analytic properties of such pre-test estimators. Among other things, he derived the bias of a single restriction pre-test estimator of β in the case of whether or not to include a regressor in a two-regressor model. Toro-Vizcarrondo (1968) extends Bancroft's example by obtaining the MSE of the same pre-test estimator.[12] Brook (1972, 1976) generalizes the results, by deriving the unweighted risk functions of the pre-test estimators of β and E(y) for the general multiple restrictions problem, as outlined here. Sclove *et al.* (1972) also derive the risk of the pre-test estimator of β in the

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[12] See also Wallace and Ashar (1972) and Wallace (1977), both of which summarise and discuss the results of these studies.

orthonormal regressor model setup and Bock *et al.* (1973) extend their analysis to the non-orthonormal case. These cases are also discussed by for instance, Wallace (1977), Judge and Bock (1978), who also further generalise this research, and Judge *et al.* (1985a). Given the breadth of this literature, we only review here that which pertains directly to this thesis; namely, the risk properties of the pre-test estimator, $X\hat{b}$.

From the aforementioned research, the risk of $X\hat{b}$, under squared error loss, is

$$\rho\left(E(y),X\hat{b}\right) = \sigma^2\left(k+(4\lambda-m)P_{20}-2\lambda P_{40}\right) , \qquad (2.2.11)$$

where

$$P_{ij} = \Pr.\left[F'_{(m+i,v+j;\lambda)} \le \left(cm(v+j)\right)/\left(v(m+i)\right)\right]$$

$$= \sum_{r=0}^{\infty} \frac{\lambda^r e^{-\lambda}}{r!} I_x\left[(m+i)/2+r;(v+j)/2\right]. \qquad (2.2.12)$$

$I_x[.;.]$ is Pearson's incomplete beta function with $x=cm/(v+cm)$; $i,j=0,1,\ldots$.

Figure 2.2.1 illustrates the risk functions of Xb, Xb^*, and $X\hat{b}$ (for $c\in(0,\infty)$). Some features are:

(a) If the restrictions are valid, $\delta=\lambda=0$, $\rho\left(E(y),X\hat{b}\right)=\sigma^2(k-mP_{20})$ and so, the pre-test risk is less than that of the unrestricted estimator but higher than that of the restricted estimator. Intuitively, if $\lambda=0$, the pre-test estimator will lead us to use the restricted estimator $100(1-\alpha)\%$ of occasions but $100\alpha\%$ of the time we will erroneously ignore the prior information. The decrease in risk is determined by the value of α.

(b) $\rho\left(E(y),Xb\right)=\rho\left(E(y),X\hat{b}\ \middle|\ c\in(0,\infty)\right)$ occurs for a value of λ, $\lambda_1\in[m/4,m/2]$. Using properties of P_{ij} these bounds can be narrowed to $B_1\le\lambda_1\le B_2$, for $v\ge2$, $B_1=m/(2(2-x))$, $B_2=m/\left[2\left(2-\min\left\{1,x\left(1+(v-2)/(m+4)\right)\right\}\right)\right]$. If $v=2$, then

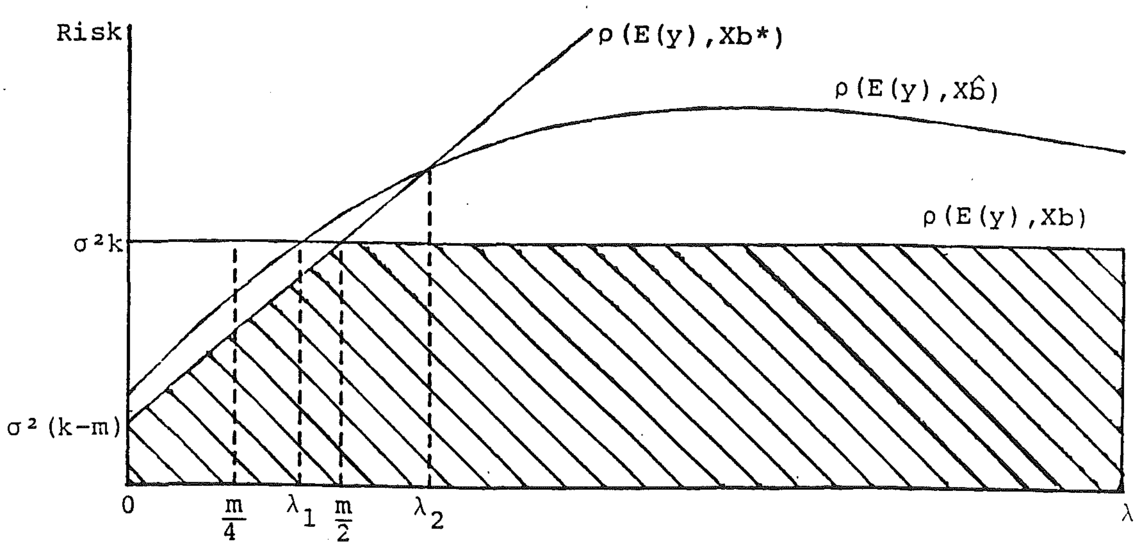

FIGURE 2.2.1: Typical Risk Functions for Xb, Xb* and $X\hat{b}$.

$\rho\left(E(y),Xb\right)=\rho\left(E(y),X\hat{b}\right)$ for $\lambda=B_1$. Note also that $B_2<m/2$ only for those values of $c<\left(v(m+4)\right)/\left(m(v-2)\right)$. (See, for instance, Judge and Bock (1978).) So, for $\lambda\in[0,\lambda_1)$ the risk of the pre-test estimator, $X\hat{b}$, is less than that of the unrestricted estimator, Xb, but higher than that of the restricted estimator, Xb*, while for $\lambda\in(\lambda_2,\infty)$, $X\hat{b}$ has smaller risk that that of Xb* but is dominated by Xb. For $\lambda\in(\lambda_1,\lambda_2)$, $X\hat{b}$ has higher risk than that of <u>both</u> Xb and Xb*. Thus, pre-testing is <u>never</u> the preferable strategy.

(c) For finite c, as δ increases, the risk of $X\hat{b}$ monotonically increases to a maximum, which occurs at a value of $\lambda>\lambda_2$, it then monotonically decreases and as $\lambda\to\infty$, $\rho\left(E(y),X\hat{b}\right)\to\rho\left(E(y),Xb\right)$. Intuitively, when the prior information is so wrong that λ is very large, then pre-testing will lead us to do the right thing; to ignore the restrictions.

(d) The smaller α is (the closer c is to ∞), the closer $\rho\left(E(y),X\hat{b}\right)$ is to $\rho\left(E(y),Xb^*\right)$ as a smaller test size increases the probability of accepting the null hypothesis. This results in a risk gain in the region to the left of λ_1 but at the cost of a (possibly) much higher risk for relatively large λ. An analogous argument can be made for large α. Clearly, from (2.2.9), if c is chosen to be zero (infinity), we will always reject (accept) the hypothesis and, the pre-test estimator degenerates to the unrestricted (restricted) estimator.

(e) Of the estimators considered, no one dominates any of the others. Cohen (1965) proves, under certain assumptions and a squared error loss function, that the pre-test estimator is inadmissible. Basically, this arises because the estimator is a discontinuous function of the test statistic, u, with a single jump at u = c. Sclove *et al.* (1972) demonstrate this inadmissibility by providing another (pre-test) estimator[13] which uniformly dominates the conventional pre-test estimator discussed here. Nevertheless, practitioners continue to report the conventional pre-test estimator and so, given the lack of dominance of either $X\hat{b}$, Xb, or Xb* and the fact that λ is rarely known, the next obvious question to ask is "is there an 'optimal' pre-test estimator?". The answer will certainly depend on the definition of 'optimal' but, more importantly, it will be linked to the choice of test size. Section 2.4 gives some attention to the literature which has addressed this issue.

We now consider the pre-test estimator of σ^2, which has received less attention in the literature than have the pre-test estimators of β and E(y). Given that σ^2 is often regarded as a nuisance parameter, this is perhaps not

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[13] Their pre-test estimator is formed by replacing b with the positive-part James-Stein estimator.

surprising. However, an estimator of σ^2 is often used as a measure of the model's "goodness of fit" and if one is interested in forming standard errors, prediction or confidence intervals or undertaking certain hypothesis tests, after pre-testing, then the pre-test estimator of σ^2 needs to be investigated. The risk functions of $\hat{\sigma}^2_j$, are derived, and numerically evaluated, in Clarke *et al.* (1987a,b).[14] They are

$$\rho(\sigma^2,\hat{\sigma}^2_L) = \sigma^4\Big[4v\lambda^2P_{80}+4v\lambda\Big((m+2)P_{60}+vP_{42}-(v+m)P_{40}\Big)+2(v+m)^2-2mv(v+m)(P_{20}-P_{02})-m(v+2)(m+2v)P_{04}+2mv^2P_{22}+mv(m+2)P_{40}\Big]\Big/\Big(v(v+m)^2\Big), \tag{2.2.13}$$

$$\rho(\sigma^2,\hat{\sigma}^2_M) = \sigma^4\Big[4(v+2)\lambda^2P_{80}+4(v+2)\lambda\Big((m+2)P_{60}+vP_{42}-(v+m+2)P_{40}\Big)+2(v+m+2)^2+m(v+2)(m+2)P_{40}-2m(v+m+2)\Big((v+2)P_{20}-vP_{02}\Big)-mv(m+2v+4)P_{04}+2mv(v+2)P_{22}\Big]\Big/\Big((v+2)(v+m+2)^2\Big), \tag{2.2.14}$$

$$\rho(\sigma^2,\hat{\sigma}^2_{ML}) = \sigma^4\Big[4\lambda^2P_{80}+4\lambda\Big((m+2)P_{60}+vP_{42}-TP_{40}\Big)+k^2+2v+m\Big((m+2)P_{40}+2(vP_{22}-TP_{20})\Big)\Big]\Big/T^2. \tag{2.2.15}$$

Note that these risk functions depend on the data only through T, k, m and, the non-centrality parameter λ. Figure 2.2.2 depicts typical risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$. The following comments are noted by the authors:

(f) As the pre-test size increases, we reject the hypothesis more

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[14] Clarke (1986, 1990) derives, and analyses, the pre-test estimator of the 'standard error of estimate' σ, after a preliminary test of linear restrictions on the coefficients; this pre-test estimator, say σ, is clearly not equal to $\left(\hat{\sigma}^2\right)^{1/2}$. We will not discuss this research here; it suffices to say that the results are found to be qualitatively similar whether one is estimating σ^2 or σ.

frequently, and so, the risk of the pre-test estimator approaches that of the unrestricted estimator. This has the effect of decreasing the maximum risk that $\hat{\sigma}_j^2$ attains but at the penalty of increasing its minimal risk value. A converse argument can be given for a decrease in the test size. These features suggest a strategy of selecting an α level which simultaneously maximizes the risk lost from reducing the modal value and minimizes the risk gain from increasing the minimum value of the risk functions. Some research along these lines is discussed in Section 2.4.

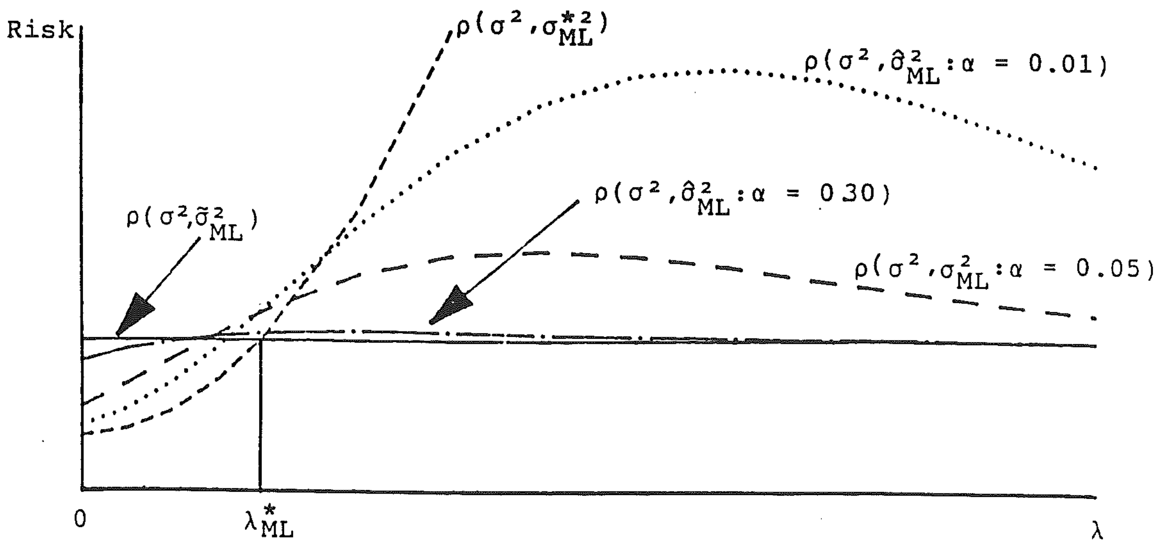

FIGURE 2.2.2: Typical risk functions for $\tilde{\sigma}_{ML}^2$, σ_{ML}^{*2} and $\hat{\sigma}_{ML}^2$.

(g) Pre-testing is never the preferred strategy, and it can be the worst alternative. Nevertheless, in realistic applied situations, involving moderate degrees of freedom and few restrictions on the regression model's parameters, naively imposing the restrictions can impose a severe risk penalty while the cost of pre-testing is relatively small.

(h) Among the component estimators of σ^2 considered, under a minimax criterion with respect to risk, those based upon the principle of minimum mean squared error are preferable, when constructing the pre-test estimator of σ^2. Clarke *et al.* (1987b) show that the pre-test estimator $\hat{\sigma}^2_M$, though composed of the minimum MSE unrestricted and restricted (when H_0 is true) estimators of σ^2, is not itself best invariant.

(i) The risk of the restricted estimator is smaller than that of the unrestricted estimator and of the pre-test estimator when the restrictions are true. The intuitive reasoning is analogous to that given in point (a). The restricted estimator continues to dominate the pre-test and unrestricted estimators for the range of $\lambda \in [0,\lambda^*_j)$, where λ^*_j is as defined previously. However, as the hypothesis error grows and approaches infinity, the risk of the restricted estimator is unbounded, while the pre-test risk approaches that of the unrestricted estimator. The common sense argument aligns with that presented in point (c) above.

For certain values of c the risk of $\hat{\sigma}^2_L$ and of $\hat{\sigma}^2_M$ approaches that of the unrestricted estimator from below. That is, the pre-test estimator can strictly dominate the unrestricted estimator. This feature, which is noted by Ohtani (1988a) rather than Clarke *et al.* (1987b), contrasts with the results found when estimating E(y) (or β) after a pre-test for linear restrictions. It does, however, also occur when estimating the error variance after a pre-test for homogeneity, in the two sample model, as we shall see in Section 2.3. Ohtani (1988a), extends the work of Clarke *et al.* (1987b), by deriving the improved estimator of the variance proposed by Stein (1964), which dominates the unrestricted estimator, $\tilde{\sigma}^2_M$. Ohtani shows that this estimator, say $\hat{\sigma}^2_S$, is in fact a pre-test estimator with a particular choice of critical value and is given by,

$$\hat{\sigma}_S^2 = \begin{cases} \tilde{\sigma}_M^2 & \text{if } u > v/(v+2) \\ \sigma_M^{*2} & \text{if } u \le v/(v+2). \end{cases}$$

Using numerical evaluations Ohtani proposes that $\hat{\sigma}_S^2$ has the minimum risk among the pre-test estimators which dominate the unrestricted estimator.

Gelfand and Dey (1988a) also consider a class of Stein (1964) estimators of the disturbance variance in a linear regression model. Among other things, they prove that $\hat{\sigma}_S^2$ dominates the unrestricted estimator under squared error loss. Further, they generalize this result by obtaining an improved estimator of σ^2 when the null hypothesis is tested against a sequence of nested hypotheses. See also Gelfand and Dey (1988b).

The above discussion assumes that the model (2.2.1) is properly specified. Rarely, in econometrics, would we believe this to be the situation: we invariably work with mis-specified regression models due to lack of data, measurement error, ignorance, or simplification (Mittelhammer (1984)). The pre-test literature, however, has paid scant attention to the sampling properties of the estimators of the parameters of a mis-specified linear regression model, after a pre-test for linear restrictions. Exceptions are Ohtani (1983), Mittelhammer (1984), Giles (1986), Ohtani (1987a), and Giles and Clarke (1989).

Ohtani (1983) derives the MSE of the pre-test predictor in the linear regression model when one of the regressors is a proxy variable. The use of a proxy variable usually arises from data constraints, and can be viewed as a mis-specification of the model. We exclude a relevant (unobservable) variable but include a somewhat irrelevant (but observable) variable. The pre-test in question is of the significance of the coefficient of the omitted variable. Assume that the true data generating process is

$$s = x\beta + z\gamma + e \; ; \quad e \sim N(0,\sigma^2 I_T) \tag{2.2.16}$$

where s and x are (T×1) vectors of observations and z is the (T×1) vector of the unobservable variable. Let p be the proxy variable for z, so the model including the proxy variable is

$$s = x\mathcal{L} + p\eta + w, \tag{2.2.17}$$

and that which excludes the unobservable variable is written as

$$s = x\mathcal{L}^* + l, \tag{2.2.18}$$

where $w=x(\beta-\mathcal{L})+z\gamma-p\eta+e$ and $l=x(\beta-\mathcal{L}^*)+z\gamma+e$. However, we assume that they are both regarded as $N(0,\sigma^2 I_T)$ random variates.

The hypothesis of interest is $H_0:\gamma=0$ (vs. $H_1:\gamma\neq0$) and, as z is unobservable, the test statistic is constructed from (2.2.18). Ohtani considers the test statistic $G=\hat{\eta}^2\left[(x'x)(p'p)-(x'p)^2\right]/(x'x)\hat{\sigma}^2$, where $\hat{\sigma}^2=(s-x\hat{\mathcal{L}}-p\hat{\eta})'(s-x\hat{\mathcal{L}}-p\hat{\eta})/(T-2)$ and $\hat{\mathcal{L}}$ and $\hat{\eta}$ are the least squares estimators of $\mathcal{L}$ and η. G has a doubly non-central F distribution with 1 and (T-2) degrees of freedom and non-centrality parameters $\left(t_\gamma^2 r_{zp.x}^2, t_\gamma^2(1-r_{zp.x}^2)\right)$, where t_γ denotes the ratio of γ to the standard error of $\hat{\gamma}$, and $r_{zp.x}$ is the partial correlation coefficient for z and p given x.

The pre-test estimator for $E\left(s\,|\,x,z\right) = x\beta+z\gamma$ (pre-test predictor) is

$$\hat{s}^{**} = \begin{cases} \hat{s}_o = x\hat{\mathcal{L}}^* & \text{if } G \leq c \\ \hat{s}_p = x\hat{\mathcal{L}}+p\hat{\eta} & \text{if } G > c \end{cases}$$

where $\hat{\mathcal{L}}^*$ is the least squares estimator of $\mathcal{L}^*$ from model (2.2.18), and c is the critical value of the test. Ohtani derives the MSE of $\hat{s}_o$, $\hat{s}_p$ and $\hat{s}^{**}$, and he numerically evaluates the risk functions for various choices of $r_{zp.x}$, and T as a function of t_γ. Among other things, he finds that there exist regions where the pre-test estimator dominates <u>both</u> $\hat{s}_o$ and $\hat{s}_p$; these regions do not exist when the true variable is used (i.e. p=z).

Though he considers a relatively simple model in his analysis, the Appendix of Ohtani (1983) derives the MSE of the pre-test predictor in the general linear regression model[15]

$$y = X\beta + Z\gamma + e \quad ; \quad e \sim N(0,\sigma^2 I_T) \tag{2.2.19}$$

where Z is a (T×p) matrix of the unobservable variables, γ is a (p×1) vector of parameters, and y, X, and e are as defined for model (2.2.1). The matrix P is the proxy matrix for Z, so the model to be estimated is

$$y = X\eta_1 + P\eta_2 + w = X_*\eta + w \tag{2.2.20}$$

where $X_*=(X,P)$, $\eta'=(\eta_1',\eta_2')$ and η_1 and η_2 are (k×1) and (p×1) respectively. The hypothesis of interest is $H\eta = h$, where H is $\left(m_*\times(k+p)\right)$, h is $(m_*\times 1)$; both H and h contain known elements. The unrestricted least squares estimator of η is $\eta_U=S_*^{-1}X_*'y$, where $S_* = X_*'X_*$, while the restricted least squares estimator of η is $\eta_R=\eta_U-S_*^{-1}H'[HS_*^{-1}H']^{-1}(H\eta_U-h)$.

The Wald (Lagrange Multiplier) test statistic in this case is $u_*=(H\eta_U-h)'[HS_*^{-1}H']^{-1}(H\eta_U-h)/m_*\hat{\sigma}_*^2$, where $\hat{\sigma}_*^2=(y-X_*\eta_U)'(y-X_*\eta_U)/v_*$, $v_*=T-(k+p)$. u_* has a doubly non-central F distribution with m_* and v_* degrees of freedom and non-centrality parameters λ_{*n} and λ_{*d}, where

$$\lambda_{*n} = (HS_*^{-1}X_*'X_{**}\beta_*-h)'[HS_*^{-1}H']^{-1}(HS_*^{-1}X_*'X_{**}\beta_*-h)/2\sigma^2 ,$$

$$\lambda_{*d} = \beta_*'X_{**}'M_*X_{**}\beta_*/2\sigma^2 ,$$

$M_*=I-X_*S_*^{-1}X_*'$, $X_{**}=(X,Z)$ and $\beta_*'=(\beta',\gamma')$. The pre-test estimator of $E(y|X,Z)=X\beta+Z\gamma$ is

$$X_*\eta_P = \begin{cases} X_*\eta_U & \text{if} \quad u_* > c \\ X_*\eta_R & \text{if} \quad u_* \le c \end{cases} .$$

Ohtani shows that

$$\rho\left(E(y),X_*\eta_p\right) = \sigma^2\left(2\lambda_{*d} + k + (4\lambda_{*n}-m_*)P_{20}^* - 2\lambda_{*n}P_{40}^*\right), \tag{2.2.21}$$

where $P_{ij}^*=\Pr.\left[F''_{(m_*+i,v_*+j;\lambda_{*n},\lambda_{*d})}<\left(cm_*(v_*+j)\right)/\left(v_*(m_*+i)\right)\right]$, $i,j=0,1,2\ldots$.

Unaware of Ohtani's results, Mittelhammer (1984) analyses a similar

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[15] Ohtani follows the approach of Toyoda (1976).

problem. He compares the risk properties of the least squares, the restricted least squares, the pre-test and the Stein rule estimators of the prediction vector when we omit relevant variables. The pre-test is for exact linear restrictions on the coefficient vector of the included regressors, as detailed in equation (2.2.2). This is a special case of Ohtani's problem where now Z is the matrix of omitted regressors, $P\eta_2=0$, and so, $X_*=X$, $v_*=v$, $H=R$, $h=r$. Then the test statistic u_* collapses to the test statistic u, as given in equation (2.2.6), which now has a doubly non-central F distribution with m and v degrees of freedom and non-centrality parameters

$$\lambda_n = \left(RS^{-1}X'Z\gamma+\delta\right)'[RS^{-1}R']^{-1}\left(RS^{-1}X'Z\gamma+\delta\right)/2\sigma^2 ,$$

$$\lambda_d=\gamma'Z'MZ\gamma/2\sigma^2 ,$$

and $M=I-XS^{-1}X'$. λ_n can be regarded as a measure of the specification error; the error arising from H_0 via δ and the bias from omitting the variables via $X'Z\gamma$. So, the pre-test estimator of E(y) is

$$X\hat{b} = \begin{cases} Xb & \text{if } u > c \\ Xb^* & \text{if } u \le c \end{cases} . \tag{2.2.22}$$

From equation (2.2.21) (or Mittelhammer (1984)), the risk of $X\hat{b}$ is

$$\rho\left(E(y),X\hat{b}\right) = \sigma^2\left(2\lambda_d+k+(4\lambda_n-m)P^d_{20}-2\lambda_n P^d_{40}\right) \tag{2.2.23}$$

where

$$P^d_{ij} = \Pr.\left[F''_{(m+i,v+j;\lambda_n,\lambda_d)}<\left(cm(v+j)\right)/\left(v(m+i)\right)\right]. \tag{2.2.24}$$

Mittelhammer shows, if the model is mis-specified in this way, that the risks of the unrestricted and restricted least squares estimators are

$$\rho\left(E(y),Xb\right) = \sigma^2\left(2\lambda_d+k\right), \tag{2.2.25}$$

and

$$\rho\left(E(y),Xb^*\right) = \sigma^2\left(2(2\lambda_n+\lambda_d)+(k-m)\right) . \tag{2.2.26}$$

Comparing the risk functions (2.2.23), (2.2.25) and (2.2.26), Mittelhammer notes the following points:

(j) The risk of Xb^* is equal to or superior to that of Xb if $\lambda_n \leq m/2$; the same condition as in the properly specified case, allowing for the redefinition of the non-centrality parameter. This is independent of the value of λ_d. Note, however, if the restrictions are true ($\delta=0$) then λ_n need not be zero. This will only result if the excluded regressors are orthogonal to those that have been included ($X'Z=0$) or $Z\gamma=0$. Consequently, imposing <u>valid</u> restrictions <u>does not</u> guarantee that Xb^* is superior to Xb. Furthermore, the use of correct prior information does not ensure that the pre-test estimator has smaller risk than the least squares estimator.

(k) Sufficient conditions for $X\hat{b}$ or Xb to have equal or superior risk are $\lambda_n \leq m/4$, if $\rho\left(E(y),X\hat{b}\right) \leq \rho\left(E(y),Xb\right)$ or $\lambda_n \geq m/2$ if $\rho\left(E(y),X\hat{b}\right) \geq \rho\left(E(y),Xb\right)$. These are equivalent to those reported when the model is properly specified and are independent of λ_d.

(l) As $\lambda_d \to \infty$ the risks of $X\hat{b}$, Xb^* and Xb are unbounded, though the risk differences are bounded (given λ_n).

Giles (1986), unaware of Ohtani (1983), considers the pre-test estimator in a regression model which is mis-specified through the inclusion of irrelevant regressors. That is $Z\gamma=0$ in model (2.2.19) but $P\eta_2$ is included in model (2.2.20) and contains irrelevant data. So, assuming $H_0: H\eta=h$ vs $H_1: H\eta \neq h$, this implies that $\lambda_d^*=0$ and $\lambda_n^*=(H\eta-h)'[HS_*^{-1}H']^{-1}(H\eta-h)/2\sigma^2$. So, when the model is over-fitted the risk functions of the estimators are the same except for a scaling of the non-centrality parameter. Giles notes that if the restrictions involve only η_1 then including extraneous regressors has no effect on the risk comparisons.

The pre-test estimator of the error variance, when the regression model is mis-specified through the omission of regressors is considered by Giles and Clarke (1989). They show that the risk functions of $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$, when regressors are wrongly excluded from the model are

$$\rho(\sigma^2,\tilde{\sigma}^2_{ML}) = \sigma^4\left(2(v+4\lambda_d)+(2\lambda_d-k)^2\right)/T^2, \tag{2.2.27}$$

$$\rho(\sigma^2,\sigma^{*2}_{ML}) = \sigma^4\left[2\left(m+v+4(\lambda_n+\lambda_d)\right)+\left(m-k+2(\lambda_n+\lambda_d)\right)^2\right]/T^2, \tag{2.2.28}$$

$$\rho(\sigma^2,\hat{\sigma}^2_{ML}) = \rho(\sigma^2,\tilde{\sigma}^2_{ML})+T^{-2}\sigma^4\left[2mvP^d_{22}-2mTP^d_{20}+4\lambda_n vP^d_{42}+\left(m(m+2)\right.\right.$$

$$\left.\left.-4\lambda_n T\right)P^d_{40}+4(m+2)\lambda_n P^d_{60}+4\lambda_n^2 P^d_{80}+4\lambda_d(mP^d_{24}+2\lambda_n P^d_{44})\right]. \tag{2.2.29}$$

The risks depend on the data only through T, k, m and, the non-centrality parameters. They numerically evaluate the risk expressions, for various choices of the arguments, as functions of the non-centrality parameters and they find, qualitatively, that the mis-specification affects the results in a similar manner to that noted by Mittelhammer (1984). In particular, imposing correct restrictions need not imply smaller risk than if the prior information was ignored or if an initial pre-test was undertaken.

2.3 Homoscedasticity Pre-test Estimators

Frequently, in applied econometric research, we wish to estimate models for which we suspect that the assumption of a scalar error covariance matrix is invalid. For example, the errors may be autocorrelated or the observations may be drawn from different populations, which may result in different error variances. In this situation, least squares is generally an unbiased but inefficient estimator of the coefficient vector; the generalized least squares estimator (GLS) is minimum variance unbiased. So, an incentive exists to test for the presence of a non-spherical error covariance matrix prior to estimating the model's parameters. In this section we consider one such case, that of pre-testing for homogeneity of the error variances in the two-sample linear model.[16] That is, we assume that model (1.3.1) comprises two samples, with T_1 and T_2 observations ($T_1+T_2=T$) respectively, which have a common location vector, β,[17] but possibly different variances:

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\beta + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \quad , \quad \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \sim N\left[0, \begin{bmatrix} \sigma_1^2 I_{T_1} & 0 \\ 0 & \sigma_2^2 I_{T_2} \end{bmatrix}\right] \tag{2.3.1}$$

or more compactly,

$$y = X\beta + e, \qquad e \sim N(0,\Sigma), \tag{2.3.2}$$

where y_i and e_i are $(T_i \times 1)$, X_i is $(T_i \times k)$ and $k<T_i$, $i = 1,2$. We consider the estimation of the parameters β and σ_1^2, given the uncertainty about whether the second sample comes from the same population as the first. If the variances are equal then the two samples may be pooled, and the usual least squares (and maximum likelihood) estimator of β, $b_A=S^{-1}X'y$, is BLUE. An unbiased estimator of σ_1^2 $(=\sigma_2^2)$ is[18] $s_A^2=(v_1s_1^2+v_2s_2^2)/(v_1+v_2)$ where $v_i=T_i-k$, $s_i^2=(y_i-X_ib_i)'(y_i-X_ib_i)/v_i$, $b_i=S_i^{-1}X_i'y_i$, $S_i=(X_i'X_i)$, $i=1,2$. We will call b_A and s_A^2 the always-pool estimators of β and σ_1^2.

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16 We will not analyse the other obvious case of pre-testing for autocorrelated errors. See Fomby and Guilkey (1978), Judge and Bock (1978), Griffiths and Beesley (1984), King and Giles (1984), Giles and Beattie (1987), and Judge *et al.* (1985a).

17 We do not consider the pre-test problem of estimating the parameters of the two-sample linear regression model after a pre-test for equality of the location vectors when the scale parameters are possibly unequal. The risk properties of this pre-test estimator have received little attention in the literature as the traditional test statistics are not exact. Two pertinent studies are Griffiths and Judge (1989) and Ozcam and Judge (1989).

18 Alternative weights could be assigned to s_1^2 and s_2^2 in the construction of such an estimator.

If the variances are unequal, a feasible GLS estimator of β is the 'two-step' Aitken estimator (2SAE) $b_N=[S_1/s_1^2+S_2/s_2^2]^{-1}[X_1'y_1/s_1^2+X_2'y_2/s_2^2]$ and an unbiased estimator of σ_1^2 is $s_N^2=s_1^2$. We will call b_N and s_N^2 the never-pool estimators of β and σ_1^2.[19]

The usual procedure, to decide which estimators of β and σ_1^2 to use, is to undertake a preliminary test of the hypothesis

$$H_0 : \sigma_1^2 = \sigma_2^2 \quad \text{vs} \quad H_1 : H_0 \text{ not true.}$$

The alternative hypothesis can be one- or two-sided depending on the researcher's prior beliefs.[20] A test statistic for homoscedasticity is $J=s_2^2/s_1^2$ (or $J^*=s_1^2/s_2^2$, depending on H_1), for which the density function is $f(J)=\phi^{-1}f\left(F_{(v_2,v_1)}\right)$, where $F_{(v_2,v_1)}$ is a central F random variate with v_2 and v_1 degrees of freedom. $\phi=(\sigma_1^2/\sigma_2^2)$ is a measure of the hypothesis error, for when H_0 is true $\phi=1$. As ϕ is typically unknown, we usually test the hypothesis under the null distribution and, assuming for simplicity the one-sided alternative $H_1:\sigma_1^2<\sigma_2^2$, we accept H_0 if $J\leq F^{\alpha}_{(v_2,v_1)}=c$ where the critical value of the test, c, corresponds to a test of size α and we have $\int_0^c dF_{(v_2,v_1)}=\Pr.\left(F_{(v_2,v_1)}\leq c\right)=(1-\alpha)$.

If the outcome of the pre-test suggests that the variances are equal ($J\leq c$) then we estimate β and σ_1^2 using the always-pool estimators b_A and s_A^2,

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[19] Using s_i^2 as the never-pool estimator of σ_i^2 and in (2.3.4), takes no account of the fact that each b_i is an estimator of the same β. One could conceivably incorporate this information to obtain a (perhaps) more efficient estimator of the error variances and of β (at least in finite samples as there is no difference asymptotically). See Judge *et al.* (1985a) for a discussion and further references. I am grateful to Mike Veall for reminding me of these alternatives. In this thesis we will follow the literature by using b_N and s_N^2 as the never-pool estimators of β and σ_1^2.

[20] An interesting discussion of this issue is contained in Greenberg (1980). See also Judge *et al.* (1985a).

respectively. Conversely, we employ the never-pool estimators, b_N and s_N^2, if J>c; i.e. when we reject H_0. After such a (pre-)testing procedure, the estimators of β and σ_1^2 actually reported are the pre-test estimators

$$b_p = \begin{cases} b_N & \text{if } J > c \\ b_A & \text{if } J \le c \end{cases}, \tag{2.3.3}$$

(or, $b_P = I_{[0,c]}(J)b_A + I_{(c,\infty)}(J)b_N$, where $I_{[.,.]}(J)$, we recall, takes the value unity if J lies within the subscripted range and zero otherwise) and

$$s_p^2 = \begin{cases} s_N^2 & \text{if } J > c \\ s_A^2 & \text{if } J \le c, \end{cases} \tag{2.3.4}$$

(or, $s_P^2 = I_{[0,c]}(J)s_A^2 + I_{(c,\infty)}(J)s_N^2$).[21]

We will consider first the risk, under squared error loss, of s_N^2, s_A^2, and s_P^2.[22] $(v_i s_i^2/\sigma_i^2) \sim \chi^2_{v_i}$, and s_i^2 is unbiased (i=1,2), so the risk of s_N^2 is

$$\rho(\sigma_1^2, s_N^2) = \mathrm{var}(\sigma_1^2, s_N^2) = 2\sigma_1^4/v_1 = 2\sigma_2^4\phi^2/v_1. \tag{2.3.5}$$

As $(v_1 s_1^2/\sigma_1^2)$ and $(v_2 s_2^2/\sigma_2^2)$ are independent Chi-square random variates, the risk of the always-pool estimator, s_A^2, is

$$\rho(\sigma_1^2, s_A^2) = \sigma_2^4\left(\phi^2(2v_1+v_2^2) - 2v_2^2\phi + v_2(v_2+2)\right)/(v_1+v_2)^2. \tag{2.3.6}$$

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[21] In the case of a two-sided alternative the pre-test estimators are $b_P' = I_{[0,c_1)}(J)b_N + I_{[c_1,c_2]}(J)b_A + I_{(c_2,\infty)}(J)b_N$

and $s_P^{2\prime} = I_{[0,c_1)}(J)s_N^2 + I_{[c_1,c_2]}(J)s_A^2 + I_{(c_2,\infty)}(J)s_N^2$,

where c_1 and c_2 are critical values such that $\int_{c_1}^{c_2} dF_{(v_2,v_1)} = (1-\alpha)$.

[22] Some of the studies have considered this problem from the aspect of pooling samples from two normal populations, rather than pooling samples relating to linear regressions. The results obtained from the former research are applicable to the latter situation by simply changing the degrees of freedom.

The always-pool estimator is only unbiased when the variances are equal. Otherwise, its bias is negative, and increases in value as σ_1^2 becomes small relative to σ_2^2. However, s_A^2 has less sampling error than the never-pool estimator, as it uses the extra v_2 degrees of freedom.

Bancroft (1944) was the first to consider the sampling properties of the pre-test estimator, s_P^2. He derives its bias and variance and evaluates its MSE, for various values of $c \in [0,\infty]$, of two small sample cases.[23] He finds that the bias of the pre-test estimator is much smaller than that of the always-pool estimator when ϕ is close to zero: that range of ϕ where the bias of the always-pool estimator is highest. Intuitively, this arises because the pre-test is doing the right thing when ϕ is small; rejecting the hypothesis. From the MSE comparisons Bancroft finds that the preliminary test with c=1 produces a MSE equal to or smaller than that of the never-pool estimator for all possible values of ϕ.

Toyoda and Wallace (1975) also examine the properties of the pre-test estimator. From their results (or Bancroft's allowing for the change in H_1) the risk of s_P^2 is

$$\rho(\sigma_1^2,s_P^2) = \rho(\sigma_1^2,s_N^2)+\sigma_2^4\Big(\phi^2 v_2\Big[2v_1(v_1+v_2)Q_{02}-(v_1+2)(v_2+2v_1)Q_{04}\Big]+2v_1v_2\phi$$

$$\cdot\Big[v_1Q_{22}-(v_1+v_2)Q_{20}\Big]+v_1v_2(v_2+2)Q_{40}\Big)\Big/\Big(v_1(v_1+v_2)^2\Big) \ , \qquad (2.3.7)$$

where, for i,j=0,1,2,...,

$$Q_{ij} = \Pr.\Big[F_{(v_2+i,v_1+j)} < \Big(v_2(v_1+j)c\Big)\Big/\Big(v_1(v_2+i)\Big)\Big]. \qquad (2.3.8)$$

Figure 2.3.1 illustrates typical risk functions for s_N^2 and s_A^2, and for s_P^2, for various values of $c\in(0,\infty)$. Note that when c=0 we always reject

[23] Bancroft assumes $\sigma_1^2 > \sigma_2^2$ if H_0 is false.

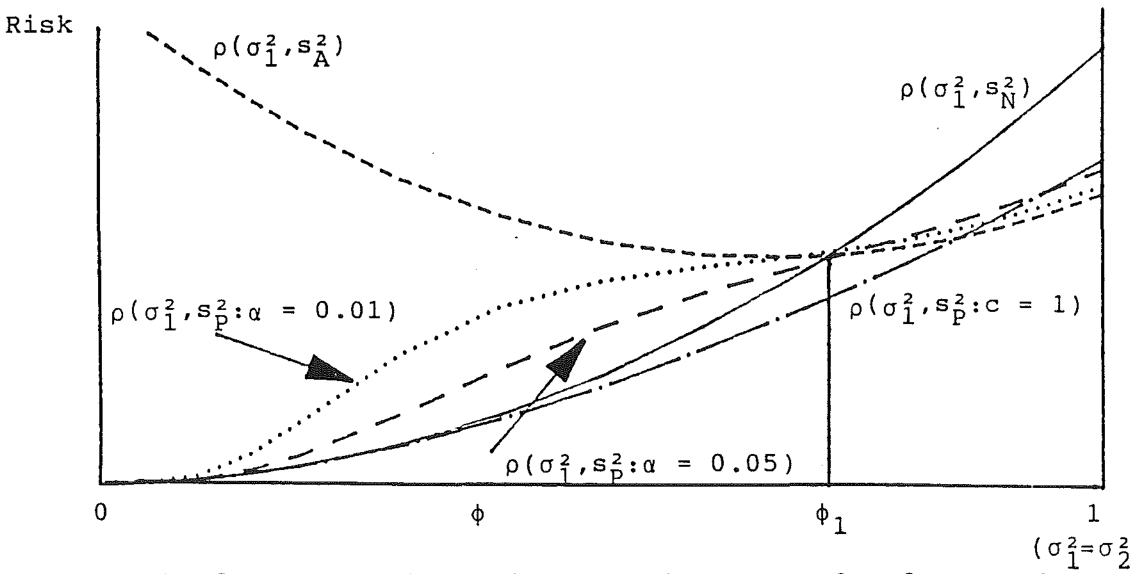

FIGURE 2.3.1: Typical risk functions for s_N^2, s_A^2, and s_P^2.

the hypothesis and so, $\rho(\sigma_1^2, s_P^2) = \rho(\sigma_1^2, s_N^2)$. Conversely, $\rho(\sigma_1^2, s_P^2) = \rho(\sigma_1^2, s_A^2)$ when $c = \infty$, so that we always accept the hypothesis. This figure highlights the following points:

(a) Comparing equations (2.3.5) and (2.3.6), there are two possible values of ϕ, ϕ_1 and ϕ_2, for which $\rho(\sigma_1^2, s_A^2)$ and $\rho(\sigma_1^2, s_N^2)$ intersect, provided that $v_1 v_2 - 4v_1 - 2v_2 \neq 0$ (see Toyoda and Wallace (1975)). In any particular case only one of these values, say ϕ_1, will lie in the interval (0,1]. If $0 < \phi < \phi_1$ then the never-pool estimator dominates the always-pool estimator. Intuitively, the variances are so different that the gain in sampling error from the extra degrees of freedom is outweighed by the bias from pooling the (unequal) variances. Alternatively, if $\phi_1 < \phi < 1$, then the always-pool estimator has smaller risk than the never-pool estimator.

(b) There exist some values of $c \in (0,2)$ such that the pre-test estimator strictly dominates the never-pool estimator for all possible values of ϕ, $0<\phi\leq 1$. Though these particular pre-test estimators do not dominate the always-pool estimator for all ϕ, they do so for a wide range of ϕ. It is only within the neighbourhood of $\phi=1$ that the risk of s_A^2 is smaller. Ohtani and Toyoda (1978) prove, for a given value of ϕ and $c \in [0,1]$, that the minimum pre-test risk occurs when $c=1$; so the never-pool estimator is inadmissible and specifically, is dominated by the pre-test estimator (at least) with $c=1$.

These features raise the question of an optimal pre-test critical value. Toyoda and Wallace (1975) show that if the objective is to maximise average efficiency, then the optimal pre-test critical value is $c=1$, and this value is independent of the degrees of freedom in each sample. On the other hand, Ohtani and Toyoda (1978) consider a minimax regret criterion and find that in this case the optimal critical value depends on the degrees of freedom and varies from about 1.7 to 2.8. Similarly, Bancroft and Han (1983) seek the critical value such that the efficiency of the pre-test estimator to that of the never-pool estimator is at least one. They find that these values are significantly higher than those implied by the traditional levels of 0.01 and 0.05. These studies will be mentioned further in Section 2.4.

To this point the discussed studies have only considered the first two moments (and hence, the risk) of pre-test estimators. Ideally, of course, we would like the exact finite sampling distribution of a pre-test estimator; not only would this enable us to consider higher order moments but it would also mean we could investigate the effects of pre-testing on interval, rather than point, estimation. The only study to date which does exactly this is Giles (1989). He derives the exact distribution of the pre-test estimator initially analysed by Bancroft (1944), s_P^2 , and finds:

(c) The pre-test estimator has a uni-modal density.

(d) As $\phi \to 1$ (H_0 is true), the distribution of s_P^2 approaches that of the always-pool estimator. This is as we would hope; as $\phi \to 1$ the pre-test should be leading us to accept the hypothesis, and so pool the samples, more often. When H_0 is true we are only rejecting the hypothesis $\alpha\%$ of occasions. On the other hand, as $\phi \to 0$, $s_P^2 \to s_N^2$ and so the distribution of the pre-test estimator approaches that of the never-pool estimator.

(e) If $\phi=1$ (H_0 is true) the probability content of a confidence interval based on the pre-test estimator is lower than that based on the always-pool estimator but higher than that based on the never-pool estimator. Intuitively, when H_0 is true the samples should be pooled, but by pre-testing there is still an $\alpha\%$ chance of not pooling the samples and reporting a confidence interval based on the never-pool estimator. A parallel converse result occurs for very small ϕ. Note that these findings, as do those reported in point (d), mirror the corresponding risk results.

Yancey *et al.* (1983) consider a related problem to that investigated here. They derive, within the framework of the orthonormal linear model, the risk functions of the pre-test estimators which result first, after a pre-test of $H_0^Y:\sigma^2=\sigma_0^2$ vs $H_1^Y:\sigma^2\neq\sigma_0^2$, where σ_0^2 is some known value and secondly, after a pre-test of $H_0^J:\sigma^2\geq\sigma_0^2$ vs $H_1^J:\sigma^2<\sigma_0^2$. They find that the risk of the inequality pre-test estimator (IPE) is equal to or less than the risk of the conventional equality pre-test estimator (CEPE), except in the neighbourhood of the equality of σ^2 and σ_0^2. When the direction of H_0^J is correct, the risk of the IPE is always equal to or less than that of the usual least squares estimator. They also consider the equality pre-test estimator which uses the minimum mean square error estimator of σ^2 as the unrestricted component (MEPE), rather than the usual least squares estimator. For the range of the parameter space $\zeta=(\sigma^2/\sigma_0^2)\in[1,\infty)$ the risk of MEPE is less than the risk of CEPE, while for $\zeta\in(0,1)$ their risks are approximately equal, though the MEPE has both a higher maximum and a lower minimum over this range. No one of the analysed pre-test estimators dominates any other.

Ohtani (1987a) extends the analysis of Yancey *et al.* (1983) by allowing for the possibility of omitted variables. He assumes that the true regression model is

$$y = X\beta + Z\gamma + \varepsilon \quad ; \quad \varepsilon \sim N(0,\sigma^2 I_T),$$

where y is a $(T\times 1)$ vector of a dependent variable, X and Z are $(T\times k)$ and $(T\times p)$ matrices of non-stochastic independent variables, β and γ are $(k\times 1)$ and $(p\times 1)$ vectors of unknown coefficients, and ε is a $(T\times 1)$ vector of normal disturbance terms, but supposes that a researcher specifies the model as

$$y = X\beta + u \quad ; \quad u = Z\gamma + \varepsilon\ ,$$

and behaves as if $u \sim N(0,\sigma^2 I_T)$ when in fact $u \sim N(Z\gamma,\sigma^2 I_T)$.

The pre-test of interest is $H_0^y:\sigma^2=\sigma_0^2$ vs $H_1^y:\sigma^2\neq\sigma_0^2$ or $H_A^y:\sigma^2>\sigma_0^2$, using the test statistic $B=e'e/\sigma_0^2$, where $e=y-X\hat{\beta}$, $\hat{\beta}=S^{-1}X'y$. Ohtani shows that $B\sim\zeta\chi'^2_{(v;\theta)}$ where $v=T-k$, $\theta=\gamma'Z'MZ\gamma/\sigma^2$, $M=I-XS^{-1}X'$, and $\zeta=\sigma^2/\sigma_0^2$. He then derives the risk, under quadratic loss, of the pre-test estimator relative to the estimator which ignores the prior information, $s^2=e'e/v$. He shows that if the alternative hypothesis is one-sided then there exists a family of pre-test estimators which strictly dominate s^2 for all possible values of ζ. This dominance is robust to the specification error. For the example that he considers (v=20) he postulates, through does not prove, that the pre-test estimator with a size of 45% has the minimum risk of this family of dominating pre-test estimators.[24] These results do not carry over to $H_0^y: \sigma^2 = \sigma_0^2$ vs $H_1^y: \sigma^2 \neq \sigma_0^2$. Then Ohtani suggests using s^2.

As we noted in the introduction to this chapter, the analytic techniques required to derive and compute the sampling properties of the estimators of the location vector in the two-sample heteroscedastic model are different from those of the estimators considered so far; in particular, the risk of this pre-test estimator is not composed of terms which involve non-central (or central) Chi-square random variates. This arises because one of its component estimators is the 2SAE, whose own sampling properties, interestingly, were derived only a decade ago (Taylor (1977, 1978)) and can be written in terms of Gauss' hypergeometric functions. So, to analyse and compute the sampling properties of this pre-test estimator we do not use the same techniques that we have for the other discussed pre-test estimators. To indicate these differences we include here a brief review of the literature that investigates the sampling properties of the estimator of β, after a pre-test for homogeneity, in the two-sample model. There has been little research on this particular pre-test problem and the studies work within the framework of either the orthonormal model (that is, they assume that $X_1'X_1' = X_2'X_2 = I_k$; see, Ohtani and Toyoda (1980), Yancey *et al.* (1984) and Judge and Yancey (1986)); or a reparameterised version of the model (2.3.1) (see, Taylor (1977, 1978), Greenberg (1980) and Mandy (1984)):

$$y = X^*\beta^* + e, \tag{2.3.9}$$

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[24] In Chapters Five and Six of this thesis we derive the critical value which results in a minimum of the pre-test risk function for some similar but more general problems. The approach used in these chapters can also be used to prove that a minimum of the pre-test risk function, considered by Ohtani, occurs when the critical value is v (whether or not we have omitted regressors). So, for the example investigated by Ohtani the optimum critical value is 20 which is equivalent to a test size of 45.8%. The proof of this proposition is available upon request.

where $X^*=XP$, $\beta^*=P^{-1}\beta$, and $P=T\times\text{diag}\left((1+\mu_i)^{-1/2}\right)$ is a non-singular matrix. The matrix T is chosen so as to diagonalise $X_1'X_1$ and $X_2'X_2$ simultaneously such that $T'X_1'X_1T=\sigma_1^2I_k$ and $T'X_2'X_2T=\sigma_2^2\Lambda_k=\sigma_2^2\text{diag}(\mu_i)$, where μ_i are the roots of the polynomial $|X_2'X_2/\sigma_2^2-\mu X_1'X_1/\sigma_1^2|=0$. $(i=1,2,\ldots,k)$

Taylor (1978) establishes the finite sample moments of the ith element of the 2SAE within the context of (2.3.9). Let this estimator be β^*_{Ni}, $i=1,2,\ldots,k$. We can write

$$\beta^*_{Ni}-\beta^*_i = (1+\mu_i)\left[\theta_iX^{*\prime}_{1i}e_1/\sigma_1^2+(1-\theta_i)X^{*\prime}_{2i}e_2/(\sigma_2^2\mu_i)\right]$$

where $\theta_i=(\sigma_1^2s_2^2)/(\sigma_1^2s_2^2+\sigma_2^2s_1^2\mu_i)$. Taylor shows that β^*_{Ni} is an unbiased estimator of β^*_i and the exact MSE of β^*_{Ni} is given by

$$\rho(\beta^*_i,\beta^*_{Ni}) = \text{var}(\beta^*_i,\beta^*_{Ni}) = (1+\mu^*_i)\frac{B(w_1,w_2+2)}{B(w_1,w_2)}\,{}_2F_1(w_1,2,w_1+w_2+2;-\mu^*_i)$$

$$+(1+1/\mu^*_i)\frac{B(w_2,w_1+2)}{B(w_2,w_1)}\,{}_2F_1(w_1,2,w_1+w_2+2;1-1/\mu^*_i)\ , \qquad (2.3.10)$$

where ${}_2F_1(a,b,c;z)$ represents Gauss' hypergeometric function; $w_j=(T_j-k)/2=v_j/2$; $\mu^*_i=v_2\mu_i/v_1=w_2\mu_i/w_1$; and $B(.,.)$ is the beta function[25,26].

The risk of the ith element of the unbiased least squares estimator of β^*, $\beta^*_A = (X^{*\prime}X^*)^{-1}X^{*\prime}y$, is

$$\rho(\beta^*_i,\beta^*_{Ai}) = \text{var}(\beta^*_i,\beta^*_{Ai}) = (1+\mu_i)(\phi^2+\mu_i)/(\phi+\mu_i)^2, \qquad (2.3.11)$$

where, as before, $\phi=\sigma_1^2/\sigma_2^2$. Taylor shows, by comparing (2.3.10) and

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25 Taylor also shows that under appropriate conditions the 2SAE is consistent and asymptotically efficient, with the same asymptotic covariance matrix as the Gauss-Markov (Aitken) estimator, which assumes Σ is known. See also Kariya (1981).

26 Note that if the regressors are orthonormal then $\mu_i = \sigma_1^2/\sigma_2^2$ for all i (see Ohtani and Toyoda (1980)). This results in a data independent risk function but, as we can see, the analytic structure is still fundamentally different from that of the estimators previously discussed.

(2.3.11), that the risk of neither estimator dominates. Nevertheless, he concludes that substantial gains can result from using the 2SAE, depending on the values of v_1, v_2, ϕ, and μ_i.

Greenberg (1980) follows Taylor's approach and derives the risk of the two-sided pre-test estimator, $\beta^{*\prime}_{Pi}$, corresponding to the ith element of b'_P, for the reparameterised model and where the test statistic is J^* rather than J. We can express $\beta^{*\prime}_{Pi}$ as

$$\beta^{*\prime}_{Pi} = I_{(0,c_1)}(J^*)\beta^*_{Ni}+I_{[c_1,c_2]}(J^*)\beta^*_{Ai} +I_{(c_2,\infty)}(J^*)\beta^*_{Ni}.$$

He shows that $\beta^{*\prime}_{Pi}$ is an unbiased estimator of β^*_i, and that its risk is

$$\rho(\beta^*_i,\beta^{*\prime}_{Pi}) = \mathrm{var}(\beta^*_i,\beta^{*\prime}_{Pi}) = \rho(\beta^*_i,\beta^*_{Ai})+$$

$$\int_{d_{2i}}^{d_{1i}} \left[\frac{\mu_i(1-\phi)^2}{(\phi+\mu_i)^2} - \frac{(1+\mu_i)^2}{\mu}\left(\theta_i - \frac{1}{1+\mu_i}\right)^2\right] f(\theta_i)d\theta_i \tag{2.3.12}$$

where $f(\theta_i)$, the marginal density of θ_i, is derived by Taylor (1978), and $d_{ji}=1/(1+c_j\mu_i/\phi)$, j=1,2. Comparing (2.3.12) with the risk functions of the previously discussed pre-test estimators directly indicates its different fundamental structure.

Greenberg shows that no one estimator, of those evaluated, strictly dominates the others. Nevertheless, his results would seem to favour the 2SAE, unless one had a very strong belief that the variances were equal. However, as Greenberg points out, the analysis implicitly assumes a uniform prior on ϕ; the results may certainly be different for a non-uniform prior distribution on ϕ.

Ohtani and Toyoda (1980) derive the risk function of the pre-test estimator, for the orthonormal model, when the alternative hypothesis is $H_1:\sigma_1^2>\sigma_2^2$. They show that in this situation the 2SAE is inadmissible, as it is dominated by the pre-test estimator when the critical value is appropriately chosen. In particular, if one adopts the criterion of minimizing average risk, they show that the optimal critical value is unity. Mandy (1984), by following Greenberg's procedures, generalises Ohtani and Toyoda's result to the non-orthonormal case. He shows that if the direction of the alternative hypothesis is correct then as we would expect, the (inequality) pre-test estimator that takes this directional information into account is superior, in terms of risk, to the two-sided (equality) pre-test estimator analysed by Greenberg (1980). However, of course, if the alternative hypothesis should be $H_1': \sigma_1^2 < \sigma_2^2$ then the inequality pre-test estimator is risk inferior to the equality pre-test estimator.

Yancey *et al.* (1984),[27] within the framework of the orthonormal model, derive the risk functions of "Stein-like always-pool and never-pool" estimators. They show that these estimators strictly dominate their traditional counterparts. They also demonstrate the inadmissibility of the traditional pre-test estimator discussed above. They show that pre-test estimators which use the Stein-like always-pool and never-pool estimators as their components are risk superior to the conventional pre-test estimator. Kennedy and Adjibolosoo (1990) also propose an alternative pre-test estimator. They weight the least squares and the 2SAE estimators by continuous functions of the data,[28] so forming a "Bayesian" (or, in the authors' terminology, a smoothed) pre-test estimator, as the weights can be

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[27] See also Judge and Yancey (1986).

[28] Recall that Cohen (1965) proves the inadmissibility of the traditional pre-test estimator because (basically) it is a discontinuous function of the test statistic (and hence, of the data). See also Zaman (1984).

related to the posterior probability that the error is homoscedastic.[29] Using ignorance priors, their Monte Carlo study shows that the smoothed pre-test estimator does not dominate the traditional pre-test estimator.

2.4 The Choice of Significance Level

In the last two sections we have presented the risk functions of various pre-test estimators. One common feature that these risk functions have is their dependence on the choice of significance level. If the test size is varied, the pre-test risk function changes, and so too, does the difference between the risk of the pre-test estimator and the risk of either of its component estimators. A second common feature is that, for any particular problem, there exists no dominating estimator; in general, the risks of the unrestricted, restricted and pre-test estimators cross somewhere in the parameter space.[30]

As the extent to which the non-sample information is true or false is unknown, these features raise the question: "Is there an optimal choice of test size such that the pre-test risk is as close as possible to that which could conceivably be achieved?". A number of studies have addressed this issue; among other things, the answer depends on the pre-test under investigation and the chosen optimality criterion.

First, we review those studies which have considered the optimal choice of test size after a pre-test for linear restrictions.[31] From Figure

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[29] Further discussions on Bayesian pre-test estimators can be found in, for instance, Zellner and Vandaele (1975), Leamer and Chamberlain (1976), Leamer (1978), Judge and Bock (1983) and Judge *et al.* (1985a). See also Roehrig (1984) for a related discussion.

[30] We did note situations where the pre-test estimator, for an appropriately chosen critical value, dominates one, but not both, of its component estimators.

[31] We assume that the model is properly specified; no research has been undertaken on this issue in a mis-specified model.

2.2.1, the minimum risk that could conceivably be achieved, for all λ, is given by the shaded area; the risk of the restricted estimator is smallest for $\lambda \in [0, m/2]$, while for $\lambda \in [m/2, \infty)$ the unrestricted estimator has the smallest risk.[32] So we desire a choice of test size which results in the risk of the pre-test estimator being as close as possible to the boundary of the shaded area, which is given by $\min\left\{\rho\left(E(y), Xb\right), \rho\left(E(y), Xb^*\right)\right\}$. As α is increased the risk of the pre-test estimator moves down (up) toward the risk of the unrestricted estimator to the right (left) of $\lambda = m/2$. So, there is a trade-off between the proximities of the pre-test risk and the minimum risk boundary. There are various ways of measuring this distance.

One possibility that is discussed in the literature is the criterion of minimax regret; for a given test size, we determine the maximum regret of $\rho\left(E(y) - X\hat{b}\right)$ from the boundary for all λ, then solve for the value of the critical value, c, which results in the minimum maximum regret. This value of c is deemed the optimal critical value. That is, $\min_c\left\{\max_\lambda\left[\rho\left(E(y), Xb\right) - \min\left[\rho\left(E(y), Xb^*\right), \rho\left(E(y), Xb\right)\right]\right]\right\}$. For the case of a single hypothesis, Sawa and Hiromatsu (1973) use this criterion to tabulate minimax regret critical values. They find an optimum value of c of about 1.8.[33] Brook (1972, 1976) by numerical integration, and for the situation of multiple restrictions, chooses values of c, say c^* ,that equate the maximum regret on either side of $\lambda = m/2$. This is a slightly different minimax regret criterion from that just considered. We define the regret function as $\text{reg}\left(E(y), X\hat{b}\right) =$

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[32] Note that James-Stein type estimators exist which have smaller risk than this boundary. See, for instance, Sclove *et al.* (1972) and Judge and Bock (1978).

[33] The restriction is whether to exclude or include a regressor. See Farebrother (1975) for further comments about this research.

$\rho\left(E(y),X\hat{b}\right)-\min\left[\rho\left(E(y),Xb\right),\rho\left(E(y),Xb^*\right)\right]$. Let λ_l (λ_u) be the value of $\lambda\le m/2$ ($>m/2$) such that $\mathrm{reg}\left(E(y),X\hat{b}\right)$ is a maximum and let $d_l(d_u)$ be that value of $\mathrm{reg}\left(E(y),X\hat{b}\right)$. The minimax regret procedure is to find the critical value, c*, which simultaneously minimises d_l and d_u, given that increasing c decreases d_l but increases d_u.

For the conditional mean forecast problem (or, when the regressors are orthonormal), Brook finds that c* is generally very close to two[34]. His results are consistent with Sawa and Hiromatsu's, and suggest that the optimal critical value, under a minimax regret criterion, is approximately two, regardless of the degrees of freedom.[35] This result can give some comfort to researchers who traditionally use the 5% significance level: two is an approximate critical value when the degrees of freedom are moderate to high, say greater than 25, and m>4.

Another way of defining the optimal critical value is as follows. Instead of searching for the maximum regret for each level of α, we could take into account the regret for each value of λ and search for the value of α which minimises their sum or average. That is, minimise the area between the pre-test risk and the minimum possible risk boundary.[36] So, we find the value of c such that $\min_c \int_0^\infty \left[\rho\left(E(y),X\hat{b}\right)-\min\left\{\rho\left(E(y),Xb^*\right),\rho\left(E(y),Xb\right)\right\}\right]d\lambda$.

This criterion is considered by Toyoda and Wallace (1976). For various

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[34] However, as we move away from orthonormality, he finds that c* decreases from two towards one.

[35] See also Wallace (1972), Toyoda and Wallace (1976), Judge and Bock (1978) and Judge *et al.* (1985a).

[36] This is, of course, equivalent to minimising the area below the risk of the pre-test estimator, as the area under the minimum possible boundary does not depend on α.

degrees of freedom, they use an iterative search procedure to compute the optimal value of the critical value. Toyoda and Wallace find that this criterion leads to a critical value of zero (i.e. use the least squares estimator) if the number of restictions is less than five. However, if the number of restrictions is large, then their results are in common with those of Brook (1972, 1976): that is, the optimal critical value is about two. See Toyoda and Wallace (1976) for an intuitive explanation of their result.[37]

This last approach is somewhat Bayesian: it assumes in effect a diffuse or uniform prior for λ. This may be giving too little weight to small λ, as the investigator must believe λ is in the neighbourhood of zero to be pre-testing at all.[38] Wallace (1977) postulates that with a strong prior on λ weighted towards zero, the minimum average risk critical value would be increased.

There has been no research into the choice of an optimal critical value, according to the aforementioned criteria, when estimating the error variance after a pre-test for exact linear restrictions. Ohtani (1988a) considers the criterion of choosing the value of c such that the maximum risk of the pre-test estimator of the error variance is a minimum. He suggests, from numerous numerical investigations, that the Stein (1964) estimator of σ^2 discussed in Section 2.2, which is a pre-test estimator with a critical value of $v/(v+2)$, is the optimal estimator according to this criterion. This is shown by Gelfand and Dey (1988a).

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[37] Note that if we adopt a minimax criterion of minimising the maximum risk then the solution is $c = 0$; that is, always use the unrestricted least squares estimator. See, for instance, Wallace and Ashar (1972) and Bock *et al.* (1973) for further discussion.

[38] This is not always the case. For instance, stepwise regression is an obvious counter example. See, for example, Wallace (1977).

We now review those studies which have considered the question of the optimal test size after a pre-test for homogeneity. Toyoda and Wallace (1975), Ohtani and Toyoda (1978) and Bancroft and Han (1983) each investigate the problem when the parameter being estimated is the error variance, σ_1^2, while Ohtani and Toyoda (1980) seek an optimal critical value for the pre-test estimator of the location vector in the orthonormal model.

Toyoda and Wallace base their choice of optimal critical value on the minimum average risk criterion, with diffuse prior, as discussed above.[39] They prove that the necessary condition for the minimum is attained when c=1 and numerically check the sufficiency and the uniqueness of this minimum. For a variety of choices of degrees of freedom Toyoda and Wallace show that this optimal critical value implies a type one error ranging from 40 to 60 percent; values which are substantially higher than the 1 and 5 percent levels traditionally used by researchers.

A minimax regret criterion is employed by Ohtani and Toyoda (1978). The regret function is defined as $reg(\sigma_1^2,s_P^2)=\rho(\sigma_1^2,s_P^2)-\min\left[\rho(\sigma_1^2,s_P^2|c=1),\ \rho(\sigma_1^2,s_A^2)\right]$. This differs from our earlier definition of the regret function because for this problem, as we noted in the previous section, there exists a family of pre-test estimators which strictly dominate the never-pool estimator. Of this family, the pre-test estimator with a critical value of one has the minimum risk. Now let $\bar{\phi}$ be the value of ϕ such that $\rho(\sigma_1^2,s_P^2)=\rho(\sigma_1^2,s_A^2)$; let d_l^* be the maximum regret for $\phi\in(0,\bar{\phi}]$; and let d_u^* be the

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[39] In fact, they frame the problem as one of maximising average efficiency; that is, $\max_c \int_0^1 \left[\min\left(\rho(\sigma_1^2,s_N^2),\rho(\sigma_1^2,s_A^2)\right)-\rho(\sigma_1^2,s_P^2)\right]d\phi$ which is equivalent to minimising average risk. Note that for this problem the bounds of integration are [0,1] rather than [0,∞], as $\phi\in[0,1]$.

maximum regret for $\phi\in[\bar{\phi},1]$. Then, the optimal critical value, c*, is that value of c for which $d_l^* = d_u^*$. When the alternative hypothesis is one-sided, Ohtani and Toyoda find that under this criterion the optimal critical value depends on the degrees of freedom and varies from about 1.7 to 2.8. This contrasts with the results of Toyoda and Wallace (1975).

Bancroft and Han (1983) investigate yet another criterion: relative efficiency of the pre-test estimator to the never-pool estimator, which is given by $\text{eff}=\left[\left(1/\rho(\sigma_1^2,s_P^2)\right)/\left(1/\rho(\sigma_1^2,s_N^2)\right)\right]$. For given values of v_1, v_2, and α, and a one-sided alternative hypothesis, they numerically solve for the maximum $\left(\text{eff}_{(\max)}\right)$ and minimum $\left(\text{eff}_{(\min)}\right)$ values of eff. For certain values of α, $\text{eff}_{(\min)}$ is larger than unity; that is, the pre-test estimator for this α value strictly dominates the never-pool estimator; and so, they suggest selecting a test size such that $\text{eff}_{(\max)}$ is the largest and $\text{eff}_{(\min)} \geq 1$. This procedure should ensure the largest gain in efficiency. Bancroft and Han find that this criterion results in optimal significance levels in the regions of 30 to 50%, depending on the values of v_1 and v_2.

Ohtani and Toyoda (1980) adopt the criterion of minimising average relative risk when they seek the optimal critical value of the pre-test for homogeneity, prior to estimating the location vector in the orthonormal model. For this model, let b_N, b_A, b_P be the never-pool, always-pool and pre-test estimators of β respectively. b_N is the 2SAE and b_A is the usual least squares estimator. The authors consider a one-sided alternative hypothesis and show, in this situation, that the 2SAE is inadmissible, and it is dominated by the pre-test estimator with a critical value of unity.[40]

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[40] This result is analogous to the one found by Toyoda and Wallace (1975) and Ohtani and Toyoda (1978) when estimating the error variance after a pre-test for homogeneity.

Accordingly, they consider a regret, or relative risk, function given by $reg(\beta, b_P) = \rho(\beta, b_P) - \min\left(\rho(\beta, b_P | c=1), \rho(\beta, b_A)\right)$ and assume that $\phi^* = 1/\phi$ is uniformly distributed over (0,1], or equivalently $\phi \geq 1$. The proper prior probability density function (pdf) of ϕ is ϕ^{-2}, as the proper prior pdf of ϕ^* is 1. So, the criterion is to minimise $G(c) = \int_1^\infty \left[reg(\beta, b_P)\right] \phi^{-2} d\phi$. Ohtani and Toyoda derive the extrema of G(c) and conclude that the optimal critical value for the pre-test is $c^*=1$. This accords with the aforementioned results of Toyoda and Wallace (1975).

So, from these studies, we see the influence of the chosen criterion on the proposed optimal test size. Nevertheless, these results suggest values of α that are significantly different from those traditionally used in practice. Further, depending on the criterion adopted, the optimal critical values may vary with the degrees of freedom. It should be noted, though, that these studies investigate the choice of critical value when the alternative hypothesis is one-sided; their results need not carry over to the case of the two-sided alternative. The papers give little, if any, attention to this issue.

2.5 Concluding Remarks

In this chapter we have reviewed the literature which studies the sampling properties of estimators of the prediction vector (or of the coefficient vector in an orthonormal model) and of the error variance, either after a pre-test for exact linear restrictions on the location vector, or else after a pre-test for homogeneity of the error variance. As we noted in the introduction to this chapter, there are many other pre-test problems investigated in the literature. For instance, a common practice is to (pre-)test for autocorrelation using (say) the Durbin-Watson statistic;

and then, depending on the outcome of the test, either assume that the errors are uncorrelated or else correct for the autocorrelation in some way. Some references for this pre-test problem are Fomby and Guilkey (1978), Judge and Bock (1978), Griffiths and Beesley (1984), King and Giles (1984), Judge *et al.* (1985a) and Giles and Beattie (1987).

Other pre-test estimators which are examined in the literature, but not so far mentioned in this thesis, are:[41]

(a) A specification pre-test estimator that results after a pre-test of Hausman's (1978) specification test (see Gourieroux and Trognon (1984) and Morey (1984)).

(b) A general ridge regression pre-test estimator which uses the least squares estimator when a regression coefficient is "large" and a ridge regression estimator when it is "small" (Srivastava and Giles (1984)). Hill and Judge (1987) also consider a pre-test estimator when multicollinearity is a possible problem.

(c) A seemingly unrelated regressions (SUR) pre-test estimator. If a pre-test indicates that the correlation between equation errors is significantly different from zero then the pre-test estimator is the generalised least squares estimator; otherwise, it is the least squares estimator (see Srivastava and Giles (1987) and Ozcam *et al.* (1988)).

(d) A mixed regression pre-test estimator. We assume in this thesis that the non-sample information is non-stochastic but, the prior beliefs may well be stochastic. Judge *et al.* (1973), Judge and Bock (1978) and Judge and Bock (1983) consider the sampling properties of the mixed estimation pre-test estimator that results from a pre-test of stochastic linear

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[41] This list is not exhaustive; it merely indicates the breadth of pre-test problems that have been investigated in econometrics.

hypothesis. This work builds on the mixed estimation problem studied by Theil and Goldberger (1961) and Theil (1963). The research of Mittelhammer (1981) and Ohtani and Honda (1984) is also worth noting, given our interest in this thesis on estimation in mis-specified models. These papers consider the mixed regression estimator when the linear model is mis-specified due to missing relevant explanatory variables.[42] To my knowledge, the mixed regression pre-test estimator has not been investigated under this form of mis-specification.

(e) Inequality pre-test estimators. The prior information on the parameters may be in the form of inequality, rather than equality, constraints. Judge and Bock (1983), Judge and Yancey (1986), Judge *et al.* (1988) and Yancey *et al.* (1989) consider the sampling properties of various pre-test estimators, as well as other estimators, after a pre-test of inequality constraints.[43]

(f) Multi-stage pre-test estimators. The literature discussed so far in this chapter considers the properties of pre-test estimators after a <u>single</u> pre-test; in practice, of course, it is more common for a researcher to undertake multiple pre-tests. For instance, the estimators of the model's parameters may be reported after, say, pre-tests for auto-correlation, homoscedasticity and linear restrictions. An analytic investigation of such multi-stage pre-test estimators is very difficult, given the (usual) non-independence of the tests.

The only paper to derive the exact risk function of a multi-stage

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[42] See also Kadiyala (1986) and Wijekoon and Trenkler (1989). Both of these studies are apparently ignorant of the earlier work of Mittelhammer (1981) and Ohtani and Honda (1984).

[43] The prior information may relate to the coefficient vector or to the scale parameter.

pre-test estimator is Ozcam and Judge (1989). This paper considers the estimation of the coefficient vector in the two-sample orthonormal heteroscedastic linear regression model when uncertainty exists on whether the variances and the location parameters across the samples are equal. The authors derive and evaluate the exact risk function of a two-stage pre-test estimator which results after a test for the equality of the variances and then a test for the equality of the location parameters.

In this thesis we consider the properties of pre-test estimators after only a single pre-test. Accordingly, we also ignore the so-called pre-test testing literature. This literature considers another aspect of multi-stage pre-testing: the effects of multiple testing on, for instance, the size and power of the tests. Some references to this literature include Gurland and McCullough (1962), Phillips and McCabe (1983), Ohtani and Toyoda (1985), Toyoda and Ohtani (1986), Ohtani (1987b,c, 1988b,c), Ohtani and Toyoda (1988) and Griffiths and Judge (1989).

We have assumed throughout this chapter that the model's disturbances are normally distributed. There is a body of literature which questions this assumption and which considers the estimation of the model's parameters under non-normal errors. We turn our attention to these issues in the next chapter.

CHAPTER THREE

NON-NORMAL DISTURBANCES: BACKGROUND

3.1 Introduction

There is a large body of literature which suggests that the traditional assumption of normally distributed regression errors may sometimes be unrealistic. Economic data series may be generated by processes whose underlying distributions have thicker tails than the normal distribution and, perhaps, infinite variances, thus increasing the frequency of outliers. In this chapter we investigate some of the issues that are raised by this possibility. First, in Section 3.2, we examine the appropriateness of the normality assumption. We follow this with a brief review of the literature which empirically questions this assumption for certain economic data series.

The possibility of non-normal disturbances has led to searches for robust estimators,[1] resulting in such estimators as the M-, L-, and R-estimators. See, for instance, Huber (1972, 1973, 1977, 1981), Hogg (1979), Bickel (1976), Mosteller and Tukey (1977), Koenker and Bassett (1978, 1982), Koenker (1982), Hampel *et al.* (1983), and Judge *et al.* (1985a).

There have also been many studies of the robustness of traditional estimators. In particular, these studies show that the least squares estimator is sensitive to the form of the underlying distribution, because it minimises squared deviations, and so gives a relatively heavy weight to

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[1] An estimator may be said to be robust if its properties are relatively invariant to the form of the underlying distribution.

the tails of the distribution.[2] Various alternative distributions to normality have been investigated: one that has received considerable attention in the literature is the spherically symmetric family of distributions (and its parent distribution, the elliptically symmetric family). This family of distributions allows for both "fat" and "thin" tails and, well known members are the normal and the multivariate Student-t distributions, as well as the Cauchy distribution. It is to this family of distributions that this thesis extends the pre-test literature discussed in the previous chapter. In Section 3.4, we provide some rationale for considering spherically symmetric regression disturbances as an alternative to the assumption of normal errors.

Several studies which analyse regression models with spherically symmetric disturbances are discussed in Section 3.5. Though an exhaustive study (or bibliography) of the literature which considers linear regression models with non-normal disturbances is beyond the scope of this thesis, mention is also made in this section of a number of papers which investigate other forms of non-normal disturbances. The studies discussed there highlight the importance of investigating the robustness of traditional estimators. Some concluding remarks appear in Section 3.6.

3.2 The Normality Assumption

The argument for normally distributed disturbances in economic models is usually based on Haavelmo's (1944) paper. He maintains that the errors can be regarded as the sum of a large number of small, independent terms and so, appealing to the Lindeberg-Lèvy central limit theorem, the distribution

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[2] Clearly, if the error distribution has an infinite variance (for example, if it is the Cauchy distribution) then the least squares estimator will have zero efficiency as it is impossible to obtain a meaningful variance estimator in this case.

of this sum will approach normality.[3] Now, there are various reasons for the inclusion of a disturbance term in a linear regression model: for instance, the net effect of excluded regressors; measurement errors; the effects of random, human behaviour; and the discrepancy between the true and the approximating model.[4] So, we require a large number of these factors, which must operate independently and linearly and be such that no particular factor dominates the others, if Haavelmo's justification is to be applicable. It is difficult to believe that this will always be the case.

It is also argued that the Lindeberg-Lèvy central limit theorem is probably not the only relevant limit theorem. Bartels (1977) suggests that there are other pertinent theorems; some of which may lead to stable non-normal distributions. Other studies suggest that non-normal disturbances may be observed even if this central limit theorem is applicable.[5] Suppose each error term, e_t $(t=1,\dots,T)$, is composed of the sum of a large number of independent finite-variance components and so, from the central limit theoerm, can be regarded as normally distributed with (say) mean zero and variance τ^2.

Now, if the variance τ^2 depends on t, then we may regard τ^2 as a random variable with pdf $f(\tau)$. So, assuming ignorance of how τ^2 changes from observation to observation, the observed joint pdf of the disturbances will be of the form

$$f(e) = \int_0^\infty (2\pi\tau^2)^{-T/2}\exp(-e'e/2\tau^2)f(\tau)d\tau. \tag{3.2.1}$$

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[3] Discussions of this issue are given, for instance, by Granger and Orr (1972), King (1979), Koenker and Bassett (1978), Koenker (1982), Judge *et al.* (1985a), and Vijverberg (1987).

[4] See, for example, Theil (1971), King (1979) and Johnston (1984).

[5] See, for example, Granger and Orr (1972) and King (1979).

The degree to which (3.2.1) varies from normality clearly depends on the mixing density $f(\tau)$. For instance, if $f(\tau)$ is an inverted gamma density then (3.2.1) is the pdf of a multivariate Student's t distribution.

From this brief discussion it is obvious that there are quite good reasons to expect non-normally distributed regression disturbances. In the next section we review some empirical studies whose authors suggest that the economic data series they investigate are non-normally distributed.

3.3 Some Empirical Evidence of Non-Normality

There is an extensive literature on the observed distributions of many economic data series. It is suggested that some of these distributions have more kurtosis (and hence fatter tails) than the normal distribution. Mandelbrot (1963a,b, 1966, 1967, 1969) suggests the class of stable Paretian distributions, which may have infinite variances,[6] for price and income series. This is supported by price-change analyses in the stock, financial and commodity markets undertaken by Fama (1963, 1965, 1970) and Sharpe (1971).

Blattberg and Gonedes (1974) maintain that a multivariate Student-t distribution also fits the data investigated by Fama (1965). They arrive at this by considering the mixture of a normal and inverted gamma distribution for the variance of the normal, as discussed in the previous section. Praetz (1972) also uses this approach in his investigation of share price changes. Following similar techniques Press (1968) models security prices and Granger and Orr (1972) consider cash flow data. Other findings of non-normal distributions are also reported by Carlson (1975) for price expectations and Praetz and Wilson (1978) for stock market returns.

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[6] See, for example, the aforementioned papers, Blattberg and Gonedes(1974) or Press (1972) for discussions of this distribution.

These results have obvious implications for the distribution of the error term in any regression model explaining such data, and there are a number of applied studies which consider regression models with non-normal disturbances. For instance, Sutradhar and Ali (1986) estimate a regression model under the assumption that the errors follow a multivariate Student-t distribution. They consider the performance of the stocks of four firms relative to the overall performance of all the stocks trading on the New York Stock Exchange.

Coursey and Nyquist (1988) estimate consumer demand equations when the error distributions have fatter and thinner tails than those of the normal distribution. They consider various distributions for the regression disturbances, including the exponential power and Student-t families of distributions. From their results Coursey and Nyquist suggest that normality of the disturbance term may be the exception; fat or thin tailed distributions may be the norm.

From this brief review it is clear that substantial empirical evidence exists to support the possibility of non-normal regression disturbances in economic models. These studies suggest, in particular, that non-normality is especially likely to be a feature of financial data series. Over the last decade there has been a significant growth in the amount of quality, empirical financial research: this research is continuing, so it is important to investigate econometric issues which may be especially pertinent to the modelling of financial data. In particular, as pre-testing is the norm in applied research, it is especially timely to consider the properties of some conventional pre-test estimators in the context of non-normal regression disturbances.

3.4 Spherically Symmetric Disturbances

There are many assumptions we could make regarding the distribution of the regression disturbances in the standard linear regression model, as alternatives to that of spherical normality. We have already mentioned three families of distributions - the spherically symmetric, stable Paretian, and exponential power families. Other choices could include the generalized t[7] or Gram-Charlier densities. We could even simply assume, as have some authors, that the distribution of the errors possesses certain properties rather than specify its exact form. For instance, Ullah *et al.* (1983) and Tracy and Srivastava (1988) assume that the disturbances are small and that they possess moments of up to the fourth order.

As we noted in the introduction to this chapter, many studies assume that the regression disturbances follow a spherically symmetric distribution. This is a sensible extension of spherical normality to investigate for a variety of reasons.[8] First, it is a class of density functions whose contours of equal density have the same spherical shape as the spherical normal. Secondly, we can generate members which have fat and thin tails relative to those of the normal density. Thirdly, all marginal and conditional distributions of a spherical random vector are also spherically distributed and have the same shape.[9]

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[7] This family includes the power exponential (or Box-Tiao), normal Laplace, and t distributions as special cases. See, for example, McDonald and Newey (1988), who consider M-estimators of the regression parameters when the disturbance distribution is the generalized t.

[8] See, for instance, Kelker (1970), Kariya and Eaton (1977), King (1979), Chmielewski (1981a), Muirhead (1982), Cacoullos and Koutras (1984), and Van Praag and Wesselman (1989) for discussions of, and further references to, this family of distributions. Note that this discussion also generally applies to the elliptically symmetric family.

[9] This latter point is not true for non-spherically symmetric members of the elliptically symmetric family. In this case, all the marginal and conditional distributions are elliptically symmetric with a common kurtosis parameter. See Van Praag and Wesselman (1989).

Finally, a subclass of the spherically symmetric family can be written in terms of a variance mixture of normal distributions;

$$f(e) = \int_0^\infty N(0,\tau^2 I_T)dG(\tau) \qquad (3.4.1)$$

where $G(\tau)$ is the distribution function of τ and is supported on $[0,\infty)$. Note that assuming that the regression disturbances follow this class of spherically symmetric distributions, which is sometimes called the class of (spherical) compound normal distributions, satisfies one of the reasons we gave in Section 3.2 for the occurrence of non-normality. Also, given the analytic form of (3.4.1), we are usually able to analytically investigate the properties of regression estimators when the disturbances follow such a distribution.

A T-dimensional random vector x is said to have a (multivariate) spherically symmetric distribution (SSD) if x and Hx have the same distribution for all T $\times$ T orthogonal matrices H. Hence, the distributions of such random variables are independent of direction from the origin and are a function only of the distance from the origin; that is, $r = (x'x)^{1/2}$. So, the joint pdf of x,[10] will be of the form

$$f(x) = \vartheta(x'x) \qquad (3.4.2)$$

with respect to the Lebesgue measure on R^T, where

$$\vartheta : [0,\infty) \to [0,\infty) \qquad (3.4.3)$$

and

$$\int_0^\infty r^{T-1}\vartheta(r^2)dr = (1/2)\Gamma(T/2)\pi^{-T/2}. \qquad (3.4.4)$$

All non-normal spherically symmetric distributions have components which are dependent but uncorrelated: the normal distribution is the only spherically

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[10] We assume in this thesis that the pdf, and the first two moments (at least), always exist. Note that the definition of a spherically (and elliptically) symmetric distribution given here is only one of a number of possibilities (see the aforementioned references).

symmetric law for which the observations are independent. Spherically symmetric distributions are members of the class of elliptically symmetric distributions. An m-dimensional random vector z is said to be elliptically symmetrically distributed (ESD) if its density function can be written as

$$f(z) = |\Omega|^{-1/2}\vartheta\left((z-\mu)'\Omega^{-1}(z-\mu)\right) \tag{3.4.5}$$

where ϑ is a positive function on $[0,\infty)$ and Ω is a positive (semi-) definite matrix. Clearly, given the distribution's symmetry, $E(z)= \mu$. So, a sperically symmetric distribution is a special case of an elliptically symmetric distribution where $\Omega = \tau^2 I_m$ and τ^2 is any positive scalar. Some examples of spherically symmetric distributions include:[11]

(a) The multivariate normal distribution, $N(0,\sigma^2 I_T)$

$$f(x) = (2\pi\sigma^2)^{-T/2}\exp\left[-x'x/(2\sigma^2)\right] . \tag{3.4.6}$$

(b) The multivariate Student-t distribution, $Mt\left[0,\left(\nu/(\nu-2)\right)\sigma^2 I_T\right]$, with a joint density function of the form

$$f(x) = c_\nu(\sigma)^{-T}\left[1 + x'x/(\nu\sigma^2)\right]^{-(T+\nu)/2}, \tag{3.4.7}$$

where σ, $\nu>0$; $-\infty<x_i<\infty$, $i=1,\ldots,T$; $c_\nu=\Gamma\left[(T+\nu)/2\right]\left[(\pi\nu)^{T/2}\Gamma(\nu/2)\right]^{-1}$ is a normalising constant; ν and σ^2 are the degrees of freedom and scale parameters of the distribution; and $\sigma_x^2=\nu\sigma^2/(\nu-2)$ is the common variance of the x_i's.

When $\nu=1$, (3.4.7) is the pdf of a multivariate Cauchy random variate, while when $\nu=\infty$, it is the pdf of a multivariate normal random variate. Further, for $\nu>2$, the elements of x are uncorrelated but not independent.

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[11] See, for instance, Lord (1954), McGraw and Wagner (1968), Goldman (1976), King (1979), Muirhead (1982), and Cacoullos and Koutras (1984). Note that the family also contains distributions that have no explicit analytic description.

(In fact, within the class of ESDs if Ω is diagonal and the elements are independent then the distribution must be the normal distribution.)

(c) The "ε-contaminated" normal distribution with $0 \le \varepsilon \le 1$ and

$$f(x) = (1-\varepsilon)(2\pi)^{-T/2}\exp(-x'x/2)+\varepsilon(2\pi\sigma^2)^{-T/2}\exp[-x'x/(2\sigma^2)] \ . \qquad (3.4.8)$$

The primary source on the properties of ESDs is Kelker (1970). King (1979) details many of the contributors to the theory of elliptically symmetric random vectors, summarises the important properties of ESDs, and surveys some of the literature. Devlin *et al.* (1976) and Chmielewski (1981a) also provide reviews and bibliographies.

3.5 Linear Regression Models with Non-Normal Errors

As we noted in the introduction to this chapter, the usual least squares estimators of the parameters of the linear regression model are sensitive to the form of the underlying distribution of the disturbances. In particular, compared with their properties under normal errors, they are no longer efficient or asymptotically efficient, though they are still unbiased and consistent estimators. If the distribution of the errors has a finite variance then b still possesses the property of minimum variance among the class of linear estimators of β. Of course, if the variance does not exist then this will not be the case .

These features, the interest in the question of the robustness of estimators, and the empirical evidence of non-normality, have led to studies which investigate the properties of estimators and tests when the regression model's disturbances are non-normally distributed. In this section we briefly review some of this literature, paying particular attention to those studies which consider linear regression models with spherical or elliptical errors.

Box (1952, 1953) considers linear regression models with spherically symmetric disturbances. In his 1952 paper he notes that the usual F test statistics have the same null distribution for all $f(e) = \vartheta(e'e)$. (See also Efron (1969).) Box (1953) asserts that those tests which are uniformly most powerful under normality assumptions maintain this property for any spherically symmetric distribution for which ϑ is a decreasing function. King (1979) proves this result for a broad class of elliptically symmetric distributions.

Now, let the T-dimensional spherical density function for the joint distribution of the errors be $f(e) = f(r^2)$, $r^2 = e'e$. Thomas (1970) proves that the usual least squares estimator of β, b, assuming that $f(r^2)$ is a decreasing function on r^2 and that the mean and variance-covariance matrix of e exist, is the linear minimum variance unbiased estimator (MVUE) and the maximum likelihood estimator of β.[12]

Thomas also shows that the Student-t, F, and Beta tests are independent of the specific spherical assumption made on the errors, e; that is, the null distributions of these test statistics are the same as under the assumption of normal errors. However, the non-null distributions, and consequently so too the power functions, of these test statistics depend on the specific spherical function f. Unaware of Thomas's work, these results are also derived by Zellner (1976), King (1979) and Singh (1987). Zellner (1976) and Singh (1987) consider the special case of multivariate Student-t (Mt) errors. Zellner (1976) proves that b is a MVUE of β while Singh (1987) shows that b is the <u>unique</u> MVUE under Mt disturbances. King (1979) generalises the results to elliptically symmetric disturbances.

Further, Thomas obtains the functional forms of these non-null distributions using polar co-ordinates and various variable transformations.

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[12] We mention here only those sections of Thomas (1970) which are of direct relevance to this thesis. This applies also to our discussion of King (1979).

Using alternative techniques to Thomas, and within the context of testing exact linear restrictions on the coefficient vector, assuming that the regression errors follow a multivariate Student-t distribution, Ullah and Phillips (1986) and Sutradhar (1988) also derive the non-null distribution of the F-test statistic, which we called u in Chapter 2. In this thesis we extend these results by assuming a compound normal distribution for the regression disturbances, and by assuming further that relevant explanatory variables have been omitted from the model's specification, all in the context of preliminary testing.

King (1979, 1980) extends many of Thomas's results. He gives necessary and sufficient conditions for the weak consistency of linear unbiased estimators of β assuming elliptically symmetric disturbances;[13] he obtains sufficient conditions for the strong consistency of the generalised least squares (GLS) estimator when the disturbance covariance matrix is known up to a scalar value; and he also determines necessary and sufficient conditions for the ordinary least squares (OLS) estimator to be strongly consistent assuming that the variance-covariance matrix of the elliptically symmetric disturbances exists and is diagonal. Further, King uses an intuitively sensible criterion to show that the GLS estimator is better than any other linear unbiased estimator of β. The criterion makes no assumption regarding the existence of the moments of the disturbances.

King proves that if <u>any</u> function of y (be it a test statistic or an estimator) is invariant to the values taken by τ^2 when $e \sim N(0,\tau^2 I_T)$ then the function has the same distribution for the wider class of elliptically symmetric distributions which assume only that $Pr(e=0)=0$ (ESD_0). So, given that τ^2 is usually unknown, this result will apply to many of the small

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[13] King defines "unbiasedness" in the traditional sense and so, assumes that the disturbances have finite first order moments.

sample (and asymptotic) tests commonly used in regression analysis. For instance, it is relevant for the usual F-test; the Durbin-Watson (1950, 1951) and the Berenblut-Webb (1973) tests for first-order autocorrelation; and tests for heteroscedastic regression disturbances proposed by Goldfeld and Quandt (1965), Ramsey (1969), and Harvey and Phillips (1974).

This result has many other implications. These tests will have the same size for any ESD_0 distribution of e. Further, confidence intervals constructed assuming normal disturbances and based on the usual Student-t statistics are equally applicable to the broader class of ESD_0 disturbances. King (1979) gives other examples.

Another consequence of importance relates to the power properties of these tests for elliptically symmetric disturbances. The result implies that all tests based on test statistics which are invariant to the scale of e under normality have the same power function for all members of the ESD_0 class as they do under the normality assumption. So, for instance, this applies to the usual tests for serial correlation and heteroscedasticity of the disturbances.

The next obvious question to ask is whether optimal power properties of normal theory tests still hold under the wider elliptical symmetry assumption. This issue is considered by Kariya (1977), Kariya and Eaton (1977) and King (1979). They show that many tests which possess optimal power properties for normally distributed disturbances are also optimal under appropriate elliptical symmetry assumptions.

King (1979) also investigates the distributions of test statistics which are not necessarily invariant to the scale of the disturbances. He confines his attention to the class of (elliptical) compound normal distributions, $ESD_N(\Sigma)$.[14] King notes that this is not a particularly restrictive assumption, especially in time series problems. Now, let $F_N(y)$ be the joint distribution function of a test statistic $s(y)$ when e is distributed $N(0,\tau^2\Sigma)$. Then its joint distribution function if e follows any other $ESD_N(\Sigma)$ law has the form

$$F(y) = \int_0^\infty F_N(y)dF(\tau). \tag{3.5.1}$$

So, when the distribution of the errors belongs to the compound normal subclass of the elliptically symmetric family then the distribution of a function of y can be viewed as a weighted average of the distribution taken by the function for different values of the scale parameter τ^2 when the distribution is $N(0,\tau^2\Sigma)$. We use this result extensively in this thesis to generalise the pre-testing literature discussed in Chapter Two.

Obviously, this has implications for the power properties of such tests. The power will be a weighted average of its powers for different values of $\tau^2 \in (0,\infty)$ under the alternative hypothesis assuming normal errors; the weights depending on the specific elliptical distribution.[15] Hence, if a test has an optimal power property under normal disturbances for all values

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[14] The form of the density function, f(e), for the class of elliptical compound normal distributions is the natural extension of (3.4.1). That is $f(e)=\int_0^\infty (2\pi\tau^2)^{-T/2}|\Sigma|^{-1/2}\exp\left[-e'\Sigma^{-1}e/2\tau^2\right]dF(\tau)$, where $F(\tau)$ is a distribution function supported on $(0,\infty)$.

[15] Sutradhar (1988) illustrates this finding. He evaluates the power function of the usual test of the significance of the slope parameter in the simple linear regression model when the disturbances follow a multivariate Student-t distribution. Sutradhar finds that the power of the test depends not only on the degrees of freedom, ν, of the Student-t distribution but also on the non-centrality parameter of the test statistic under normality, which reflects the hypothesis error. Specifically, if the hypothesis is very false then the power of the test when ν is finite is less than that for $\nu = \infty$, the degrees of freedom corresponding to normal errors. However, the powers are very close for small values of the non-centrality parameter.

of $\tau^2 \in (0,\infty)$ then it will also possess this property under elliptical errors. So the UMPI size-α test we used in Chapter Two to test the validity of m exact linear restrictions on the coefficient vector maintains its optimal power property when the error distribution is spherically symmetric.

Clearly, if $F_N(y)$ in (3.5.1) is independent of τ^2 then $F(y) = F_N(y)$. So, if we assume a properly specified regression model, then the test statistic J, which we use to test for homogeneity of the variances in the two-sample linear regression model has the same null and the same non-null distributions under the wider assumption of elliptically symmetric disturbances as it does under normality. Chmielewski (1981b), ignorant of King (1979), uses a variant of this result to prove that the usual normal theory test statistics for testing $H_0:\tau_i^2=\ldots=\tau_j^2=\tau^2$ against $H_1:\tau_i^2\neq\tau_j^2$ for at least one pair of (i,j), $i\neq j$, have both null and non-null distributions which remain invariant for the class of spherically symmetric distributions. In Chapter Six we present an alternative proof of this result for one of the particular problems that we investigate in this thesis.

As we noted above, Zellner (1976) considers the linear regression model assuming that the error vector follows a multivariate Student-t distribution, $Mt(0,\sigma_e^2 I_T)$. $\sigma_e^2=\nu\sigma^2/(\nu-2)$ is the common variance of the elements of e, where ν and σ^2 are, respectively, the degrees of freedom and the scale parameters of the Mt distribution. Under this assumption, the marginal distribution of each error term is univariate Student-t and the marginal distributions have thicker tails than under a normality assumption for small values of ν. As $\nu\to\infty$, the pdf approaches a normal form, while when $\nu=1$, the pdf is Cauchy.

Aside from proving that b is the MLE[16] and a MVUE of β,[17] Zellner derives the MLE of σ^2, which is $\tilde{\sigma}^2_{ML}=(y-Xb)'(y-Xb)/T=e'Me/T$. Further, let q be a positive scalar, then within the class of estimators of the form $qe'Me$ the minimal MSE estimator of σ^2 is, assuming $\nu>4$, $\tilde{\sigma}^2_M=\left[(\nu-4)e'Me\right]/\left[\nu(v+2)\right]$. Correspondingly, the minimal MSE estimator for σ^2_e is $\tilde{\sigma}^2_{eM}=\left[(\nu-4)e'Me\right]/\left[(\nu-2)(v+2)\right]$. Unbiased estimators for σ^2_e and σ^2 are, respectively, $e'Me/v$ and $(\nu-2)e'Me/(\nu v)$. Zellner also shows that a maximum of the likelihood function with respect to β, σ^2, and ν does not exist. That is, while we can derive the MLE's of β and σ^2 for any given ν, MLE's of β, σ^2 and ν do not exist.

In addition, Zellner investigates various aspects of the model within a Bayesian framework. He shows that the joint posterior density of β and σ^2 depends on the error distribution assumption. However, the marginal posterior of β is the same as under a normality assumption. Chib *et al.* (1988) extend Zellner's Bayesian analysis to linear regression models with elliptical errors. They investigate the prediction problem and show that the Bayesian prediction density under the elliptical assumption is identical to that which would be obtained under normally distributed errors. So, assuming normality when the true distribution is in fact elliptical will not result in incorrect predictive inferences.

Ullah and Zinde-Walsh (1984) analyse the robustness of the Lagrange multiplier (LM), likelihood ratio (LR), and Wald (W) tests for testing linear restrictions in a linear regression model with Mt disturbances, though they note that their results extend to the class of spherically

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[16] Zellner shows that b is the MLE for all likelihood functions which are monotonically decreasing functions of $(y-X\beta)'(y-X\beta)$.

[17] Recall that Singh (1987) proves that b is the unique MVUE.

symmetric regression disturbances. Let LM_N, LR_N, and W_N denote, respectively, the LM, LR, and W test statistics under normal errors and LM_t, LR_t, and W_t denote the corresponding test statistics for the Mt disturbances. They show that

$$LR_t=LR_N, \quad LM_t=g^{-1}LM_N, \quad W_t=gW_N; \quad 0 < g = (T+\nu)/(T+\nu+2) < 1.$$

So, in small samples the LR test is robust, while the LM and W tests are not. The LM and W tests are robust only in large samples; as $T\to\infty$, $g\to1$. Further, $g\to1$ as $\nu\to\infty$, as we would expect: when $\nu=\infty$ the errors are normally distributed.

Andrews and Phillips (1987) consider optimal median-unbiased estimation in a linear regression model with $ESD_N(\Sigma)$ disturbances[18]. They show that the generalised least squares estimator is best for any monotone loss function. A monotone loss function is one which is non-decreasing the more we over- or under-estimate the parameter of interest. So, an estimator which is optimal with respect to the class of monotone loss functions has a more concentrated distribution around the estimand than has any other considered estimator. Andrews and Phillips also propose, assuming normality, a best median-unbiased estimator of the error variance.

A number of studies investigate the properties of minimax and Stein estimators of location parameters under non-normality. Strawderman (1974), Berger (1975), Brandwein (1979), and Brandwein and Strawderman (1978, 1980), for example, consider minimax estimators which dominate the usual best invariant estimator when the distribution is spherically symmetric.

Ullah *et al.* (1983) investigate the conditions under which a general class of shrinkage estimators, that includes both a Stein-type estimator and

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[18] An estimator $\bar{\varphi}$ of φ is median unbiased if Pr.$(\bar{\varphi}\geq\varphi)\geq1/2$ and Pr.$(\varphi\leq\varphi)\geq1/2$. If Pr.$(\varphi=\varphi)=0)$, as is usually the case, this condition simplifies to Pr.$(\bar{\varphi}>\varphi)$=Pr.$(\varphi<\varphi)=1/2$.

a ridge-type adaptive estimator as special cases, dominate the least squares estimator when the disturbances are small and possess moments of up to the fourth order. Non-normal distributions are obtained by varying the measures of skewness and kurtosis. They find that the conditions under which these estimators dominate the usual least squares estimator depend on the degree of non-normality and can be quite different from those conditions which result under normality.

Tracy and Srivastava (1988), assuming the same assumptions on the disturbances as Ullah *et al.* (1983), develop a family of Stein-like mixed regression estimators and compare their (approximate) risk performance to that of the traditional mixed regression and Stein rule estimators. See also Srivastava and Chandra (1985).

Some other studies which investigate the sampling properties of Stein-type estimators under non-normal errors include Dey and Berger (1983), Shinozaki (1984), Judge *et al.* (1985b) and Miyazaki *et al.* (1986). Shinozaki (1984), within the context of the K-mean problem,[19] demonstrates an explicit James-Stein estimator that dominates the least squares estimator under squared error loss when the coordinates of the estimator b are independently, identically, and symmetrically distributed.[20]

Judge *et al.* (1985b) consider the estimation of K (K>3) location parameters of an orthonormal linear regression model when the regression

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[19] Recall that the problem of estimating K means is equivalent to that of estimating K location parameters in the linear regression model with orthonormal regressors.

[20] As we noted in the previous chapter, James and Stein (1961), for the K-mean problem, derive a family of estimators that, assuming normality, dominate the usual least squares (maximum likelihood) estimator. They also show that the assumption of normality is unnecessary but, in this case, they could not give explicit dominating estimators.

disturbances are distributed as Mt. They investigate the sampling properties of the James-Stein estimator, and its positive-part counterpart. In particular, Judge *et al.* show that the James-Stein estimator is minimax and they give the conditions under which it will dominate the least squares estimator. Using Monte Carlo techniques they compare the (empirical) risks of the least squares estimator, the positive-part James-Stein estimator, and two members of the robust L-estimator family. The risk characteristics of the Stein-type estimators under Mt errors are found to be similar to those assuming normality. Further, there are regions of the parameter space where there exists significant risk gains from using the Stein-type estimators instead of the least-squares or the L-estimators. From their Monte Carlo study Judge *et al.* also report that the estimator proposed by Stein (1981) has smaller risk than the traditional robust and Stein-type estimators .[21]

A comparison of the empirical risks of Stein-type estimators, the maximum likelihood estimator and the L-estimators is also undertaken by Miyazaki *et al.* (1986). In this study they assume independent, identically distributed univariate Student-t disturbances. If the distributional assumption is correct their Monte Carlo study suggests that the risk characteristics of the traditional Stein-like estimators assuming univariate Student-t errors are similar to those observed under normality. Miyazaki *et al.* find that the risk comparisons produce results which are qualitatively similar to those in their aforementioned 1985 study (Judge *et al.* (1985b)), which assumed Mt regression disturbances. In addition, this investigation finds that if the degrees of freedom parameter of the univariate Student-t distribution is mis-specified then, as we would expect, the risk function is

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[21] Stein's 1981 estimator has been shown to have lower risk than traditional Stein estimators when the distribution is possibly heavy tailed. See, for instance, Dey and Berger (1983) and Miyazaki *et al.* (1986).

higher virtually everywhere in the parameter space than if the correct assumption is made. They further show that if the degrees of freedom parameter is mis-specified then the Stein estimators are no longer minimax.

These 1985 and 1986 investigations of Judge, Miyazaki and Yancey[22] suggest that imposing the additional assumption of independent errors, as opposed to uncorrelated but dependent disturbances, has little qualitative effect on the risk characteristics of the analysed estimators. This contrasts with, for example, the findings of Phillips and Hajivassiliou (1987) and Lye (1990).

Phillips and Hajivassiliou investigate the distribution of the t-ratio when the sample is drawn from a standard Cauchy (0,1) population. Aside from undefined second moments this assumption implies that the elements are, in particular, independent of each other. For this case, they find that the distribution of the t-ratio is bimodal, even asymptotically. This is different from the distribution of the t-ratio under a multivariate Cauchy distribution which, from Thomas (1970), Zellner (1976) and King (1979), is t with n-1 degrees of freedom, as it is under normality. Phillips and Hajivassiliou suggest that this example highlights the implications of the differences between lack of correlation and independence in non-normal populations. This difference does not appear to be reflected in the risk functions, given the findings of Judge, Miyazaki and Yancey: additional research is obviously required.[23]

We have discussed a number of studies which assume Mt or univariate t regression disturbances. One common feature among them is the assumption that ν, the degrees of freedom parameter of the distribution, is known.

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[22] These studies are also discussed in Judge and Yancey (1986).

[23] I am grateful to Peter Phillips for raising this issue.

Obviously, in realistic applied situations the researcher would rarely know the value of ν. So, to provide estimates of the error variance, for instance, we require an estimate of ν from the data. Zellner (1976) shows that, assuming Mt disturbances, we cannot maximise the likelihood function with respect to β, σ^2 and ν. This has led Sutradhar and Ali (1986) and Singh (1988) to propose estimators for ν. Sutradhar and Ali show, among other things, the consistency of their estimator.

3.6 Summary and Concluding Remarks

From the discussion in this chapter it is clear that there are several good reasons for expecting non-normal regression disturbances in a linear regression model explaining economic data. The empirical evidence we presented supported these arguments. In particular, it would seem that non-normality may be the norm, rather than the exception, for economic models involving financial data.

The possibility of non-normal regression errors has resulted in two streams of research. The first concentrates on the development of robust estimators and tests; that is, estimators and tests whose properties are relatively invariant to the form of the error distribution. The second investigates the properties of traditional estimators and tests under alternative error distribution assumptions to that of normality. There are, of course, many assumptions we could make, and many have been considered in the literature as we noted in our discussion.

In this thesis we assume the disturbance distribution is spherically symmetric. In this chapter we gave several reasons to explain our selection of this family of distributions and we briefly reviewed a number of studies which investigate the properties of estimators and tests when the regression model's disturbances are spherically or elliptically distributed. In addition, we mentioned some of the papers which consider non-normality assumptions other than those of spherical or elliptical symmetry. Aside from highlighting the need to study the robustness of such estimators and tests, these studies illustrate the strong interest in this area of research. It is interesting to note that, to the best of our knowledge, none of this research considers the sampling properties of conventional pre-test estimators assuming non-normal disturbances. Hence the motivation for this thesis.

CHAPTER FOUR

SMALL SAMPLE PROPERTIES OF THE PRE-TEST LINEAR RESTRICTIONS PREDICTOR ESTIMATOR UNDER MIS-SPECIFICATION

4.1 Introduction

In this chapter we consider some finite sample properties of estimators of the conditional forecast of y (say, predictor estimators) in the linear regression model after a pre-test for exact linear restrictions. We recall from the literature review in Chapter Two that to avoid data dependent risk functions we assume that the parameter of interest is the conditional forecast of y, rather than the location vector β. In terms of the relevance of the resulting risk functions to those of the location vector β this is equivalent to assuming that the regressors are orthonormal. Of course, in reality the regressors are rarely orthonormal and we should remember that the mapping from the conditional mean (or orthonormal regressors) case to that of considering the unweighted risk of the estimators of β when the regressors are not orthonormal is not direct and is significantly more complicated. Nevertheless, the research from studies such as Wallace (1972), Brook (1972, 1976), Bock *et al.* (1973), Yancey *et al.* (1973), and Judge and Bock (1978) suggests that the qualitative risk characteristics observed for the orthonormal regressors case carry over to the more general problem of non-orthonormal regressors.

We investigate the properties of the estimators when the regression model is possibly mis-specified in two ways. First, there may be relevant regressors that have been omitted from the design matrix and secondly, possibly simultaneously, the standard assumption of normal disturbances should be merely one of spherical symmetry. In fact from our discussion in Chapter Three we know that this possible mis-specification of the distribution of the regression disturbances should have little impact on the qualitative properties of the risk function of the predictor pre-test estimator. This is because the test statistic, u, is valid under the null, it is still a UMPI size-α test, and its non-null distribution can be regarded as a weighted average of its non-null distribution under normality; the form of the variance mixing distribution determines the weights.

We shall see this in Section 4.2 where we establish the framework within which we work and where we derive some necessary preliminary results. In particular, we derive the non-null distribution of the test statistic, u. In so doing, we define certain notation which we also use in Chapters Five and Six as well as in this chapter.

We follow, in Section 4.3, with the derivations of the exact risk functions of the unrestricted, the restricted and the pre-test estimators. As an interesting aside, we also briefly consider the bias functions. Obviously, the exact risk function will depend on the variance mixing distribution and so, to illustrate this, we consider the special case of multivariate Student-t regression disturbances. This distribution arises by assuming that the variance mixing distribution is of the inverted gamma form.

From the general results we derive the exact risk functions of the unrestricted, the restricted and the pre-test estimators when the regression disturbances are multivariate Student-t. We numerically evaluate these risk functions and then compare them for various values of the degrees of freedom parameter ν: recall that the value of ν indicates the departure from normality as the smaller ν is the fatter are the tails of the marginal distributions of the errors relative to those under a normality assumption. Given that our interest is from a mis-specification (or robustness) viewpoint, we compare the risk functions in the space that would be used under the normality assumption.

So, the work in this chapter extends the current literature by deriving the exact risk function of the pre-test estimator when the model is possibly mis-specified in two ways. It should be noted that, to the best of our knowledge, the exact sampling properties of the conventional pre-test estimator have not previously been derived, assuming even a correctly specified design matrix, for any type of non-normal disturbances. Accordingly, we devote Section 4.4 to comparing the risk functions assuming that there are no omitted regressors and then investigate the more general problem in Section 4.5. In these sections we include only some representative results of the numerical evaluations. Further cases are given in the appendix to this chapter. Finally, some concluding remarks are given in Section 4.6.

4.2 The Model Framework, Estimators and Some Preliminary Results

Suppose that the process generating the $(T\times 1)$ vector of observations on the dependent variable y is

$$y = X\beta + Z\gamma + e \ , \tag{4.2.1}$$

where X and Z are $(T\times k)$ and $(T\times p)$ full rank matrices of non-stochastic variables, and β and γ are $(k\times 1)$ and $(p\times 1)$ vectors of unknown parameters respectively. We assume that the $(T\times 1)$ vector of disturbances e is distributed according to the laws of the class of (spherical) compound normal distributions with a zero finite mean vector and a finite scalar variance-covariance matrix, $E(ee')=\sigma_e^2 I_T$: we write $e \sim SSD_N(0,I_T)$.[1] σ_e^2 is the common variance of the e_i's, $i=1,\ldots,T$.

Recall that this class of distributions is a subclass of the family of spherically symmetric distributions[2] which can be expressed as a variance mixture of normal distributions. That is, we can write

$$f(e) = \int_0^\infty f_N(e) f(\tau) d\tau \;, \tag{4.2.2}$$

where $f(e)$ is the pdf of e when $e \sim N(0,\tau^2 I_T)$ and $f(\tau)$ is the pdf of τ and is supported on $[0,\infty)$. So, $E(ee')=\int_0^\infty E_N(ee')f(\tau)d\tau=E(\tau^2)I_T$: that is, $\sigma_e^2=E(\tau^2)$.

The errors are uncorrelated but are dependent; independence is a feature if and only if the underlying distribution is normal. Further, the marginal distribution of the errors may have fatter or thinner tails than that which would result under a normality assumption.

Now suppose that the researcher specifies the model

$$y = X\beta + u \quad ; \quad u \sim N(0,\sigma_u^2 I_T) \tag{4.2.3}$$

as the true generating process. He proceeds assuming (4.2.3) to be properly specified when in fact $u \sim SSD_N(Z\gamma, I_T)$. Note that $\sigma_u^2 I_T = E(ee')=E(\tau^2)I_T$. In addition, we assume that the investigator has (uncertain) extraneous prior information about the parameters β which he can express as m (<k) exact linearly independent restrictions

$$R\beta = r, \tag{4.2.4}$$

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[1] We require the existence of the first two moments if risk is to be a meaningful basis for the comparison of the estimators.

[2] King (1979), for instance, shows that the assumption of the compound normal distribution rather than the wider SSD class is not particularly restrictive, especially in time series problems.

where R is an $(m\times k)$ known full rank matrix, and r is an $(m\times 1)$ vector of known non-stochastic elements. Following the notation of Chapter Two we let $\delta=R\beta-r$ represent an $(m\times 1)$ specification error vector of the prior information. If the restrictions are valid then $R\beta=r$ and $\delta=0$.

Under the assumptions of (4.2.3) the unrestricted and restricted least squares (and maximum likelihood) estimators of β are respectively,

$$\left.\begin{aligned} b &= S^{-1}X'y \\ \text{and}\qquad b^* &= b + S^{-1}R'[RS^{-1}R']^{-1}(r-Rb)\ . \end{aligned}\right\} \tag{4.2.5}$$

Note, from our discussion in Chapter Three and assuming that there are no omitted regressors, that b and b* are still the MLE's under the spherical assumption. Of course, they are only the MLE's of β in (4.2.1) if X and Z are orthogonal.

The researcher, uncertain of the validity of the restrictions, undertakes a pre-test of

$$H_0 : \delta = 0 \quad \text{vs.} \quad H_1 : \delta \neq 0 \tag{4.2.6}$$

using the traditional Wald (and Lagrange Multiplier) test statistic

$$u = (Rb-r)'[RS^{-1}R']^{-1}(Rb-r)v/m(y-Xb)'(y-Xb)\ . \tag{4.2.7}$$

u is a central F random variate with m and v degrees of freedom under the null hypothesis when there are no excluded regressors and the disturbances are spherically symmetric.[3] However, this property no longer holds if the model is mis-specified: when variables are omitted, the distribution of u depends not only on m, v and the degree of mis-specification but it depends also on the variance mixing distribution. This is seen in Theorem 4.2.1 and Corollary 4.2.1 below.

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[3] King (1979) shows that this property carries over to the wider class of elliptical symmetry.

Theorem 4.2.1

Under our assumptions, the density function of u, given by (4.2.7) is

$$f(u) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\theta_n^r/r!\right)\left(\theta_d^s/s!\right) m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\left(v+mu\right)^{(\frac{m+v}{2})+r+s}}$$

$$\cdot \int_0^{\infty} e^{-(\theta_n+\theta_d)/\tau^2}\left(\tau^2\right)^{-(r+s)} f(\tau)d\tau \ , \qquad (4.2.8)$$

where

$$\left.\begin{aligned} \theta_n &= (\Lambda+\delta)'[RS^{-1}R']^{-1}(\Lambda+\delta)/2 \ , \\ \theta_d &= \gamma' Z' M Z\gamma/2 \ , \end{aligned}\right\} \qquad (4.2.9)$$

and $\Lambda=RS^{-1}X'Z\gamma$ and $\delta=R\beta-r$.

Proof.

From Chapter Three $f(u)=\int_0^{\infty} f_N(u)f(\tau)d\tau$ where $f_N(u)$ is the joint density function of u when $e\sim N(0,\tau^2 I_T)$. $f_N(u)$ is well known to be a doubly non-central F density with m and v degrees of freedom and non-centrality parameters $\lambda_{n\tau}=\theta_n/\tau^2$ and $\lambda_{d\tau}=\theta_d/\tau^2$ (see, for instance, Ohtani (1983), Mittelhammer (1984), and Giles (1986)). The density function of a doubly non-central F random variate, $F''_{(m,v;\lambda_{n\tau},\lambda_{d\tau})}$ is (Johnson and Kotz (1970)),

$$f_N(u) = e^{-(\lambda_{n\tau}+\lambda_{d\tau})}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\lambda_{n\tau}^r/r!\right)\left(\lambda_{d\tau}^s/s!\right) m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\left(v+mu\right)^{\frac{1}{2}(m+v)+r+s}} .$$

So,

$$f(u) = \int_0^\infty e^{-(\theta_n+\theta_d)/\tau^2} \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\theta_n^r/r!\right)\left(\theta_d^s/s!\right) m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\left(v+mu\right)^{\frac{1}{2}(m+v)+r+s}}$$

$$\cdot\left(\tau^2\right)^{-(r+s)} f(\tau)d\tau \;. \qquad (4.2.10)$$

(4.2.8) follows directly from (4.2.10). #

Corollary 4.2.1

Under the null hypothesis

$$f(u) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(\theta_{n0}^r/r!\right)\left(\theta_d^s/s!\right) m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\left(v+mu\right)^{\frac{1}{2}(m+v)+r+s}}$$

$$\cdot\int_0^\infty e^{-(\theta_{n0}+\theta_d)/\tau^2}\left(\tau^2\right)^{-(r+s)} f(\tau)d\tau \;, \qquad (4.2.11)$$

where $\theta_{n0} = \Lambda'[RS^{-1}R']^{-1}\Lambda/2$.

Proof.

$\theta_n=\theta_{n0}$ when $\delta=0$, and so (4.2.11) follows from Theorem 1. #

Thus, if the model is mis-specified by the omission of regressors then, the null distribution depends on the degree of model mis-specification (through $Z\gamma$), the collinearity between the omitted and included variables,[4] and the variance mixing distribution $f(\tau)$ as well as the degrees of freedom, m and v. Obviously, from (4.2.8), we can regard the distribution of u as a weighted average of the values of a doubly non-central F distribution, $F''_{(m,v;\theta_n/\tau^2,\theta_d/\tau^2)}$, which is the distribution of u under normality. The weights are determined by the specific form of $f(\tau)$. Of course, if the regression disturbances are in fact normally distributed, $N(0,\sigma^2 I_T)$, then u is a doubly non-central F random variate under both the null and the alternative hypothesis. The only difference between the null and the non-null distribution is the numerator non-centrality parameter, which is θ_{n0}/σ^2 under the null and θ_n/σ^2 under the alternative.

As we illustrate our results using the multivariate Student-t distribution with degrees of freedom parameter ν, and scale parameter σ^2, Corollary 4.2.2 derives the non-null distribution of u assuming that the regression disturbances are Mt. This corollary extends the results of Ullah and Phillips (1986) and Sutradhar (1988), both of whom derive the non-null distribution of u assuming Mt errors in a properly specified model.

<u>Corollary 4.2.2</u>

If $e \sim Mt\left(0, \frac{\sigma^2 \nu}{\nu-2} I_T\right)$ then

$$f_{Mt}(u) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(2\lambda_n/\nu\right)^r \left(2\lambda_d/\nu\right)^s m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{r!s!\left[1+2(\lambda_n+\lambda_d)/\nu\right]^{\frac{\nu}{2}+r+s} B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)}$$

---

[4] θ_d is a maximum, *ceteris paribus*, when the included and the excluded regressors are orthogonal. Intuitively, in this situation the included regressors do not capture any of the effect of the excluded variables, and so the specification error is at its worst. In this case there is no specification bias even though $Z\gamma$ is non-zero as there is no impact on the value of $X\beta$. So, $\Lambda=0$ and $\theta_n=0$ under the null.

$$\cdot \frac{\Gamma\left(\frac{\nu}{2}+r+s\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(v+mu\right)^{\frac{m+v}{2}+r+s}}, \tag{4.2.12}$$

where $\lambda_n=\theta_n/\sigma^2$ and $\lambda_d = \theta_d/\sigma^2$.[5]

Proof.

$e \sim Mt(0,\frac{\sigma^2\nu}{\nu-2} I_T)$ when $f(\tau)$ is an inverted gamma density function (see, for example, Zellner (1971, 1976)). So,

$$f(\tau) = \left[2/\Gamma\left(\frac{\nu}{2}\right)\right]\left(\nu\sigma^2/2\right)^{\nu/2}\tau^{-(\nu+1)}e^{-\nu\sigma^2/2\tau^2}, \tag{4.2.13}$$

$E(ee')=\nu\sigma^2/(\nu-2)I_T$ as $E(\tau^2)=\nu\sigma^2/(\nu-2)$ from (4.2.13). Then, using Theorem 4.2.1 and the inverted gamma density, we have

$$f_{Mt}(u) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(\theta_n^r/r!\right)\left(\theta_d^s/s!\right)m^{\frac{m}{2}+r}\,v^{\frac{v}{2}+s}\,u^{\frac{m}{2}+r-1}\,2(\nu\sigma^2/2)^{\nu/2}}{\left(v+mu\right)^{\frac{(m+v)}{2}+r+s} B\left(\frac{m}{2}+r\,;\frac{v}{2}+s\right)\Gamma\left(\frac{\nu}{2}\right)}$$

$$\cdot\int_0^\infty e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]/2\tau^2}\left(\tau^2\right)^{-(r+s+\nu/2+1/2)} f(\tau)d\tau. \tag{4.2.14}$$

Let $\tau^2 = 1/z$ so that the integral in (4.2.14) becomes

$$\frac{1}{2}\int_0^\infty e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]z/2} z^{(r+s+\nu/2-1)}dz$$

---

[5] We define λ_n and λ_d as the ratio of θ_n and θ_d to the scale parameter σ^2 to remain consistent with the notation used in Chapter Two and to enable us to compare the risk functions of the estimators for different ν.

$$= \frac{1}{2}\left[\frac{2}{[2(\theta_n+\theta_d)+\nu\sigma^2]}\right]^{r+s+\nu/2} \int_0^\infty e^{-t} t^{(r+s+\nu/2-1)} dt$$

with the change of variable $t=[2(\theta_n+\theta_d) + \nu\sigma^2]z/2$. Now $\int_0^\infty e^{-t}t^{f-1}dt=\Gamma(f)$ so (4.2.14) becomes

$$f_{Mt}(u) = \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{\theta_n^r \theta_d^s m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{r!s!\left(v+mu\right)^{\frac{(m+v)}{2}+r+s} B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)}$$

$$\cdot\left(\frac{\nu\sigma^2}{2}\right)^{\frac{\nu}{2}} \cdot \frac{\Gamma\left(\frac{\nu}{2}+r+s\right)}{\Gamma\left(\frac{\nu}{2}\right)}\left[\frac{2}{2(\theta_n+\theta_d)+\nu\sigma^2}\right]^{r+s+\nu/2} .$$

Collecting terms and allowing for the change from θ_n to λ_n and θ_d to λ_d completes the proof. #

Finally, when the design matrix is properly specified $\theta_d=0$ and $\theta_n=\delta'[RS^{-1}R']^{-1}\delta/2\equiv\theta$. In this case (4.2.9) and (4.2.12) collapse, respectively, to

$$f(u) = \sum_{r=0}^\infty \frac{\theta^r m^{\frac{m}{2}+r} v^{\frac{v}{2}} u^{\frac{m}{2}+r-1}}{r!\left(v+mu\right)^{\frac{m+v}{2}+r} B\left(\frac{m}{2}+r;\frac{v}{2}\right)} \cdot \int_0^\infty e^{-\theta/\tau^2}\left(\tau^2\right)^{-r} f(\tau)d\tau , \quad (4.2.15)$$

and

$$f_{Mt}(u) = \sum_{r=0}^\infty \frac{\left(2\lambda/\nu\right)^r \Gamma\left(\frac{\nu}{2}+r\right) m^{\frac{m}{2}+r} v^{\frac{v}{2}} u^{\frac{m}{2}+r-1}}{r!\left(1+2\lambda/\nu\right)^{\frac{\nu}{2}+r} B\left(\frac{m}{2}+r;\frac{v}{2}\right)\Gamma\left(\frac{\nu}{2}\right)\left(v+mu\right)^{\frac{m+v}{2}+r}}, \quad (4.2.16)$$

where $\lambda=\theta/\sigma^2$. Note, first, that (4.2.16) is equivalent to the expression derived by Ullah and Phillips (1986) and Sutradhar (1988). Secondly, note that both (4.2.15) and (4.2.16) collapse to a non-central F pdf when the errors are normally distributed, $N(0,\sigma^2 I_T)$.[6]

Ullah and Phillips (1986) derive (4.2.16) by noting the result that if $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, then we can write $e=\sqrt{\nu}a/q$ where a and q are independently distributed as $N(0,\sigma^2 I_T)$ and χ^2_ν respectively. This contrasts with the approach we employ of regarding the error vector e as being randomly drawn from a multivariate normal distribution with a random standard deviation generated from the inverted gamma distribution.[7] It does not matter which approach is used, given the relationship between the χ^2, inverted gamma and gamma pdf's (see, for example, Zellner (1971, pp.369-373)). Nevertheless, we could use the approach of Ullah and Phillips when the errors are Mt to derive the distributions of the test statistics, and the bias and the risk functions of the estimators. Given this, we illustrate their procedure in Appendix 4.1, at the end of this chapter, where we reprove Corollary 4.2.2 using this alternative route.

In this section we have established the non-null distribution of the test statistic, u, which we use to test the validity of the linear restrictions, and we have shown that this distribution collapses to those that have been reported in the literature in certain special cases. In the next section we derive the exact bias and the exact risk functions of the unrestricted, the restricted and the pre-test estimators of E(y) for the situations under consideration here.

4.3. **The Bias and Risk Functions**

Our criterion for evaluating the performance of the estimators is

---

6 Recall that if a random variable $X \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$ then $X \underset{\nu\to\infty}{\sim} N(0,\sigma^2 I_T)$.

7 Sutradhar (1988) also uses this approach.

risk under squared error loss, which we defined in equation (1.3.3) as $\rho\left(\theta,\hat{\theta}(y)\right)=E\left[\left(\hat{\theta}(y)-\theta\right)'\left(\hat{\theta}(y)-\theta\right)\right]=tr\left\{cov\left(\hat{\theta}(y)\right)+bias\left(\hat{\theta}(y)\right)bias\left(\hat{\theta}(y)\right)'\right\}$, where $\hat{\theta}(y)$ is some estimator of the parameter vector θ. This criterion has the advantage of allowing the bias and the variance of an estimator to be traded off but the penalty is that we require the existence of the first two moments of the estimator's distribution, and so accordingly, of the distribution of the regression disturbances. This implies that we cannot consider, for instance, distributions which have infinite variances, such as the Cauchy distribution. We discuss this issue further in the concluding chapter of this thesis, Chapter Seven. So in this chapter, and in the next two chapters, we limit our attention to those SSD_N's which have a finite mean and variance.

We consider the unrestricted least squares estimator Xb; the restricted least squares estimator Xb*; and the pre-test estimator $X\hat{b}$, which were defined in Chapter Two. For completeness their definitions are repeated here:

$$Xb = XS^{-1}X'y \ ; \ Xb^* = Xb + XS^{-1}R'[RS^{-1}R']^{-1}(r-Rb) \ ;$$

$$X\hat{b} = \begin{cases} Xb & \text{if } u > c \\ Xb^* & \text{if } u \le c \end{cases} = I_{[0,c]}(u)Xb^* + I_{(c,\infty)}(u)Xb. \qquad (4.3.1)$$

The pre-test estimator, $X\hat{b}$, arises after a pre-test of the validity of the prior information which we express as

$$H_0 : \delta = 0 \quad vs \quad H_1 : \delta \neq 0 \ , \qquad (4.3.2)$$

and it is either the unrestricted or the restricted estimator depending on whether or not we reject or accept the null hypothesis. Obviously, the pre-test estimator depends on the data, the hypothesis, and the significance level of the test.

Prior to deriving the risk functions of the predictor estimators, we now turn our attention to a brief consideration of their bias functions. We use the traditional definition of the bias of an estimator $\hat{\theta}(y)$ of θ, which is $\text{bias}\left(\hat{\theta}(y)\right)=E\left(\hat{\theta}(y)\right)-\theta$.

Theorem 4.3.1

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $SSD_N(0,I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\text{bias}(Xb) = -MZ\gamma \ , \tag{4.3.3}$$

$$\text{bias}(Xb^*) = -MZ\gamma-XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda) \ , \tag{4.3.4}$$

$$\text{bias}(X\hat{b}) = -MZ\gamma-XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)\int_0^\infty P_{20}^{d\tau}f(\tau)d\tau \ , \tag{4.3.5}$$

where, we recall, $\delta=R\beta-r$ which is a measure of the hypothesis error, $\Lambda=RS^{-1}X'Z\gamma$ represents the effects of the omitted regressors, and $P_{ij}^{d\tau}$ is

$$\begin{aligned} P_{ij}^{d\tau} &= \Pr.\left[F''_{(m+i,v+j;\lambda_{n\tau},\lambda_{d\tau})} < \left(cm(v+j)\right)/\left(v(m+i)\right)\right] \\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{-(\lambda_{n\tau}+\lambda_{d\tau})}\left(\lambda_{n\tau}^r/r!\right)\left(\lambda_{d\tau}^s/s!\right)I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right], \end{aligned} \tag{4.3.6}$$

where $I_x(.;.)$ is Pearson's incomplete beta function with $x=cm/(v+cm)$, and $i,j,=0,1,2,\dots$. $P_{ij}^{d\tau}$ is a function of τ as $\lambda_{n\tau}=\theta_n/\tau^2$ and $\lambda_{d\tau}=\theta_d/\tau^2$.

Proof.

To establish (4.3.3) and (4.3.4) we merely need to note that $E(Xb)=X\beta+XS^{-1}X'Z\gamma$ and $E(Xb^*)=X\beta+XS^{-1}X'Z\gamma-XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)$. Now, to derive the bias of $X\hat{b}$ we have that $X\hat{b}=Xb+(Xb^*-Xb)I_{[0,c]}(u)$ $=Xb-XS^{-1}R'[RS^{-1}R']^{-1}(Rb-r)I_{[0,c]}(u)$, as $Xb^*=Xb-XS^{-1}R'[RS^{-1}R']^{-1}(Rb-r)$. So,

$\text{bias}(X\hat{b})=-MZ\gamma-E\left[XS^{-1}R'[RS^{-1}R']^{-1}(Rb-r)I_{[0,c]}(u)\right]=-MZ\gamma-E(Q)$. We write[8], $Rb-r=(R\beta-r)+RS^{-1}X'Z\gamma+RS^{-1}X'e=R(\beta-\beta_0)+RS^{-1}X'Z\gamma+RS^{-1}X'e$ where β_0 is any solution of $R\beta=r$. Then, let $A=RS^{-1}X'$, $B=[RS^{-1}R']^{-1}$, $d=X(\beta-\beta_0)$ and note that $AX=R$, so that

$$
\begin{aligned}
Rb-r &= AX(\beta-\beta_0) + AZ\gamma + Ae \\
&= A(d + Z\gamma + e) \\
&= Ae_1,
\end{aligned}
\tag{4.3.7}
$$

where $e_1=d+Z\gamma+e \sim SSD_N(d+Z\gamma, I_T)$.

We use the same procedure to rewrite the test statistic, u, as

$$
\begin{aligned}
u &= \frac{v(Rb-r)'[RS^{-1}R']^{-1}(Rb-r)}{m(y-Xb)'(y-Xb)} \\
&= \frac{ve_1'Ce_1}{me_2'Me_2}
\end{aligned}
\tag{4.3.8}
$$

where $e_2=Z\gamma+e \sim SSD_N(Z\gamma, I_T)$ and $C=A'BA$ is a symmetric, idempotent matrix of rank m. So, $E(Q)=E\left[A'BAe_1I_{[0,c]}(ve_1'Ce_1/me_2'Me_2)\right]=E\left[Ce_1I_{[0,c]}(ve_1'Ce_1/me_2'Me_2)\right]$. As C is symmetric, idempotent we can express it as $C=LL'$ such that $L'L=I_m$, which gives $E(Q)=\tau LE\left[(e^*/\tau)I_{[0,c]}\left((ve^{*\prime}e^*/\tau^2)/(me_2'Me_2/\tau^2)\right)\right]$ where $e^*=L'e_1$.

Let $E_N(Q)$ be the E(Q) when $e\sim N(0,\tau^2I_T)$. It is straightforward to show, under these assumptions, that $e^*\sim N\left(L'(d+Z\gamma),\tau^2I_m\right)$ and that the quadratic forms $(e^{*\prime}e^*/\tau^2)=(e_1'Ce_1/\tau^2)$ and $(e_2'Me_2/\tau^2)$ are independent and that both are non-central Chi-square random variates with, respectively, m and v degrees of freedom and non-centrality parameters $\lambda_{n\tau}$ and $\lambda_{d\tau}$. So, using Theorem 1 of Judge and Bock (1978, p.321),

$$
E_N(Q) = \tau LL'(d+Z\gamma)P_{20}^{d\tau}/\tau
$$

---

[8] We follow here the approach of Ullah and Phillips (1986).

$$= XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)P_{20}^{d\tau} . \qquad (4.3.9)$$

So, $\text{bias}(X\hat{b})=-MZ\gamma-XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)\int_0^\infty P_{20}^{d\tau}f(\tau)d\tau.$ #

The ways in which these results simplify in certain special cases are given in the following corollaries. Corollary 4.3.1 considers the situation of a properly specified design matrix; and Corollaries 4.3.2 and 4.3.3 derive the bias functions of the pre-test estimator $X\hat{b}$ when the errors are, respectively, Mt and normal. Note that we need only consider the bias of $X\hat{b}$ for specific $f(\tau)$ as the bias functions of Xb and Xb* are independent of τ and so are the same for all members of the SSD family.

Corollary 4.3.1

If there are no omitted regressors ($Z\gamma=0$) and the regression disturbances are $SSD_N(0,I_T)$ then

$$\text{bias}_0(Xb) = 0 , \qquad (4.3.10)$$

$$\text{bias}_0(Xb^*) = -XS^{-1}R'[RS^{-1}R']^{-1}\delta , \qquad (4.3.11)$$

$$\text{bias}_0(X\hat{b}) = -XS^{-1}R'[RS^{-1}R']^{-1}\delta\int_0^\infty P_{20}^{\tau}f(\tau)d\tau. \qquad (4.3.12)$$

where $$P_{ij}^{\tau} = \Pr.\left[F'_{(m+i,v+j;\lambda_\tau)} \le \left(cm(v+j)\right)/\left(v(m+i)\right)\right] , \qquad (4.3.13)$$

$\lambda_\tau=\theta/\tau^2$ and $\theta=\delta'[RS^{-1}R']^{-1}\delta/2$, $i,j=0,1,2,\ldots$.

Proof.

$Z\gamma=0$, so $\Lambda=0$ and (4.3.10), (4.3.11) and (4.3.12) follow directly from, respectively, (4.3.3), (4.3.4) and (4.3.5). #

Corollary 4.3.2

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $Mt(0,\sigma^2\nu/(\nu-2)I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\text{bias}_{Mt}(X\hat{b}) = -MZ\gamma - XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)P^{d}_{202}\,, \tag{4.3.14}$$

and if there are no omitted regressors, $Z\gamma = 0$, then

$$\text{bias}_{OMt}(X\hat{b}) = -XS^{-1}R'[RS^{-1}R']^{-1}\delta P_{202}\,, \tag{4.3.15}$$

where

$$P^{d}_{ijn} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} \frac{\left(2\lambda_n/\nu\right)^r\left(2\lambda_d/\nu\right)^s\Gamma\left(\frac{\nu}{2}+r+s+n-2\right)}{r!s!\left[1+2(\lambda_n+\lambda_d)/\nu\right]^{\frac{\nu}{2}+r+s+n-2}\Gamma\left(\frac{\nu}{2}+n-2\right)}$$

$$\cdot\ I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]\,, \tag{4.3.16}$$

and

$$P_{ijn} = \sum_{r=0}^{\infty}\frac{\left(2\lambda/\nu\right)^r\Gamma\left(\frac{\nu}{2}+r+n-2\right)}{r!\left(1+2\lambda/\nu\right)^{\frac{\nu}{2}+r+n-2}\Gamma\left(\frac{v}{2}+n-2\right)} I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)\right] \tag{4.3.17}$$

$i,j,n=0,1,2,\ldots$.

<u>Proof</u>.

$e\sim Mt(0,\sigma^2\nu/(\nu-2)I_T)$ when $f(\tau)$ is an inverted gamma (IG) density function. So, let $P^{dt}_{ij} = \int_0^\infty P^{d\tau}_{ij}f(\tau)d\tau$ when $\tau\sim IG$ then, using (4.2.12) and (4.3.6), we have

$$P^{dt}_{ij} = \int_0^\infty \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{-(\lambda_{n\tau}+\lambda_{d\tau})}\left(\lambda^r_{n\tau}/r!\right)\left(\lambda^s_{d\tau}/s!\right)I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]$$

$$\cdot\ \frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\nu\sigma^2}{2}\right)^{\nu/2}\tau^{-(\nu+1)}e^{-\nu\sigma^2/2\tau^2}d\tau$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta^r_n\theta^s_d}{r!s!\Gamma\left(\frac{\nu}{2}\right)}I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]2(\nu\sigma^2/2)^{\nu/2}$$

$$\cdot \int_0^\infty e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]/2\tau^2} \left(\tau^2\right)^{-(r+s+\nu/2+1/2)} d\tau \, .$$

From the proof to Corollary 4.2.2 we showed that

$$\int_0^\infty e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]/2\tau^2} \left(\tau^2\right)^{-(r+s+\nu/2+1/2)} d\tau$$

$$= \frac{1}{2}\left[\frac{2}{\left(2(\theta_n+\theta_d)+\nu\sigma^2\right)}\right]^{r+s+\nu/2} \Gamma\left(\frac{\nu}{2}+r+s\right) , \tag{4.3.18}$$

and so

$$P^{dt}_{ij} = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{\left(2\theta_n/\nu\sigma^2\right)^r \left(2\theta_d/\nu\sigma^2\right)^s \Gamma\left(\frac{\nu}{2}+r+s\right)}{r!s!\left[1+2(\theta_n+\theta_d)/\nu\sigma^2\right]^{\frac{\nu}{2}+r+s} \Gamma\left(\frac{\nu}{2}\right)}$$

$$\cdot\; I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]$$

$$= P^{d}_{ij2} \, , \tag{4.3.19}$$

using (4.3.16), as $\lambda_n=\theta_n/\sigma^2$ and $\lambda_d=\theta_d/\sigma^2$.[9] Thus, $\int_0^\infty P^{d\tau}_{20} f(\tau)d\tau = P^{d}_{202}$ when τ~IG and so, as τ only enters (4.3.5) via this term (4.3.14) follows directly.

Now, we can write P^{d}_{ijn} as

$$P^{d}_{ijn} = \sum_{r=0}^{\infty} \frac{\left(2\lambda_n/\nu\right)^r \Gamma\left(\frac{\nu}{2}+r+n-2\right) I_x\left[\frac{1}{2}(m+i)+r;\frac{1}{2}(v+j)\right]}{r!\left[1+2(\lambda_n+\lambda_d)/\nu\right]^{\frac{\nu}{2}+r+n-2} \Gamma\left(\frac{\nu}{2}+n-2\right)}$$

---

9 Although we could use the definition of P^{dt}_{ij} as given by (4.3.19) it is more helpful at a later stage to employ the definition of P^{d}_{ijn} as given by (4.3.16).

$$+\sum_{r=0}^{\infty}\sum_{s=1}^{\infty}\frac{\left(2\lambda_n/\nu\right)^r\left(2\lambda_d/\nu\right)^s\Gamma\left(\frac{\nu}{2}+r+s+n-2\right)}{r!s!\left[1+2(\lambda_n+\lambda_d)/\nu\right]^{\frac{\nu}{2}+r+s+n-2}\Gamma\left(\frac{\nu}{2}+n-2\right)}$$

$$.\ I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]\ , \qquad (4.3.20)$$

and so P_{ijn} follows when $Z\gamma=0$ as then $\lambda_d=0$, and $\lambda_n=\lambda$. When the design matrix is properly specified $\Lambda=0$ and making these appropriate substitutions into (4.3.14) gives (4.3.15) directly. #

Corollary 4.3.3

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are normally distributed as $N(0,\sigma^2 I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\text{bias}_N(X\hat{b}) = -MZ\gamma - XS^{-1}R'[RS^{-1}R']^{-1}(\delta+\Lambda)P^d_{20}\ , \qquad (4.3.21)$$

and if there are no omitted regressors, $Z\gamma=0$, then

$$\text{bias}_{0N}(X\hat{b}) = -XS^{-1}R'[RS^{-1}R']^{-1}\delta P_{20}\ , \qquad (4.3.22)$$

where

$$P^d_{ij} = \Pr.\left[F''_{(m+i,v+j;\lambda_n,\lambda_d)} \le \left(cm(v+j)\right)/\left(v(m+i)\right)\right]$$

and

$$P_{ij} = \Pr.\left[F'_{(m+i,v+j;\lambda)} \le \left(cm(v+j)\right)/\left(v(m+i)\right)\right]\ .$$

Proof.

(4.3.21) and (4.3.22) can be obtained from Corollary 4.3.2 as $e\sim N(0,\sigma^2 I_T)$ when $\nu=\infty$. In this case $\lim_{\nu\to\infty}P^d_{ijn}=P^d_{ij}$ and $\lim_{\nu\to\infty}P_{ijn}=P_{ij}$. We only need show that $\lim_{\nu\to\infty}P^d_{ijn}=P^d_{ij}$ as the other limit follows an analogous proof. For this result to hold we require that

$$L \equiv \lim_{\nu\to\infty} \frac{\left(\frac{\nu}{2}\right)^{-(r+s)} \Gamma\left(\frac{\nu}{2}+r+s+n-2\right)}{\left[1+\frac{(\lambda_n+\lambda_d)}{(\nu/2)}\right]^{\frac{\nu}{2}+r+s+n-2} \Gamma\left(\frac{\nu}{2}+n-2\right)} = e^{-(\lambda_n+\lambda_d)} .$$

Now, let $z = \nu/2$ and let $w = n-2$ so that

$$L = \lim_{z\to\infty} \frac{\Gamma(z+w+r+s)z^{-(r+s)}}{\left[1+\frac{(\lambda_n+\lambda_d)}{z}\right]^{z}\left[1+\frac{(\lambda_n+\lambda_d)}{z}\right]^{w+r+s} \Gamma(z+w)} .$$

Then, observe that

$$\lim_{z\to\infty} \left[1+\frac{(\lambda_n+\lambda_d)}{z}\right]^{z} = e^{(\lambda_n+\lambda_d)} ,$$

$$\lim_{z\to\infty} \left[1+\frac{(\lambda_n+\lambda_d)}{z}\right]^{w+r+s} = 1 ,$$

$$\lim_{z\to\infty} \left[\frac{\Gamma(z+w+r+s)}{z^{r+s}\Gamma(z+w)}\right] = \lim_{z\to\infty} \left[\frac{(z+w+r+s-1)(z+w+r+s-2)\ldots(z+w)\Gamma(z+w)}{z^{r+s}\Gamma(z+w)}\right]$$

$$= \lim_{z\to\infty} \left[\left(1+\frac{w+r+s-1}{z}\right)\left(1+\frac{w+r+s-2}{z}\right)\ldots\left(1+\frac{w}{z}\right)\right]$$

$$= 1 ,$$

which proves the result. #

The bias functions of Xb and Xb* do not depend on the specific variance mixing distribution and, in fact, are the same for all members of the $SSD_N(0,I_T)$ family. Intuitively, this is not surprising as $E(e)=0$. However, from Theorem 4.3.1 and its associated corollaries, the bias of $X\hat{b}$ is affected by $f(\tau)$ through $\int_0^\infty P_{20}^{d\tau} f(\tau)d\tau$. This is intuitively reasonable as we form the pre-test estimator by weighting each of its component estimators, with the weights depending on the non-null distribution of u, which we showed in Section 4.2 depends on τ.

If the design matrix excludes relevant regressors then Xb, Xb* and $X\hat{b}$ are biased. Xb is unbiased only when the model is properly specified ($Z\gamma=0$) or the included and the excluded regressors are orthogonal ($X'Z=0$). Otherwise the bias of Xb is of the opposite sign to that of $Z\gamma$.

Xb* and $X\hat{b}$ are unbiased only when $Z\gamma=0$, or $X'Z=0$, and the linear restrictions are valid ($\delta=0$). That is, Xb* and $X\hat{b}$ are biased even if $\delta=0$, if the model is mis-specified. Note that if $\lambda_n=0$, i.e. $\delta+\Lambda=0$, then $\text{bias}(Xb)=\text{bias}(Xb^*)=\text{bias}(X\hat{b})=-MZ\gamma$. The bias of $X\hat{b}$ depends on all of the arguments in the model; X, Z, β, γ, R, r, τ^2, θ_n, θ_d, m, v, and α. In particular, when $\alpha=0$ ($c=\infty$) we will always accept H_0, the pre-test estimator collapses to the restricted estimator, $p_{ij}^{d\tau}=1$, and so $\text{bias}(X\hat{b})=\text{bias}(Xb^*)$. Alternatively, $\text{bias}(X\hat{b})\to\text{bias}(Xb)$ as $\alpha\to0$, as we reject H_0 more frequently.

Corollary 4.3.2 gives the bias functions of $X\hat{b}$ when the regression disturbances are distributed as $Mt(0,\sigma_e^2 I_T)$. These functions depend on the scale parameter, σ^2, and the degrees of freedom, ν, of the Mt distribution. Accordingly, in particular, they will vary for different values of ν.

We now turn our attention to the risk functions of the estimators.

<u>Theorem 4.3.2</u>

Under the assumptions of Theorem 4.3.1

$$\rho\left(E(y),Xb\right) = kE(\tau^2) + 2\theta_d , \tag{4.3.23}$$

$$\rho\left(E(y),Xb^*\right) = (k-m)E(\tau^2) + 2(\theta_d + \theta_n) , \tag{4.3.24}$$

$$\rho\left(E(y),X\hat{b}\right) = kE(\tau^2) + 2\theta_d + 4\theta_n\int_0^\infty P_{20}^{d\tau}f(\tau)d\tau$$

$$- m\int_0^\infty \tau^2 P_{20}^{d\tau} f(\tau)d\tau - 2\theta_n \int_0^\infty P_{40}^{d\tau} f(\tau)d\tau. \qquad (4.3.25)$$

Proof.

$V(Xb)=\int_0^\infty V_N(Xb)f(\tau)d\tau$ where $V_N(Xb)$ is the $V(Xb)$ when $e\sim N(0,\sigma^2 I_T)$. It is straightforward to show that $V_N(Xb)=\tau^2 XS^{-1}X'$, and so $V(Xb)=XS^{-1}X'E(\tau^2)$. (4.3.23) follows as $tr\left(V(Xb)\right)=kE(\tau^2)$ and $tr\left(bias(Xb)bias(Xb)'\right)=\gamma' Z' MZ\gamma$.

Similarly, let $V_N(Xb^*)=V(Xb)$ when $e\sim N(0,\tau^2 I_T)$, then $V(Xb^*)=\int_0^\infty V_N(Xb^*)f(\tau)d\tau=\tau^2\left(XS^{-1}X'-A'BA\right)$. So, $V(Xb^*)=E(\tau^2)\left(XS^{-1}X'-A'BA\right)$ and $tr\left(V(Xb^*)\right)=(k-m)E(\tau^2)$. (4.3.24) follows as $tr\left(bias(Xb^*)bias(Xb^*)'\right)=\gamma' Z' mZ\gamma+(\delta+\Lambda)'[RS^{-1}R']^{-1}(\delta+\Lambda)$.

To establish $\rho\left(E(y),X\hat{b}\right)$ we write $X\hat{b}=Xb-A'B(Rb-r)I_{[0,c]}(u)$, so that $X\hat{b}-E(y)=\left(Xb-E(y)\right)-A'B(Rb-r)I_{[0,c]}(u)$ and

$$\rho\left(E(y),X\hat{b}\right) = E\left[\left(X\hat{b}-E(y)\right)'\left(X\hat{b}-E(y)\right)\right]$$

$$= \rho\left(E(y),Xb\right) + E\left\{\left[(Rb-r)'B(Rb-r) - 2\left(Xb-E(y)\right)'A'B(Rb-r)\right]I_{[0,c]}(u)\right\}$$

$$= \rho\left(E(y),Xb\right) + E\left\{\left[2(\delta+\Lambda)'B(Rb-r) - (Rb-r)'B(Rb-r)\right]I_{[0,c]}(u)\right\}$$

as $\left(Xb-E(y)\right)'A' = \left[(Rb-r)-\left((R\beta-r)+AZ\gamma\right)\right]'$. We now adopt the notation and definitions used in the proof to Theorem 4.3.1. So,

$$\rho\left(E(y),X\hat{b}\right)= \rho\left(E(y),Xb\right) + E\Big\{2\tau(d+Z\gamma)'L(e^*/\tau)I_{[0,c]}\left[(ve^{*\prime}e^*/\tau^2)/\right.$$

$$\left.(me_2'Me_2/\tau^2)\right]-\tau^2(e^{*\prime}e^*/\tau^2)I_{[0,c]}\left[(ve^{*\prime}e^*/\tau^2)/(me_2'Me_2/\tau^2)\right]\Big\}$$

$$= \rho\left(E(y),Xb\right) + E\{G\}$$

$$= \rho\left(E(y),Xb\right) + \int_0^\infty E_N\{G\}f(\tau)d\tau , \qquad (4.3.26)$$

where $E_N\{G\}$ is $E\{G\}$ when $e\sim N(0,\tau^2 I_T)$. In this case $e^*/\tau\sim N\left(L'(d+Z\gamma)/\tau,I_m\right)$

and $E_N\left((e^*/\tau)I_{[0,c]}\left[(ve^{*\prime}e^*/\tau^2)/(me_2'Me_2/\tau^2)\right]\right)=L'(d+Z\gamma)P_{20}^{d\tau}/\tau$, using Theorem 1 of Judge and Bock (1978, p.321); and $E_N\left((e^{*\prime}e^*/\tau^2)I_{[0,c]}\left[(ve^{*\prime}e^*/\tau^2)/(me_2'Me_2/\tau^2)\right]\right)=mP_{20}^{d\tau}+2\lambda_{n\tau}P_{40}^{d\tau}$, using Lemma 1 of Clarke *et al.* (1987a). So,

$$E_N\{G\} = 2(d+Z\gamma)'LL'(d+Z\gamma)P_{20}^{d\tau} - \tau^2 mP_{20}^{d\tau} - 2\tau^2\lambda_{n\tau}P_{40}^{d\tau}$$

$$= (4\theta_n-\tau^2 m)P_{20}^{d\tau} - 2\theta_n P_{40}^{d\tau} , \qquad (4.3.27)$$

as $\lambda_{n\tau}=\theta_n/\tau^2$ and $(d+Z\gamma)'LL'(d+Z\gamma)=2\theta_n$. Substituting (4.3.27) into (4.3.26) completes the proof. #

The risk functions depend on the hypothesis error through δ and hence θ_n; and the specification error through Λ, and so θ_n, and θ_d. Corollary 4.3.4 derives the risk functions of Xb, Xb^* and $X\hat{b}$ for the special case of no omitted regressors.

We note that the risk functions of Xb and Xb^*, as well as that of $X\hat{b}$, in contrast to the bias functions, are determined by the specific form of $f(\tau)$. That $\rho\left(E(y),Xb\right)$ and $\rho\left(E(y),Xb^*\right)$ also depend on $f(\tau)$ is not surprising as we require the variance-covariance matrix of e to derive these functions, and $E(ee')=E(\tau^2)I_T$. Corollaries 4.3.5 and 4.3.6 evaluate the risk functions of Xb, Xb^* and $X\hat{b}$ for the special cases of Mt and normal errors.

<u>Corollary 4.3.4</u>

If there are no omitted regressors $(Z\gamma=0)$ and the regression disturbances are $SSD_N(0,I_T)$ then

$$\rho_0\left(E(y),Xb\right) = kE(\tau^2) , \qquad (4.3.28)$$

$$\rho_0\left(E(y),Xb^*\right) = (k-m)E(\tau^2) + 2\theta , \qquad (4.3.29)$$

$$\rho_0\left(E(y),X\hat{b}\right) = kE(\tau^2)+ 4\theta\int_0^\infty P_{20}^{\tau}f(\tau)d\tau - 2\theta\int_0^\infty P_{40}^{\tau}f(\tau)d\tau$$

$$- m\int_0^\infty \tau^2 P_{20}^{\tau} f(\tau)d\tau \ . \qquad (4.3.30)$$

Proof.

If $Z\gamma=0$, $\Lambda=0$, so $\theta_d=0$, $\theta_n=\theta$, $P_{ij}^{d\tau}=P_{ij}^{\tau}$ and (4.3.23), (4.3.24) and (4.3.24) collapse, respectively, to (4.3.28), (4.3.29) and (4.3.30). #

Corollary 4.3.5

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $Mt(0,\sigma^2\nu/(\nu-2)I_T)$, and the pre-test is of H_0 in (4.3.2), then for $\nu>2$

$$\rho_{Mt}\left(E(y),Xb\right) = \sigma^2\left[k\nu + 2\lambda_d(\nu-2)\right]/(\nu-2) \ , \qquad (4.3.31)$$

$$\rho_{Mt}\left(E(y),Xb^*\right) = \sigma^2\left[(k-m)\nu + 2(\lambda_d+\lambda_n)(\nu-2)\right]/(\nu-2) \ , \qquad (4.3.32)$$

$$\rho_{Mt}\left(E(y),X\hat{b}\right) = \sigma^2\left[k\nu-m\nu P_{201}^{d} + 2\lambda_d(\nu-2)\right.$$

$$\left. + 2\lambda_n(\nu-2)(2P_{202}^{d}-P_{402}^{d})\right]/(\nu-2) \ . \qquad (4.3.33)$$

If there are no omitted regressions, $Z\gamma=0$, then

$$\rho_{OMt}\left(E(y),Xb\right) = \sigma^2 k\nu/(\nu-2) \ , \qquad (4.3.34)$$

$$\rho_{OMt}\left(E(y),Xb^*\right) = \sigma^2\left[(k-m)\nu + 2\lambda(\nu-2)\right]/(\nu-2) \ , \qquad (4.3.35)$$

$$\rho_{OMt}\left(E(y),X\hat{b}\right) = \sigma^2\left[k\nu-m\nu P_{201} + 2\lambda(\nu-2)(2P_{202}-P_{402})\right]/(\nu-2). \qquad (4.3.36)$$

Proof.

When $\tau\sim IG$ with scale parameter σ^2 and degrees of freedom parameter ν then $E(\tau^2)=\int_0^\infty \tau^2 f(\tau)d\tau=\nu\sigma^2/(\nu-2)$. Substituting this into (4.3.23) and (4.3.24) and noting that $\lambda_d=\theta_d/\sigma^2$ and $\lambda_n=\theta_n/\sigma^2$ gives (4.3.31) and (4.3.32).

(4.3.33) follows from (4.3.25) as $E(\tau^2)=\nu\sigma^2/(\nu-2)$, $\int_0^\infty P_{ij}^{d\tau} f(\tau)d\tau=P_{ij2}^{d}$

from (4.3.19), and

$$\int_0^\infty \tau^2 P_{ij}^{d\tau} f(\tau)d\tau = \frac{\nu\sigma^2}{\nu-2} P_{ij1}^{d} . \tag{4.3.37}$$

To prove (4.3.37) we write

$$\int_0^\infty \tau^2 P_{ij}^{d\tau} f(\tau)d\tau = \int_0^\infty \sum_{r=0}^\infty \sum_{s=0}^\infty e^{-(\lambda_{n\tau}+\lambda_{d\tau})} \left(\lambda_{n\tau}^r/r!\right)\left(\lambda_{n\tau}^s/s!\right)$$

$$.I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]\frac{2}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu\sigma^2}{2}\right)^{\nu/2}\left(\tau^2\right)^{\frac{1}{2}-\frac{\nu}{2}} e^{-\nu\tau^2/2\tau^2} d\tau$$

$$= \sum_{r=0}^\infty \sum_{s=0}^\infty (\theta_n^r/r!)(\theta_n^s/s!)I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]$$

$$\cdot \frac{2}{\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{\nu\sigma^2}{2}\right)^{\nu/2} \int_0^\infty \left(\tau^2\right)^{-(\frac{\nu}{2}+r+s-\frac{1}{2})} e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]/2\tau^2} d\tau .$$

Now,

$$\int_0^\infty \left(\tau^2\right)^{-(\frac{\nu}{2}+r+s-\frac{1}{2})} e^{-[2(\theta_n+\theta_d)+\nu\sigma^2]/2\tau^2} d\tau$$

$$= \frac{1}{2}\left[\frac{2}{2(\theta_n+\theta_d)+\nu\sigma^2}\right]^{\frac{\nu}{2}+r+s-1} \Gamma\left(\frac{\nu}{2}+r+s-1\right) \tag{4.3.38}$$

and so

$$\int_0^\infty (\tau^2)P_{ij}^{d\tau} f(\tau)d\tau = \sum_{r=0}^\infty \sum_{s=0}^\infty \frac{\left(2\theta_n/\sigma^2\nu\right)^r\left(2\theta_d/\sigma^2\nu\right)^s\Gamma\left(\frac{\nu}{2}+r+s-1\right)}{r!s!\Gamma\left(\frac{\nu}{2}\right)}$$

$$\cdot \left(\frac{\nu\sigma^2}{2}\right)^{\frac{\nu}{2}+r+s}\left[\frac{2}{2(\theta_n+\theta_d)+\nu\sigma^2}\right]^{\frac{\nu}{2}+r+s-1} I_x\left[\tfrac{1}{2}(m+i)+r;\tfrac{1}{2}(v+j)+s\right]$$

from which (4.3.38) follows as $\lambda_n=\theta_n/\sigma^2$, $\lambda_d=\theta_d/\sigma^2$, and $\Gamma\left(\frac{\nu}{2}\right)=\left(\frac{\nu}{2}-1\right)\Gamma\left(\frac{\nu}{2}-1\right)$.

(4.3.31), (4.3.32) and (4.3.33) collapse to (4.3.34), (4.3.35) and (4.3.36) when $Z\gamma=0$, as then $\lambda_d=0$, $\lambda_n=\lambda$, and $P_{ijn}^{d}=P_{ijn}$. #

Corollary 4.3.6

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $N(0,\sigma^2 I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\rho_N\left(E(y),Xb\right) = \sigma^2(k + 2\lambda_d) , \tag{4.3.39}$$

$$\rho_N\left(E(y),Xb^*\right) = \sigma^2\left[(k-m) + 2(\lambda_d+\lambda_n)\right] , \tag{4.3.40}$$

$$\rho_N\left(E(y),X\hat{b}\right) = \sigma^2\left[k+2\lambda_d - mP^d_{20} + 2\lambda_n(2P^d_{20}-P^d_{40})\right]. \tag{4.3.41}$$

If there are no omitted regressors, $Z\gamma=0$, then

$$\rho_{ON}\left(E(y),Xb\right) = \sigma^2 k , \tag{4.3.42}$$

$$\rho_{ON}\left(E(y),Xb^*\right) = \sigma^2(k-m+2\lambda) , \tag{4.3.43}$$

$$\rho_{ON}\left(E(y),X\hat{b}\right) = \sigma^2\left[k-mP_{20} + 2\lambda(2P_{20}-P_{40})\right] . \tag{4.3.44}$$

Proof.

Let $\nu=\infty$ in Corollary 4.3.5 and note that in this case $P^d_{ijn}=P^d_{ij}$ and $P_{ijn}=P_{ij}$. #

Remarks

(i) $\rho_N\left(E(y),Xb\right)$, $\rho_N\left(E(y),Xb^*\right)$, $\rho_N\left(E(y),X\hat{b}\right)$, $\rho_{ON}\left(E(y),Xb\right)$, $\rho_{ON}\left(E(y),Xb^*\right)$ and $\rho_{ON}\left(E(y),X\hat{b}\right)$ the risk functions given by (2.2.3), (2.2.4), (2.2.11), (2.2.25), (2.2.26) and (2.2.23), respectively, in Chapter Two.

(ii) When $\alpha=1$, $c=0$, $P^{d\tau}_{ij}=0$, so we always reject H_0 and the risk of the pre-test estimator equals that of the unrestricted estimator, because the pre-test will always lead us to select Xb. Conversely, the smaller α is (the closer c is to ∞), the nearer the pre-test risk is to that of the restricted estimator as a smaller test size increases the probability of accepting the null hypothesis.

Given the mathematical complexities of the risk functions, it is more illuminatory to consider the risks with the aid of numerical evaluations rather than attempting to glean their details solely from the expressions. Consequently, we defer further discussion of the risks until the next sections. There we investigate the special case of multivariate Student-t regression disturbances. Under this assumption we have numerically evaluated the risks for various values of α, m, k, T, and ν as functions of λ_d and λ_n, and with the aid of these evaluations we investigate the properties of the risk functions. Note, however, that we will keep the discussion as general as possible, using the Mt evaluations to illustrate features of the results.

4.4 Comparisons of the Risk Functions when the Regressors are Correctly Specified

In this section we compare the risk functions of Xb, Xb* and $X\hat{b}$ when there are no omitted regressors. To aid this discussion we assume Mt errors and then we numerically evaluate the risk expressions (4.3.35), (4.3.36) and (4.3.37) for various choices of ν, α, m,k, and v (and hence, T) as functions of λ.[10]

A wide selection of values of the arguments was investigated: v=10, 16, 20, 30; k = 4, 5; m=1, 2, 3, 4, 5; α=0.01, 0.05, 0.30, 0.50, 0.75, 0.90, ν=5, 10, 100, 1000, 10000, and ∞; and $\lambda\in[0,3(0.1);3,20(0.5)]$. The FORTRAN

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[10] The following discussion is in terms of the risk of the estimators. To eliminate the scale parameter σ^2 we consider for the numerical evaluations, risk relative to σ^2 and parameterise with respect to λ rather than with respect to θ. So, the relative risk of an estimator $X\bar{b}$ of E(y) is $R\left(E(y),X\bar{b}\right)=\rho\left(E(y),X\bar{b}\right)/\sigma^2$. Equivalently, and without loss of generality, the evaluations could be interpreted as depicting risk functions when $\sigma^2=1$.

computer programs were specially written and were executed on an 80386 personal computer. Davies' (1980) algorithm was used to evaluate the P_{ij}'s and the subroutines GAMMLN and BETAI from Press *et al.* (1986) were employed to obtain the P_{ijn}'s. It is impossible to include all of these results in this thesis. Accordingly, a small, but representative, sample is given in Tables A4.2.1 to A4.2.4 of Appendix 4.2 of this chapter. Figure 4.4.1 and Figure 4.4.2 graphically depict typical risk functions of Xb, Xb* and $X\hat{b}$ ($0<\alpha<1$) for Mt regression disturbances when $\nu=\infty$ and, in contrast, for a small value of ν, respectively. We recall that the disturbances are normal if $\nu=\infty$, so Figure 4.4.1 is identical to Figure 2.2.1 of Chapter Two. Figures 4.4.3 to 4.4.6 graphically present the relevant results of Table A4.2.2.[11] We have omitted legends from these figures to avoid cluttering them. The following legend is applicable to each of the figures.

Legend for Figures 4.4.3 to 4.4.6

$R_0(E(y), Xb)$	$R_0(E(y), Xb^*)$	$R_0(E(y), X\hat{b})$ $\alpha = 0.01$
$R_0(E(y), X\hat{b})$ $\alpha = 0.05$	$R_0(E(y), X\hat{b})$ $\alpha = 0.30$	$R_0(E(y), X\hat{b})$ $\alpha = 0.75$

[11] The tables in Appendix 4.2 are presented in terms of the case of omitted regressors. So, the relevant results for this discussion are those for when $\lambda_d=0$.

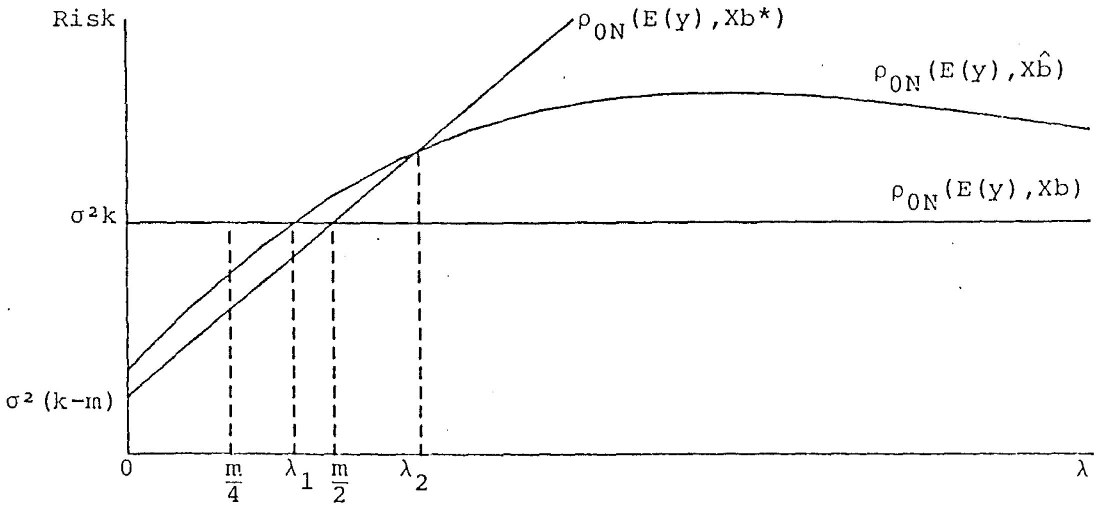

FIGURE 4.4.1: Typical Risk Functions for Xb, Xb* and $X\hat{b}$ when $e \sim N(0,\sigma^2 I_T)$ (i.e. $\nu = \infty$).

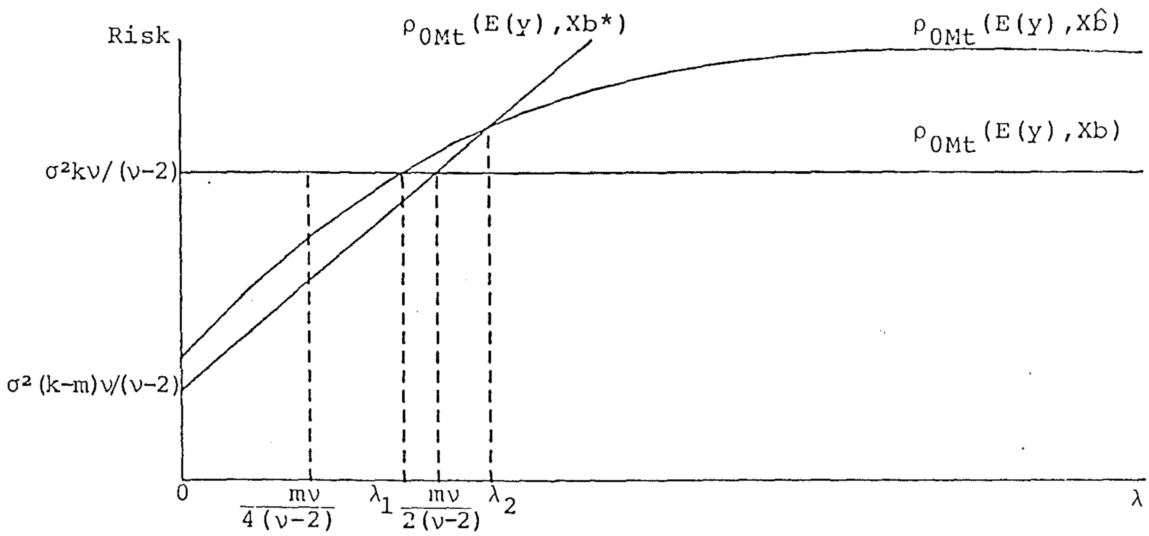

FIGURE 4.4.2: Typical Risk Functions for Xb, Xb* and $X\hat{b}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, and ν is small.

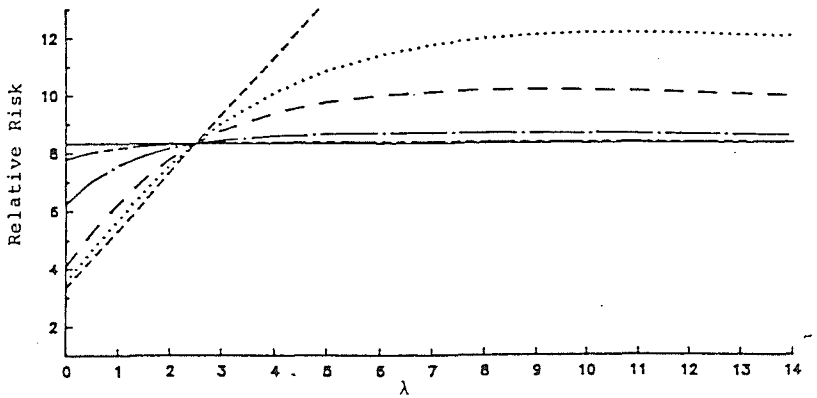

FIGURE 4.4.3: Relative risk functions for Xb, Xb* and $X\hat{b}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 5$.

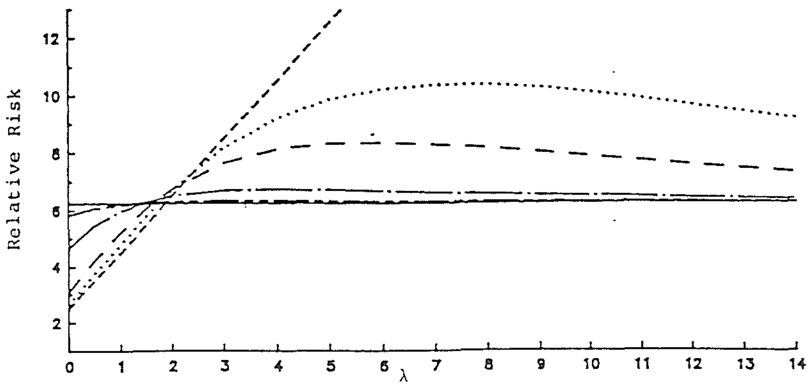

FIGURE 4.4.4: Relative risk functions for Xb, Xb*, and $X\hat{b}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 10$.

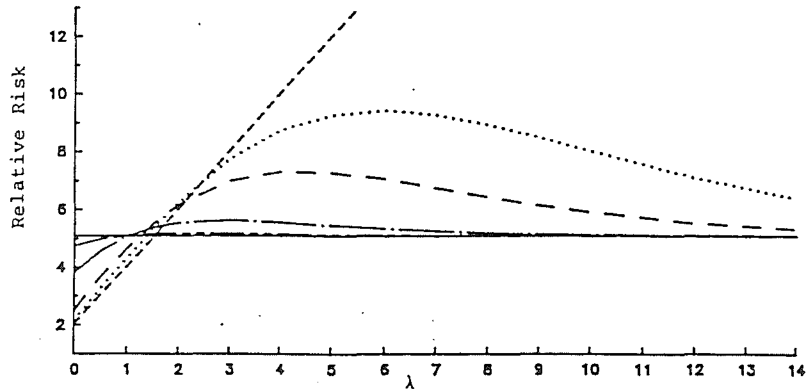

FIGURE 4.4.5: Relative risk functions for Xb, Xb*, and $X\hat{b}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 100$.

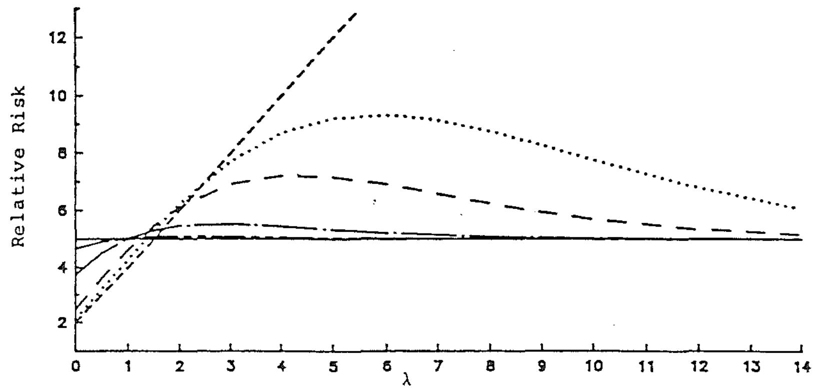

FIGURE 4.4.6: Relative risk functions for Xb, Xb*, and $X\hat{b}$ when $e \sim N(0, \sigma^2 I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = \infty$.

The following comments, nevertheless, are based on the full set of results:

(a) When the null hypothesis is true $(\delta=\theta=\lambda=0)$, $\rho_0\left(E(y),Xb\right)=kE(\tau^2)>$ $\rho_0\left(E(y),X\hat{b}\,|\,\delta=0\right)=E(\tau^2)(k-m)I_x\left[(m+2)/2;v/2\right] > \rho_0\left(E(y),Xb^*\,|\,\delta=0\right)=(k-m)E(\tau^2)$, as $P^{\tau}_{20}=I_x\left[(m+2)/2;v/2\right]$ when $\delta=0$, and $0<P^{\tau}_{ij}<1$ for $0<\alpha<1$ and $i,j=0,1,2,\ldots$.

(b) Equality of the risks of Xb and Xb* occurs when $\theta=mE(\tau^2)/2\equiv\theta^*\equiv\lambda^*\sigma^2$. If $e\sim N(0,\sigma^2 I_T)$ then, as we noted in Chapter Two, $\lambda^*_N=m/2$ (see Figure 4.2.1); while if $e\sim Mt\left(0,\nu\sigma^2/(\nu-2)I_T\right)$ then $\lambda^*_{Mt}=m\nu/\left(2(\nu-2)\right)>\lambda^*_N$.

So, if we assume normality when in fact the distribution of the errors belongs to the wider class of SSD_N, then there is a range of θ over which we would choose the incorrect estimator. For example, if $E(\tau^2)>\sigma^2$ (that is, the marginal distribution of e has fatter tails than under normality) then, to minimise risk, we would select Xb* for $\theta<\theta^*$ but assuming normal disturbances implies that we would incorrectly select Xb for $\theta\in(m\sigma^2/2,\theta^*)$ (or equivalently, $\lambda\in(\lambda^*_N,\lambda^*)$).

(c) $\rho\left(E(y),X\hat{b}\right)$ is a function of θ and so the difference between the risk of $X\hat{b}$ and that of either Xb or Xb* is a function of θ. The difference between the risk of Xb and $X\hat{b}$, $\rho_0\left(E(y),Xb\right)-\rho_0\left(E(y),X\hat{b}\right)$, is

$$= -4\theta\int_0^{\infty}P^{\tau}_{20}f(\tau)d\tau + 2\theta\int_0^{\infty}P^{\tau}_{40}f(\tau)d\tau + m\int_0^{\infty}\tau^2P^{\tau}_{20}f(\tau)d\tau$$

$$= 2\theta\int_0^{\infty}\left(P^{\tau}_{40}-P^{\tau}_{20}\right)f(\tau)d\tau + \int_0^{\infty}P^{\tau}_{20}(m\tau^2-2\theta)f(\tau)d\tau \; . \qquad (4.4.1)$$

$\left(P^{\tau}_{40}-P^{\tau}_{20}\right)<0$ (Judge and Bock (1973, p.73)) and $f(\tau)$ is a proper pdf, so a sufficient condition for Xb to be risk superior to $X\hat{b}$ is for

$$(m\tau^2 - 2\theta) < 0 \quad ; \quad \forall\ \tau \; . \qquad (4.4.2)$$

In particular, equation (4.4.2) implies that $\rho_0\left(E(y),Xb\right)-\rho_0\left(E(y),X\hat{b}\right)<0$, if $\theta>mE(\tau^2)/2$, or equivalently, if $\lambda>\lambda^*$.

Alternatively, we can write equation (4.4.1) as

$$\rho_0\left[E(y),Xb\right]-\rho_0\left[E(y),X\hat{b}\right]=\int_0^\infty P_{20}^{\tau}(m\tau^2-4\theta)f(\tau)d\tau+2\theta\int_0^\infty P_{40}^{\tau}f(\tau)d\tau. \tag{4.4.3}$$

A sufficient condition for equation (4.4.3) to be positive, implying that $X\hat{b}$ is risk superior to Xb, is

$$(m\tau^2 - 4\theta) > 0 \ ; \quad \forall\ \tau \ . \tag{4.4.4}$$

In particular, equation (4.4.4) implies that $\rho_0\left[E(y),Xb\right]-\rho_0\left[E(y),X\hat{b}\right]>0$, if $\theta<mE(\tau^2)/4$, or equivalently, if $\lambda<mE(\tau^2)/(4\sigma^2)$.

So, the equality of the risks of Xb and $X\hat{b}$ occurs for a value of θ, say θ_1, within the following bounds:

$$\frac{mE(\tau^2)}{4} \le \theta_1 \le \theta^* = \frac{mE(\tau^2)}{2} \ . \tag{4.4.5}$$

Note that if $e\sim N(0,\sigma^2 I_T)$ then $E(\tau^2)=\sigma^2$ and equation (4.4.5) collapses to the bounds discussed in the existing literature, i.e. $\frac{m}{4}\le\lambda_1\le\lambda_N^*=\frac{m}{2}$. Alternatively, if $e\sim Mt\left(0,\nu\sigma^2/(\nu-2)I_T\right)$ then equation (4.4.5) is $\frac{m\nu\sigma^2}{4(\nu-2)}\le\theta_1\le\theta_{Mt}^*=\frac{m\nu\sigma^2}{2(\nu-2)}$, or equivalently, $\frac{m\nu}{4(\nu-2)}\le\lambda_1\le\lambda_{Mt}^*=\frac{m\nu}{2(\nu-2)}$. These features are illustrated in Figure 4.4.1 and Figure 4.4.2, respectively.[12]

These results imply first, that pre-testing is <u>never</u> the preferable strategy; the risk of Xb or Xb* will always be smaller than that of $X\hat{b}$. Secondly, if $\theta_1\ne\theta^*$ then there is a range of θ for which the risk of $X\hat{b}$ is higher than that of <u>both</u> Xb and Xb*. Thirdly if $E(\tau^2)>\sigma^2$ (the marginal distribution of e has fatter tails than under a normality assumption) then the bounds given by equation (4.4.5) are wider than they would be if the errors were normally distributed. This also implies that the pre-test

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[12] We could conceivably also extend the (narrower) bounds $[B_1,B_2]$ discussed by Judge and Bock (1978) to the situation of SSD_N errors (see Chapter Two). We have not pursued this issue.

estimator has smaller risk than the unrestricted least squares estimator for a wider θ-range. The converse result applies when $E(\tau^2)<\sigma^2$.

(d) So, the risk of Xb^* is smaller than that of $X\hat{b}$ for (at least) $\theta \le \theta^*$; their risks are equal for a θ value, θ_2, of at least θ^*; and then for $\theta > \theta_2$ the risk of Xb^* can be infinitely higher than that of $X\hat{b}$.[13] Thus, the risk function of $X\hat{b}$ monotonically increases for a value of $\theta>\theta_2$ and then monotonically decreases to approach the risk of Xb. Intuitively, when the prior information is so wrong that θ is very large, then pre-testing will lead us to do the right thing; to ignore the restrictions.

(e) Comparing Figures 4.4.3 to 4.4.6, a decrease in the value of ν from the normal errors case ($\nu=\infty$) causes an upward shift of the estimator risk functions, a decrease in the rate at which the risk of the pre-test estimator approaches that of the unrestricted estimator, and an increase in the risk gain of the restricted estimator over the unrestricted estimator for all λ such that $R_0\left(E(y),Xb^*\right)<R_0\left(E(y),Xb\right)$. For the unrestricted and the restricted estimators these changes occur because of the increase in the estimators' variances as ν decreases (the marginal distribution has fatter tails). For the pre-test estimator, the increase in its variance and its absolute bias (for relatively large λ) both contribute to the observed differences. Our numerical evaluations suggest that, in general, the difference between an estimator's risk under the normality assumption and that under the Mt assumption is relatively insignificant for $\nu\ge 100$.

(f) *Ceteris paribus*, an increase in the value of v causes the risk functions to shift upwards; an increase in the number of restrictions increases

[13] In the diagrams $\lambda_2=\theta_2/\sigma^2$.

the maximum risk of the pre-test estimator and increases the range of λ over which we would prefer the restricted estimator: $\lambda^*=mE(\tau^2)/(2\sigma^2)$ increases with m. These features are evident from the tables in Appendix 4.2.

(g) If we increase the size of the pre-test, we reject H_0 more often and so we give greater weight to Xb when forming $X\hat{b}$. This reduces the modal value of the risk function of $X\hat{b}$ but at the expense of an increase in the minimum value of this risk. This features raises the question of an 'optimal' test size. We discussed several studies in Chapter Two which consider this question under the assumption of normal errors. We noted that the optimal test size depends on the chosen optimality criterion. Given (f) above, it is unclear as to how a departure from normality would affect the choice of an optimal critical value. Such a study is beyond the scope of this thesis and this issue remains for future research.

In this section we have compared the risk functions of Xb, Xb* and $X\hat{b}$ when the design matrix is properly specified. We have generalized many of the results reported in the literature for the case of normal errors, and we found that the features of the risk functions for SSD_N errors are (qualitatively) similar to those we observed in Chapter Two for normal errors. In particular, pre-testing is never the preferable strategy. There are, however, implications for the choice of estimator if we incorrectly assume normality: there is a θ-range over which we choose the wrong estimator (if θ were known). In the next section we extend this discussion further by allowing for the possibility of omitted regressors.

4.5 Comparisons of the Risk Functions when Relevant Variables are Excluded

In this section we consider the risk functions of Xb, Xb* and $X\hat{b}$ when the design matrix is missing relevant explanatory variables. As in the previous section, we have undertaken numerical evaluations of the relative risk functions, assuming Mt errors, for various choices of ν, α, m, k and v (and hence T) as functions of λ_n and λ_d.

Given that the mis-specification adds another dimension to the problem, that of λ_d, we narrowed the range of values of the other arguments of the problem from those which were investigated in the previous section. We considered v=10, 16, 20, 30; k=4, 5; m=1, 3; α=0.01, 0.05, 0.30, 0.50, 0.75; ν=5, 10, 100, 1000,∞; $\lambda_n \in [0,5(0.5);5,10(1.0);10,20(2.0)]$; and $\lambda_d \in [0,5(0.5);5,10(1.0);10,20(2.0)]$. We used Davies' (1980) algorithm and the subroutines GAMMLN and BETAI from Press *et al.* (1986) to assist with the evaluations of P^d_{ij} and P^d_{ijn}, respectively. FORTRAN computer programs were executed on a VAX 6230 computer.

A typical sample of the results is given in Tables A4.2.1 to A4.2.4 of Appendix 4.2 of this chapter. These tables give the risks of Xb, Xb* and $X\hat{b}$ as functions of λ_n for a given value of λ_d. We recall that λ_d is a measure of the mis-specification error, while λ_n depends on both the hypothesis error and the mis-specification error: when λ_d=0, $\lambda_n=\lambda$. Figures 4.5.1 to 4.5.4 graphically present the risk functions from Table A4.2.2 when λ_d=10 and $\lambda_n \in [0,14]$. We are also interested in the risk functions as λ_d varies. Accordingly, Table A4.2.5, and Figures 4.5.5 to 4.5.8 illustrate the risk functions in this dimension, for various values of λ_n.[14] The legend for the figures follows.

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[14] We could illustrate the results using a risk surface. However, such figures are confusing when there is more than one risk surface. So, we have opted to present two-dimensional cross-sections of the risk surface.

Legend for Figures 4.5.1 to 4.5.8

$R(E(y),Xb)$	$R(E(y),Xb^*)$	$R(E(y),X\hat{b})$ $\alpha = 0.01$
$R(E(y),X\hat{b})$ $\alpha = 0.05$	$R(E(y),X\hat{b})$ $\alpha = 0.30$	$R(E(y),X\hat{b})$ $\alpha = 0.75$

(a) The difference between the risk functions of Xb and Xb* is given by

$$\rho\left(E(y),Xb\right) - \rho\left(E(y),Xb^*\right) = mE(\tau^2) - 2\theta_n \qquad (4.5.1)$$

which is independent of θ_d. So, allowing for the redefinition of θ to θ_n, the risks of Xb and Xb* are equal for the same value of θ_n i.e. $\theta_n^*=mE(\tau^2)/2$. We note directly though, that $\theta_n\neq 0$ when H_0 is true ($\delta=0$) unless θ_d is simultaneously zero or X and Z are orthogonal. Let θ_{n0} be that value of θ_n for which H_0 is true. Then,

$$\theta_{n0} = \Lambda'[RS^{-1}R']^{-1}\Lambda/2 \qquad (4.5.2)$$

where, we recall, $\Lambda=RS^{-1}X'Z\gamma$ is a measure of the mis-specification bias. So, using equation (4.5.1), if $\theta_{n0}>mE(\tau^2)/2$ then the risk of the restricted estimator is greater than that of the unrestricted estimator, even though the prior information is perfectly valid. When the model is mis-specified in this way the use of prior information (even if it is correct) does not guarantee a reduction in the risk of estimating E(y). This is consistent with Mittelhammer (1984) for the case of normal errors.

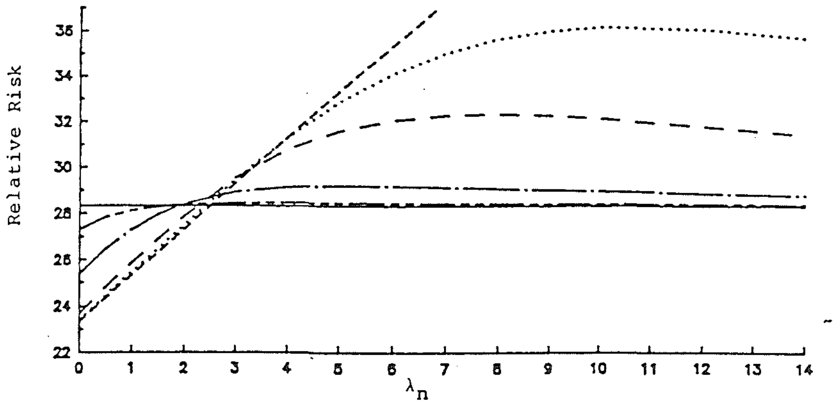

FIGURE 4.5.1: Relative risk functions for Xb, Xb*, and X$\hat{b}$ when e ~ $Mt(0,\nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 5, and λ_d = 10.

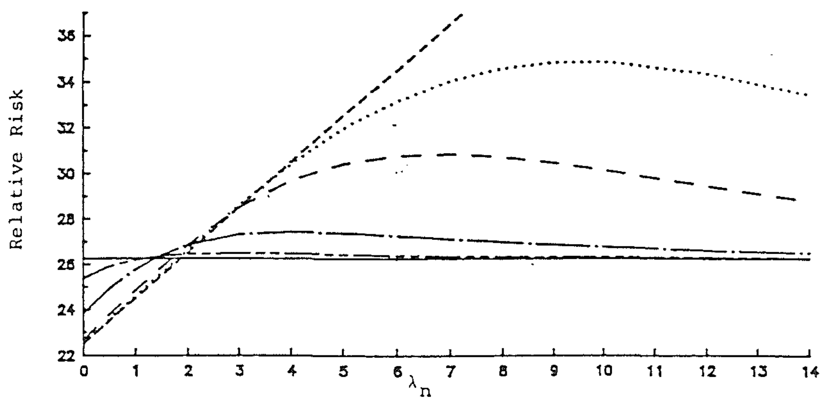

FIGURE 4.5.2: Relative risk functions for Xb, Xb*, and X$\hat{b}$ when e ~ $Mt(0,\nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 10, and λ_d = 10.

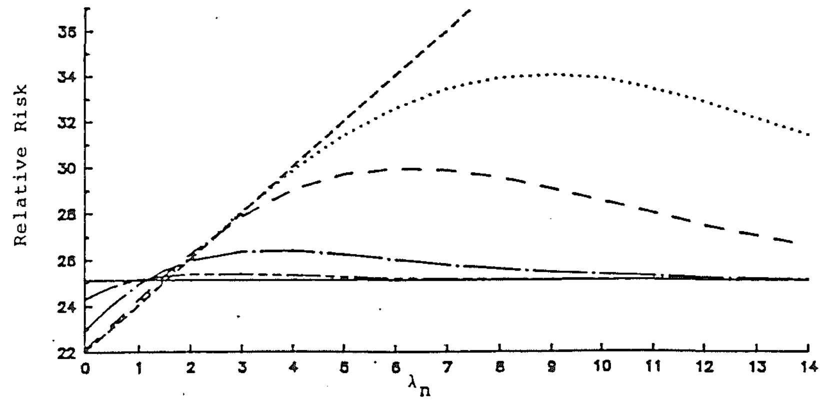

FIGURE 4.5.3: Relative risk functions for Xb, Xb*, and X$\hat{b}$ when e ~ $Mt(0,\nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 100, and λ_d = 10.

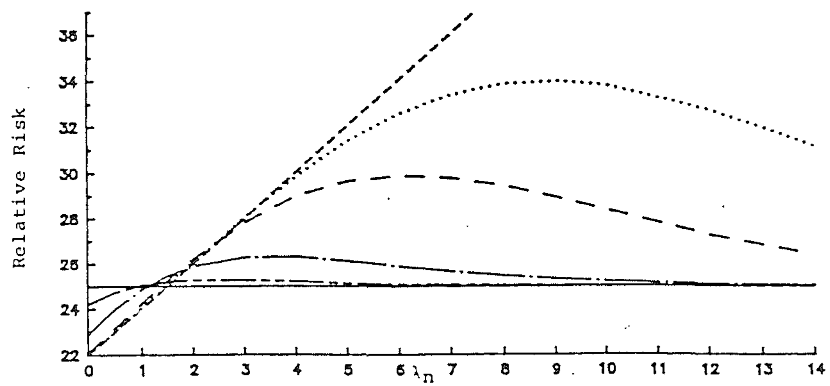

FIGURE 4.5.4: Relative risk functions for Xb, Xb*, and X$\hat{b}$ when e ~ $N(0,\sigma^2 I_T)$, T = 30, k = 5, m = 3, ν = ∞, and λ_d = 10.

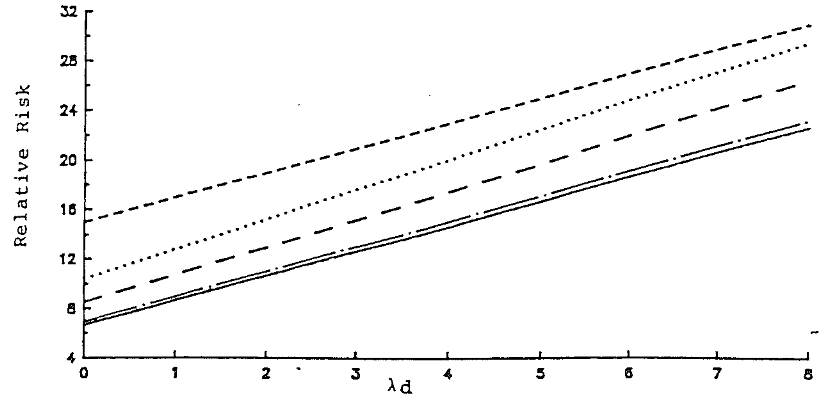

FIGURE 4.5.5: Relative risk functions for Xb, Xb*, and $X\hat{b}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 5, and $\lambda_n = 5$.

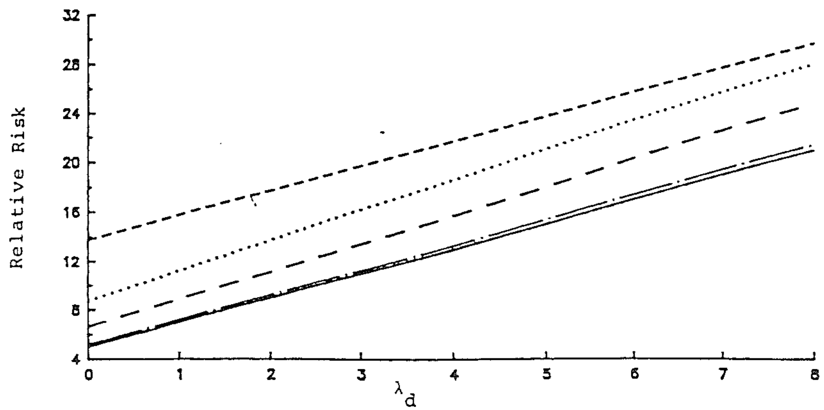

FIGURE 4.5.6: Relative risk functions for Xb, Xb*, and $X\hat{b}$ when $e \sim Mt(0, \sigma\nu^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 10, and $\lambda_n = 5$.

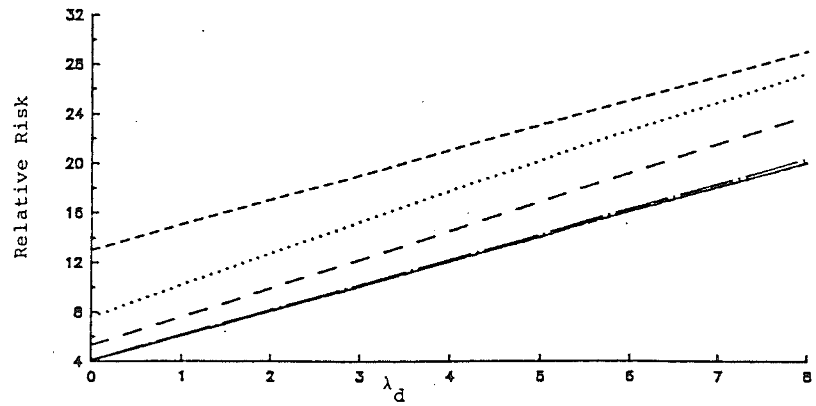

FIGURE 4.5.7: Relative risk functions for Xb, Xb* and $X\hat{b}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 100, and $\lambda_n = 5$.

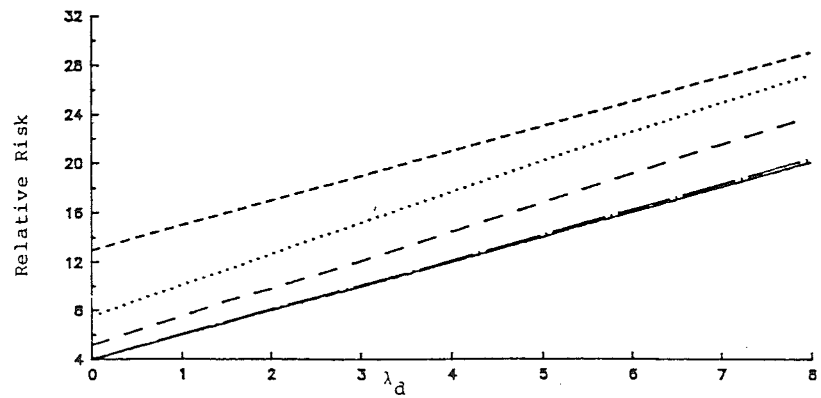

FIGURE 4.5.8: Relative risk functions for Xb, Xb* and $X\hat{b}$ when $e \sim N(0, \sigma^2 I_T)$, T = 20, k = 4, m = 1, ν = ∞, and $\lambda_n = 5$.

Further, Xb^* may have smaller risk than Xb even if δ is significantly large. This will occur if Λ and δ are of opposite sign such that the two biases mitigate each other.

Given that we do not usually know θ_n and θ_d and that $\rho\left(E(y),Xb^*\right)$ is unbounded as $\theta_n\to\infty$ one may suggest that we should always ignore the prior information if we believe our model is also mis-specified, for at least the risk of Xb is bounded as $\theta_n\to\infty$. Unfortunately, however, the risks of both Xb and Xb^* are unbounded as $\theta_d\to\infty$ (though their risk difference is bounded as (4.5.1) does not depend on θ_d). Table A4.2.5 and the figures contained in this section illustrate these results, which are analogous to those which we reported in Chapter Two for the case of normal errors.

(b) $$\rho\left(E(y),Xb\right)-\rho\left(E(y),X\hat{b}\right)=4\theta_n\int_0^\infty P_{20}^{d\tau}f(\tau)d\tau+m\int_0^\infty \tau^2P_{20}^{d\tau}f(\tau)d\tau+2\theta_n\int_0^\infty P_{40}^{d\tau}f(\tau)d\tau$$

$$=\int_0^\infty P_{20}^{d\tau}(m\tau^2-2\theta_n)f(\tau)d\tau + 2\theta_n\int_0^\infty \left(P_{40}^{d\tau}-P_{20}^{d\tau}\right)f(\tau)d\tau . \qquad (4.5.3)$$

Now, $\left(P_{40}^{d\tau}-P_{20}^{d\tau}\right)<0$, (see, for instance, Mittelhammer (1984)) so a sufficient condition for Xb to have smaller risk than $X\hat{b}$ is for

$$(m\tau^2-2\theta_n) < 0 \ ; \ \forall\ \tau . \qquad (4.5.4)$$

In particular, if $\theta_n>mE(\tau^2)/2$ then Xb is risk superior to $X\hat{b}$.

Alternatively, we can write equation (4.5.3) as

$$\int_0^\infty P_{20}^{d\tau}(m\tau^2-4\theta_n)f(\tau)d\tau + 2\theta_n\int_0^\infty P_{40}^{d\tau}f(\tau)d\tau . \qquad (4.5.5)$$

A sufficient condition for (4.5.5) to be positive is for

$$m\tau^2 - 4\theta_n > 0 \ ; \quad \forall\ \tau . \qquad (4.5.6)$$

In particular, if $\theta_n<mE(\tau^2)/4$ then $X\hat{b}$ is risk superior to Xb.

So, the equality of the risks of Xb and $X\hat{b}$ occurs for a value of θ, θ_{n1} within the bounds $\frac{mE(\tau^2)}{4}\le\theta_{n1}\le\frac{mE(\tau^2)}{2}$ or, equivalently,

$$\frac{mE(\tau^2)}{4\sigma^2} \le \lambda_{n1} \le \frac{mE(\tau^2)}{2\sigma^2} \qquad (4.5.7)$$

These bounds are identical to those we reported in the previous section for the case where $\lambda_d=0$, and collapse to those identified by Mittelhammer (1984) when $e\sim N(0,\sigma^2 I_T)$.

We note, though, as does Mittelhammer, that the use of perfectly correct prior linear constraints does not guarantee that the pre-test estimator will have smaller risk than the unrestricted estimator. If the degree of mis-specification is such that $\theta_{n0}<\theta_{n1}$ then $\rho\left(E(y),X\hat{b}\right)<\rho\left(E(y),Xb\right)$. However, the converse will result if $\theta_{n0}>\theta_{n1}$.

(c) The relative risk functions of Xb, Xb* and $X\hat{b}$ for a given value of λ_d are qualitatively similar to the case when $\lambda_d=0$ (see, for instance, Figures 4.5.1 to 4.5.4). We see that an increase in λ_d causes an upward shift of the estimator risk functions; an increase in the maximum regret of the risk function of $X\hat{b}$ from that of Xb for $\lambda_n>\lambda_n^*$, while the regret from Xb* when $\lambda_n=0$ decreases.

(d) For a given value of θ_n the risk functions of Xb, Xb* and $X\hat{b}$ are unbounded as $\theta_d\to\infty$. However, as we noted in (a), the risk difference $\rho\left(E(y),Xb\right)-\rho\left(E(y),Xb^*\right)$ is bounded and equal to $mE(\tau^2)-2\theta_n$. So, if $\theta_n \gtreqless mE(\tau^2)/2$ then the difference is $\lesseqgtr 0$.

Similarly, $\rho\left(E(y),Xb\right)-\rho\left(E(y),X\hat{b}\right)$ is bounded, and is equal to $\left(mE(\tau^2)-2\theta_n\right)$ when $\theta_d=\infty$. Using equation (4.5.3) this bound arises as $P_{ij}^{d\tau}\to 1$ when $\theta_d\to\infty$.[15] So, the risk difference $\rho\left(E(y),Xb^*\right)-\rho\left(E(y),X\hat{b}\right)=0$ when $\theta_d=\infty$, for any fixed value of θ_n. These features are evident in Figures 4.5.1 to 4.5.4 where $\lambda_n=5>m\nu/\left(2(\nu-2)\right)$ and so the risk of the unrestricted estimator lies below that of both the pre-test and the restricted estimators.

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[15] See Mittelhammer (1984).

In this section we have seen that the results we discussed in Chapter Two for normal errors and mis-specification of the design matrix qualitatively carry over to the broader case of SSD_N errors. In particular, once we admit the possibility of omitted variables and given that the values of θ_n and θ_d are usually unknown, the appropriate choice of estimator is unclear.

4.6 Concluding Remarks

In this chapter we have considered some finite sample properties of estimators of the conditional forecast of y in a mis-specified linear regression model after a pre-test for exact linear restrictions. We assumed that the model mis-specification involves omitted variables from the design matrix and an incorrectly specified error distribution. As we postulated in the introduction to this chapter we found that the mis-specification of the distribution of the regression disturbances has little impact on the qualitative properties of the risk function of the predictor pre-test estimator. However, there are quantitative effects as we depart from normality in the way we investigated.

For instance, if the marginal distribution of the errors has fatter tails than under normality then the risk functions shift upwards, there is an increase in the range over which we prefer Xb* to Xb, and there is a decrease in the rate at which the risk of the pre-test estimator approaches that of the unrestricted estimator. These features result whether or not the design matrix is mis-specified.

We only briefly considered the question of the choice of an optimal test size. The effect on optimal test size of departures from normality is still to be resolved, as is the impact of a mis-specified design matrix on the choice of test size. This remains for future research.

APPENDIX 4.1

ALTERNATIVE PROOF OF COROLLARY 4.2.2

In this appendix we illustrate the procedure employed by Ullah and Phillips (1986) by reproving Corollary 4.2.2, which we recall derives the non-null distribution of the test statistic, u, when the regression disturbances e are $Mt\left(0,\frac{\sigma^2\nu}{\nu-2}I_T\right)$. For completeness we now repeat the statement of Corollary 4.2.2 followed by the alternative proof.

Corollary 4.2.2

If $e\sim Mt\left(0,\sigma^2\nu/(\nu-2)I_T\right)$ then

$$f_{Mt}(u)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(2\lambda_n/\nu\right)^r\left(2\lambda_d/\nu\right)^s m^{\frac{m}{2}+r}\,v^{\frac{v}{2}+s}\,u^{\frac{m}{2}+r-1}\;\Gamma\left(\frac{\nu}{2}+r+s\right)}{r!s!\left[1+2(\lambda_n+\lambda_d)/\nu\right]^{\frac{\nu}{2}+r+s}B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\Gamma\left(\frac{\nu}{2}\right)\left(v+mu\right)^{\frac{m+v}{2}+r+s}}. \tag{A4.1}$$

Alternative Proof.

In the proof to Theorem 4.3.1 we show that

$$u=\frac{v(Rb-r)'[RS^{-1}R']^{-1}(Rb-r)}{m(y-Xb)'(y-Xb)}=\frac{v(e+d+Z\gamma)'C(e+d+Z\gamma)}{m(e+Z\gamma)'M(e+Z\gamma)}$$

where $d=X(\beta-\beta_0)$, β_0 being any solution to $R\beta=r$; $C=A'BA$ is a symmetric, idempotent matrix of rank m, $A=RS^{-1}X'$, $B=[RS^{-1}R']^{-1}$, $M=I_T-XS^{-1}X'$ is a symmetric, idempotent matrix of rank v, and $S=(X'X)$.

Now, $e\sim Mt\left(0,\frac{\sigma^2\nu}{\nu-2}I_T\right)$ so we can write $e=\sqrt{\nu}a/q$, where a and q^2 are independently distributed as $N(0,\sigma^2 I_T)$ and χ^2_ν respectively. Then

$$u = \frac{v\left[\frac{\sqrt{\nu}a}{q}+d+Z\gamma\right]' C\left[\frac{\sqrt{\nu}a}{q}+d+Z\gamma\right]}{m\left[\frac{\sqrt{\nu}a}{q}+Z\gamma\right]' M\left[\frac{\sqrt{\nu}a}{q}+Z\gamma\right]}$$

$$= \frac{v(b+d_1)' C(b+d_1)}{mb'Mb}, \qquad \text{(A4.2)}$$

where $b=a+qZ\gamma/\sqrt{\nu}$ and $d_1=qd/\sqrt{\nu}$. To obtain the density of (A4.2) we observe that, given q^2, $b\sim N(qZ\gamma/\sqrt{\nu},\sigma^2 I_T)$ and so $b'Mb/\sigma^2\sim\chi^{2\prime}_{v;\lambda_{dt}}$ where $\lambda_{dt}=\frac{q^2\gamma' Z'MZ\gamma}{2\nu\sigma^2}=\frac{q^2}{\nu}\lambda_d$. Similarly, given q^2, $(b+d_1)\sim N(qZ\gamma/\sqrt{\nu}+d_1,\sigma^2 I_T)$ and $(b+d_1)'C(b+d_1)/\sigma^2\sim\chi^{2\prime}_{m;\lambda_{nt}}$ where $\lambda_{nt}=\frac{1}{2\sigma^2}(qZ\gamma/\sqrt{\nu}+d_1)'C(qZ\gamma/\sqrt{\nu}+d_1)=\frac{q^2}{\nu}\lambda_n$. The quadratic forms $(b+d_1)'C(b+d_1)/\sigma^2$ and $b'Mb/\sigma^2$ are independent, given q^2, and so, (A4.2) has a doubly non-central F distribution with m and v degrees of freedom and non-centrality parameters λ_{nt} and λ_{dt}. This density is well known as (Johnson and Kotz (1970, p.197))

$$f(u|q^2) = e^{-(\lambda_{nt}+\lambda_{dt})}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(\lambda_{nt}^{r}/r!\right)\left(\lambda_{dt}^{s}/s!\right)m^{\frac{m}{2}+r}v^{\frac{v}{2}+s}u^{\frac{m}{2}+r-1}}{B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\left(v+mu\right)^{\frac{m+v}{2}+r+s}}.$$

The unconditional density of u is then obtained by noting that

$f(u) = \int_0^\infty f(u|q^2)f(q^2)dq^2$, where $f(q^2) = 2^{-\nu/2}\left[\Gamma\left(\frac{\nu}{2}\right)\right]^{-1}e^{-q^2/2}(q^2)^{\nu/2-1}$. So,

$$f(u) = \int_0^\infty e^{-(q^2\lambda_n/\nu+q^2\lambda_d/\nu)}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\left(q^2\lambda_n/\nu\right)^r\left(q^2\lambda_d/\nu\right)^s}{r!s!}$$

$$\cdot\frac{m^{\frac{m}{2}+r}v^{\frac{v}{2}+s}u^{\frac{m}{2}+r-1}e^{-q^2/2}(q^2)^{\nu/2-1}}{\left(v+mu\right)^{\frac{m+v}{2}+r+s}B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)2^{\frac{\nu}{2}}\Gamma\left(\frac{\nu}{2}\right)}dq^2$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda_n/\nu)^r (\lambda_d/\nu)^s m^{\frac{m}{2}+r} v^{\frac{v}{2}+s} u^{\frac{m}{2}+r-1}}{r!s!\left(v+mu\right)^{\frac{m+v}{2}+r+s} B\left(\frac{m}{2}+r;\frac{v}{2}+s\right) 2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)}$$

$$\cdot \int_0^{\infty} e^{-q^2\left(\nu+2(\lambda_n+\lambda_d)\right)/(2\nu)} \left(q^2\right)^{\nu/2+r+s-1} dq^2 \; .$$

Let $t=q^2\left(\nu+2(\lambda_n+\lambda_d)\right)/(2\nu)$ then

$$f(u) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(\lambda_n/\nu)^r (\lambda_d/\nu)^s (2\nu)^{\frac{\nu}{2}+r+s} m^{\frac{m}{2}+r} v^{\frac{v}{2}+2} u^{\frac{m}{2}+r-1}}{r!s!2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right) \left[\nu+2(\lambda_n+\lambda_d)\right]^{\frac{\nu}{2}+r+s} \left(v+mu\right)^{\frac{m+v}{2}+r+s}}$$

$$\cdot \left[B\left(\frac{m}{2}+r;\frac{v}{2}+s\right)\right]^{-1} \int_0^{\infty} e^{-t} t^{\nu/2+r+s-1} dt$$

and so by collecting terms and by noting that $\int_0^{\infty} e^{-t} t^{f-1} dt = \Gamma(f)$ (A4.1) follows directly.

APPENDIX 4.2

TABLES OF RELATIVE RISKS OF Xb, Xb^* AND $X\hat{b}$

In this Appendix we give a sample of the numerical evaluations of the relative risks of Xb, Xb^* and $X\hat{b}$ (α=0.01, 0.05, 0.30 and 0.75). The relative risks of the estimators, for given values of λ_d, as a function of λ_n, are given in Tables A4.2.1, A4.2.2, A4.2.3, and A4.2.4. We recall that λ_d is a measure of the specification error and that λ_n depends on both the specification error and the hypothesis error. When $\lambda_d=0$ there are no omitted regressors, $\lambda_n=\lambda$ which will be zero when the prior information is valid. So, for nonzero λ_d, $\lambda_n>0$ even if H_0 is true.

In these tables we consider $\lambda_d=0,1,3,10$; $\lambda_n=[0,2(0.5);2,10(1.0);10,14(2.0)]$ and $\nu=5,10,100,\infty$. For each of these values of λ_n, λ_d and ν, each table gives the relative risks of the estimators for different values of v, k and m. Tables A4.2.1 and A4.2.2 both consider v=30 and k=5, and m=1 and m=3 respectively. The case of v=16 and k=4 is given in Tables A4.2.3 and A4.2.4, with m=1 in the former table and m=3 in the latter table.

Table A4.2.5 presents the relative risks for given values of λ_n, as a function of λ_d. We consider v=16, k=4, m=1; $\nu=5,10,100,\infty$; $\lambda_n=0,1,5$ and $\lambda_d=[0,2(0.5);2,10(1.0);10,14(2.0)]$.

TABLE A4.2.1: Relative Risks of Xb, Xb* and Xδ

$v = 30,\ k = 5,\ m = 1.$

Estimator	λ_n = 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$\nu=5, \lambda_d=0$															
Xb	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333	8.333
Xb*	6.667	7.667	8.667	9.667	10.667	12.667	14.667	16.667	18.667	20.667	22.667	24.667	26.667	30.667	34.667
Xδ:α=0.01	6.795	7.871	8.802	9.567	10.176	11.006	11.465	11.686	11.758	11.740	11.671	11.571	11.454	11.207	10.963
Xδ:α=0.05	7.108	8.088	8.782	9.260	9.581	9.925	10.041	10.045	9.995	9.919	9.837	9.751	9.668	9.515	9.382
Xδ:α=0.30	7.958	8.311	8.491	8.583	8.628	8.654	8.642	8.621	8.595	8.572	8.553	8.532	8.515	8.487	8.469
Xδ:α=0.75	8.319	8.333	8.340	8.343	8.344	8.344	8.343	8.343	8.341	8.340	8.340	8.340	8.340	8.338	8.338
$\nu=10, \lambda_d=0$															
Xb	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250	6.250
Xb*	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
Xδ:α=0.01	5.096	6.174	7.109	7.874	8.470	9.218	9.521	9.544	9.406	9.186	8.932	8.670	8.421	7.978	7.621
Xδ:α=0.05	5.331	6.309	6.992	7.440	7.711	7.914	7.870	7.725	7.550	7.377	7.218	7.078	6.959	6.775	6.639
Xδ:α=0.30	5.968	6.317	6.481	6.549	6.567	6.539	6.489	6.443	6.401	6.370	6.347	6.326	6.312	6.294	6.278
Xδ:α=0.75	6.239	6.253	6.259	6.260	6.261	6.260	6.257	6.255	6.254	6.252	6.252	6.251	6.250	6.250	6.250
$\nu=100, \lambda_d=0$															
Xb	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102	5.102
Xb*	4.082	5.082	6.082	7.082	8.082	10.082	12.082	14.082	16.082	18.082	20.082	22.082	24.082	28.082	32.082
Xδ:α=0.01	4.160	5.239	6.180	6.948	7.535	8.197	8.315	8.084	7.676	7.210	6.764	6.374	6.052	5.604	5.352
Xδ:α=0.05	4.352	5.329	6.001	6.418	6.637	6.681	6.458	6.159	5.876	5.646	5.472	5.349	5.262	5.170	5.129
Xδ:α=0.30	4.872	5.218	5.365	5.407	5.395	5.317	5.240	5.184	5.150	5.129	5.116	5.113	5.106	5.102	5.102
Xδ:α=0.75	5.093	5.107	5.112	5.112	5.111	5.108	5.105	5.104	5.102	5.102	5.102	5.102	5.102	5.102	5.102
$\nu=\infty, \lambda_d=0$															
Xb	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
Xb*	4.000	5.000	6.000	7.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	28.000	32.000
Xδ:α=0.01	4.077	5.156	6.098	6.867	7.452	8.104	8.197	7.931	7.484	6.989	6.519	6.120	5.802	5.383	5.170
Xδ:α=0.05	4.265	5.242	5.912	6.326	6.538	6.563	6.316	5.997	5.705	5.475	5.309	5.193	5.118	5.042	5.014
Xδ:α=0.30	4.775	5.119	5.265	5.304	5.288	5.205	5.125	5.070	5.038	5.020	5.011	5.007	5.004	5.000	5.000
Xδ:α=0.75	4.921	5.045	5.090	5.096	5.086	5.055	5.030	5.016	5.009	5.005	5.003	5.000	5.000	5.000	5.000
$\nu=5, \lambda_d=1$															
Xb	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333	10.333
Xb*	8.667	9.667	10.667	11.667	12.667	14.667	16.667	18.667	20.667	22.667	24.667	26.667	28.667	32.667	36.667
Xδ:α=0.01	8.782	9.860	10.816	11.617	12.264	13.158	13.655	13.892	13.968	13.945	13.866	13.755	13.624	13.352	13.086
Xδ:α=0.05	9.082	10.085	10.812	11.319	11.662	12.026	12.146	12.146	12.087	12.005	11.910	11.816	11.726	11.561	11.423
Xδ:α=0.30	9.940	10.316	10.507	10.603	10.650	10.673	10.658	10.634	10.608	10.583	10.560	10.542	10.522	10.496	10.468
Xδ:α=0.75	10.318	10.334	10.340	10.343	10.344	10.344	10.344	10.342	10.341	10.341	10.340	10.340	10.340	10.337	10.337
$\nu=10, \lambda_d=1$															
Xb	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250	8.250
Xb*	7.000	8.000	9.000	10.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	31.000	35.000
Xδ:α=0.01	7.083	8.162	9.119	9.920	10.558	11.385	11.742	11.790	11.656	11.427	11.155	10.873	10.603	10.118	9.727
Xδ:α=0.05	7.305	8.306	9.023	9.503	9.801	10.031	9.993	9.841	9.651	9.464	9.292	9.142	9.011	8.811	8.666
Xδ:α=0.30	7.951	8.324	8.500	8.573	8.591	8.561	8.505	8.455	8.411	8.379	8.351	8.333	8.316	8.294	8.284
Xδ:α=0.75	8.238	8.254	8.260	8.261	8.261	8.259	8.257	8.255	8.254	8.252	8.252	8.252	8.250	8.250	8.250
$\nu=100, \lambda_d=1$															
Xb	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102	7.102
Xb*	6.082	7.082	8.082	9.082	10.082	12.082	14.082	16.082	18.082	20.082	22.082	24.082	26.082	30.082	34.082
Xδ:α=0.01	6.147	7.226	8.185	8.988	9.619	10.379	10.574	10.387	9.988	9.505	9.025	8.596	8.231	7.713	7.415
Xδ:α=0.05	6.326	7.325	8.032	8.485	8.736	8.818	8.602	8.289	7.985	7.729	7.536	7.394	7.295	7.183	7.138
Xδ:α=0.30	6.855	7.225	7.386	7.433	7.422	7.340	7.255	7.193	7.156	7.131	7.119	7.111	7.106	7.102	7.102
Xδ:α=0.75	7.092	7.108	7.113	7.113	7.112	7.109	7.105	7.104	7.102	7.102	7.102	7.102	7.102	7.102	7.102
$\nu=\infty, \lambda_d=1$															
Xb	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000
Xb*	6.000	7.000	8.000	9.000	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	26.000	30.000	34.000
Xδ:α=0.01	6.064	7.143	8.102	8.906	9.537	10.288	10.463	10.243	9.805	9.291	8.786	8.343	7.979	7.485	7.221
Xδ:α=0.05	6.239	7.238	7.943	8.394	8.639	8.703	8.464	8.132	7.815	7.557	7.369	7.235	7.145	7.054	7.019
Xδ:α=0.30	6.757	7.128	7.287	7.330	7.317	7.229	7.141	7.080	7.044	7.022	7.011	7.007	7.004	7.000	7.000
Xδ:α=0.75	6.914	7.049	7.098	7.105	7.095	7.061	7.034	7.018	7.009	7.005	7.003	7.000	7.000	7.000	7.000
$\nu=5, \lambda_d=3$															
Xb	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333	14.333
Xb*	12.667	13.667	14.667	15.667	16.667	18.667	20.667	22.667	24.667	26.667	28.667	30.667	32.667	36.667	40.667
Xδ:α=0.01	12.762	13.838	14.829	15.694	16.416	17.448	18.037	18.320	18.410	18.381	18.281	18.145	17.990	17.659	17.344
Xδ:α=0.05	13.037	14.076	14.864	15.429	15.817	16.233	16.365	16.356	16.280	16.178	16.068	15.955	15.848	15.657	15.497
Xδ:α=0.30	13.906	14.328	14.540	14.645	14.693	14.712	14.692	14.660	14.629	14.601	14.574	14.551	14.535	14.502	14.479
Xδ:α=0.75	14.316	14.334	14.342	14.345	14.345	14.345	14.344	14.342	14.342	14.342	14.341	14.339	14.339	14.337	14.337
$\nu=10, \lambda_d=3$															
Xb	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250	12.250
Xb*	11.000	12.000	13.000	14.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	29.000	31.000	35.000	39.000
Xδ:α=0.01	11.063	12.138	13.127	13.990	14.706	15.693	16.174	16.294	16.182	15.943	15.638	15.318	15.004	14.434	13.969
Xδ:α=0.05	11.260	12.296	13.074	13.619	13.971	14.267	14.246	14.081	13.868	13.651	13.453	13.275	13.123	12.890	12.719
Xδ:α=0.30	11.916	12.337	12.538	12.620	12.640	12.603	12.540	12.479	12.432	12.392	12.361	12.342	12.323	12.298	12.284
Xδ:α=0.75	12.226	12.255	12.261	12.263	12.263	12.260	12.258	12.255	12.254	12.253	12.253	12.253	12.250	12.250	12.250
$\nu=100, \lambda_d=3$															
Xb	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102	11.102
Xb*	10.082	11.082	12.082	13.082	14.082	16.082	18.082	20.082	22.082	24.082	26.082	28.082	30.082	34.082	38.082
Xδ:α=0.01	10.128	11.200	12.188	13.049	13.760	14.705	15.066	14.988	14.632	14.135	13.601	13.093	12.651	11.982	11.570
Xδ:α=0.05	10.281	11.315	12.084	12.607	12.922	13.091	12.901	12.569	12.223	11.919	11.678	11.500	11.370	11.221	11.151
Xδ:α=0.30	10.820	11.240	11.428	11.487	11.479	11.387	11.289	11.217	11.167	11.141	11.125	11.114	11.110	11.102	11.102
Xδ:α=0.75	11.090	11.109	11.115	11.115	11.114	11.110	11.107	11.104	11.102	11.102	11.102	11.102	11.102	11.102	11.102
$\nu=\infty, \lambda_d=3$															
Xb	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000
Xb*	10.000	11.000	12.000	13.000	14.000	16.000	18.000	20.000	22.000	24.000	26.000	28.000	30.000	34.000	38.000
Xδ:α=0.01	10.044	11.117	12.104	12.965	13.676	14.615	14.963	14.858	14.468	13.937	13.375	12.848	12.391	11.732	11.355
Xδ:α=0.05	10.194	11.227	11.995	12.516	12.826	12.980	12.769	12.418	12.056	11.746	11.505	11.330	11.213	11.082	11.030
Xδ:α=0.30	10.723	11.143	11.329	11.385	11.373	11.275	11.173	11.101	11.055	11.030	11.017	11.007	11.004	11.000	11.000
Xδ:α=0.75	10.900	11.057	11.115	11.125	11.112	11.073	11.041	11.022	11.011	11.005	11.003	11.000	11.000	11.000	11.000
$\nu=5, \lambda_d=10$															
Xb	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333	28.333
Xb*	26.667	27.667	28.667	29.667	30.667	32.667	34.667	36.667	38.667	40.667	42.667	44.667	46.667	50.667	54.667
Xδ:α=0.01	26.723	27.773	28.806	29.792	30.704	32.217	33.253	33.853	34.112	34.135	34.013	33.803	33.550	33.004	32.482
Xδ:α=0.05	26.928	28.014	28.953	29.711	30.285	30.962	31.203	31.202	31.080	30.910	30.723	30.540	30.368	30.069	29.826
Xδ:α=0.30	27.792	28.360	28.656	28.799	28.859	28.867	28.822	28.768	28.717	28.672	28.637	28.604	28.576	28.533	28.499
Xδ:α=0.75	28.309	28.337	28.348	28.351	28.352	28.350	28.348	28.346	28.345	28.342	28.341	28.339	28.339	28.337	28.337
$\nu=10, \lambda_d=10$															
Xb	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250	26.250
Xb*	25.000	26.000	27.000	28.000	29.000	31.000	33.000	35.000	37.000	39.000	41.000	43.000	45.000	49.000	53.000
Xδ:α=0.01	25.027	26.073	27.100	28.080	28.983	30.473	31.471	31.991	32.125	31.980	31.657	31.240	30.784	29.886	29.110
Xδ:α=0.05	25.154	26.238	27.168	27.911	28.463	29.071	29.196	29.048	28.770	28.456	28.148	27.869	27.623	27.237	26.966
Xδ:α=0.30	25.802	26.375	26.667	26.793	26.828	26.775	26.677	26.584	26.511	26.453	26.407	26.375	26.347	26.312	26.291
Xδ:α=0.75	26.229	26.258	26.268	26.270	26.270	26.265	26.261	26.259	26.256	26.255	26.253	26.253	26.253	26.250	26.250
$\nu=100, \lambda_d=10$															
Xb	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102	25.102
Xb*	24.082	25.082	26.082	27.082	28.082	30.082	32.082	34.082	36.082	38.082	40.082	42.082	44.082	48.082	52.082
Xδ:α=0.01	24.095	25.136	26.158	27.130	28.024	29.492	30.448	30.887	30.883	30.546	29.998	29.340	28.662	27.436	26.519
Xδ:α=0.05	24.179	25.261	26.180	26.907	27.440	27.977	27.984	27.682	27.247	26.790	26.381	26.040	25.774	25.425	25.252
Xδ:α=0.30	24.705	25.285	25.571	25.680	25.687	25.573	25.427	25.309	25.227	25.177	25.145	25.126	25.116	25.107	25.102
Xδ:α=0.75	25.083	25.113	25.122	25.123	25.120	25.114	25.108	25.106	25.104	25.102	25.102	25.102	25.102	25.102	25.102
$\nu=\infty, \lambda_d=10$															
Xb	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000
Xb*	24.000	25.000	26.000	27.000	28.000	30.000	32.000	34.000	36.000	38.000	40.000	42.000	44.000	48.000	52.000
Xδ:α=0.01	24.012	25.053	26.074	27.045	27.939	29.403	30.355	30.785	30.764	30.400	29.823	29.139	28.433	27.170	26.255
Xδ:α=0.05	24.092	25.174	26.092	26.817	27.349	27.876	27.869	27.549	27.093	26.623	26.202	25.861	25.593	25.266	25.109
Xδ:α=0.30	24.608	25.188	25.474	25.580	25.585	25.462	25.312	25.191	25.112	25.060	25.035	25.019	25.010	25.005	25.000
Xδ:α=0.75	24.847	25.084	25.178	25.197	25.182	25.123	25.072	25.039	25.019	25.011	25.006	25.003	25.000	25.000	25.000

TABLE A4.2.2: Relative Risks of Xb, Xb* and $X\hat{b}$

$v = 30,\ k = 5,\ m = 3$

Estimator	λ_n = 0.0	0.5	1.0	1.5	2.0
$\nu=5, \lambda_d=0$					
Xb	8.333	8.333	8.333	8.333	8.333
Xb*	3.333	4.333	5.333	6.333	7.333
$X\hat{b}:\alpha=0.01$	3.529	4.631	5.688	6.667	7.552
$X\hat{b}:\alpha=0.05$	4.098	5.260	6.263	7.100	7.783
$X\hat{b}:\alpha=0.30$	6.240	7.021	7.550	7.912	8.161
$X\hat{b}:\alpha=0.75$	7.782	8.023	8.162	8.247	8.300
$\nu=10, \lambda_d=0$					
Xb	6.250	6.250	6.250	6.250	6.250
Xb*	2.500	3.500	4.500	5.500	6.500
$X\hat{b}:\alpha=0.01$	2.647	3.750	4.813	5.804	6.701
$X\hat{b}:\alpha=0.05$	3.073	4.237	5.242	6.075	6.743
$X\hat{b}:\alpha=0.30$	4.680	5.459	5.972	6.303	6.509
$X\hat{b}:\alpha=0.75$	5.836	6.075	6.207	6.277	6.314
$\nu=100, \lambda_d=0$					
Xb	5.102	5.102	5.102	5.102	5.102
Xb*	2.041	3.041	4.041	5.041	6.041
$X\hat{b}:\alpha=0.01$	2.161	3.265	4.334	5.337	6.251
$X\hat{b}:\alpha=0.05$	2.509	3.673	4.681	5.514	6.168
$X\hat{b}:\alpha=0.30$	3.820	4.596	5.095	5.394	5.554
$X\hat{b}:\alpha=0.75$	4.764	5.001	5.124	5.179	5.199
$\nu=\infty, \lambda_d=0$					
Xb	5.000	5.000	5.000	5.000	5.000
Xb*	2.000	3.000	4.000	5.000	6.000
$X\hat{b}:\alpha=0.01$	2.118	3.222	4.291	5.296	6.212
$X\hat{b}:\alpha=0.05$	2.459	3.623	4.631	5.464	6.118
$X\hat{b}:\alpha=0.30$	3.744	4.519	5.016	5.312	5.467
$X\hat{b}:\alpha=0.75$	4.669	4.906	5.027	5.081	5.098
$\nu=5, \lambda_d=1$					
Xb	10.333	10.333	10.333	10.333	10.333
Xb*	5.333	6.333	7.333	8.333	9.333
$X\hat{b}:\alpha=0.01$	5.503	6.592	7.653	8.654	9.572
$X\hat{b}:\alpha=0.05$	6.027	7.191	8.221	9.095	9.818
$X\hat{b}:\alpha=0.30$	8.136	8.966	9.533	9.920	10.185
$X\hat{b}:\alpha=0.75$	9.739	10.007	10.160	10.251	10.306
$\nu=10, \lambda_d=1$					
Xb	8.250	8.250	8.250	8.250	8.250
Xb*	4.500	5.500	6.500	7.500	8.500
$X\hat{b}:\alpha=0.01$	4.620	5.710	6.773	7.782	8.709
$X\hat{b}:\alpha=0.05$	5.003	6.166	7.195	8.066	8.778
$X\hat{b}:\alpha=0.30$	6.576	7.403	7.958	8.318	8.544
$X\hat{b}:\alpha=0.75$	7.794	8.060	8.206	8.284	8.324
$\nu=100, \lambda_d=1$					
Xb	7.102	7.102	7.102	7.102	7.102
Xb*	4.041	5.041	6.041	7.041	8.041
$X\hat{b}:\alpha=0.01$	4.134	5.224	6.290	7.305	8.244
$X\hat{b}:\alpha=0.05$	4.438	5.601	6.630	7.500	8.202
$X\hat{b}:\alpha=0.30$	5.717	6.542	7.084	7.417	7.601
$X\hat{b}:\alpha=0.75$	6.722	6.987	7.126	7.190	7.213
$\nu=\infty, \lambda_d=1$					
Xb	7.000	7.000	7.000	7.000	7.000
Xb*	4.000	5.000	6.000	7.000	8.000
$X\hat{b}:\alpha=0.01$	4.091	5.181	6.248	7.263	8.204
$X\hat{b}:\alpha=0.05$	4.388	5.551	6.580	7.449	8.150
$X\hat{b}:\alpha=0.30$	5.640	6.466	7.006	7.336	7.515
$X\hat{b}:\alpha=0.75$	6.627	6.892	7.029	7.092	7.112
$\nu=5, \lambda_d=3$					
Xb	14.333	14.333	14.333	14.333	14.333
Xb*	9.333	10.333	11.333	12.333	13.333
$X\hat{b}:\alpha=0.01$	9.464	10.533	11.592	12.618	13.586
$X\hat{b}:\alpha=0.05$	9.912	11.069	12.134	13.072	13.872
$X\hat{b}:\alpha=0.30$	11.940	12.853	13.494	13.937	14.240
$X\hat{b}:\alpha=0.75$	13.651	13.973	14.154	14.259	14.321
$\nu=10, \lambda_d=3$					
Xb	12.250	12.250	12.250	12.250	12.250
Xb*	8.500	9.500	10.500	11.500	12.500
$X\hat{b}:\alpha=0.01$	8.583	9.650	10.708	11.734	12.706
$X\hat{b}:\alpha=0.05$	8.889	10.043	11.104	12.038	12.829
$X\hat{b}:\alpha=0.30$	10.380	11.293	11.925	12.347	12.616
$X\hat{b}:\alpha=0.75$	11.706	12.028	12.205	12.299	12.346
$\nu=100, \lambda_d=3$					
Xb	11.102	11.102	11.102	11.102	11.102
Xb*	8.041	9.041	10.041	11.041	12.041
$X\hat{b}:\alpha=0.01$	8.097	9.163	10.220	11.247	12.222
$X\hat{b}:\alpha=0.05$	8.325	9.478	10.534	11.464	12.247
$X\hat{b}:\alpha=0.30$	9.521	10.433	11.058	11.458	11.692
$X\hat{b}:\alpha=0.75$	10.634	10.957	11.129	11.211	11.242
$\nu=\infty, \lambda_d=3$					
Xb	11.000	11.000	11.000	11.000	11.000
Xb*	8.000	9.000	10.000	11.000	12.000
$X\hat{b}:\alpha=0.01$	8.054	9.120	10.176	11.204	12.179
$X\hat{b}:\alpha=0.05$	8.275	9.428	10.484	11.412	12.194
$X\hat{b}:\alpha=0.30$	9.444	10.357	10.980	11.378	11.609
$X\hat{b}:\alpha=0.75$	10.539	10.862	11.033	11.114	11.143
$\nu=5, \lambda_d=10$					
Xb	28.333	28.333	28.333	28.333	28.333
Xb*	23.333	24.333	25.333	26.333	27.333
$X\hat{b}:\alpha=0.01$	23.401	24.429	25.461	26.491	27.514
$X\hat{b}:\alpha=0.05$	23.681	24.782	25.867	26.910	27.888
$X\hat{b}:\alpha=0.30$	25.378	26.462	27.321	27.966	28.428
$X\hat{b}:\alpha=0.75$	27.324	27.837	28.133	28.302	28.395
$\nu=10, \lambda_d=10$					
Xb	26.250	26.250	26.250	26.250	26.250
Xb*	22.500	23.500	24.500	25.500	26.500
$X\hat{b}:\alpha=0.01$	22.527	23.551	24.578	25.605	26.622
$X\hat{b}:\alpha=0.05$	22.673	23.768	24.844	25.877	26.843
$X\hat{b}:\alpha=0.30$	23.820	24.914	25.772	26.407	26.852
$X\hat{b}:\alpha=0.75$	25.375	25.896	26.197	26.361	26.442
$\nu=100, \lambda_d=10$					
Xb	25.102	25.102	25.102	25.102	25.102
Xb*	24.082	25.082	26.082	27.082	28.082
$X\hat{b}:\alpha=0.01$	24.095	25.136	26.158	27.130	28.024
$X\hat{b}:\alpha=0.05$	24.179	25.261	26.180	26.907	27.440
$X\hat{b}:\alpha=0.30$	24.705	25.285	25.571	25.680	25.687
$X\hat{b}:\alpha=0.75$	25.083	25.113	25.122	25.123	25.120
$\nu=\infty, \lambda_d=10$					
Xb	25.000	25.000	25.000	25.000	25.000
Xb*	24.000	25.000	26.000	27.000	28.000
$X\hat{b}:\alpha=0.01$	24.012	25.053	26.074	27.045	27.939
$X\hat{b}:\alpha=0.05$	24.092	25.174	26.092	26.817	27.348
$X\hat{b}:\alpha=0.30$	24.608	25.188	25.474	25.580	25.585
$X\hat{b}:\alpha=0.75$	24.847	25.084	25.178	25.197	25.182

Estimator	3.0	4.0	5.0	6.0	7.0
$\nu=5, \lambda_d=0$					
Xb	8.333	8.333	8.333	8.333	8.333
Xb*	9.333	11.333	13.333	15.333	17.333
$X\hat{b}:\alpha=0.01$	9.013	10.098	10.870	11.404	11.760
$X\hat{b}:\alpha=0.05$	8.773	9.394	9.776	10.000	10.117
$X\hat{b}:\alpha=0.30$	8.454	8.598	8.665	8.694	8.701
$X\hat{b}:\alpha=0.75$	8.355	8.378	8.388	8.391	8.389
$\nu=10, \lambda_d=0$					
Xb	6.250	6.250	6.250	6.250	6.250
Xb*	8.500	10.500	12.500	14.500	16.500
$X\hat{b}:\alpha=0.01$	8.171	9.209	9.868	10.234	10.381
$X\hat{b}:\alpha=0.05$	7.652	8.133	8.330	8.356	8.282
$X\hat{b}:\alpha=0.30$	6.699	6.736	6.709	6.661	6.609
$X\hat{b}:\alpha=0.75$	6.336	6.333	6.320	6.309	6.300
$\nu=100, \lambda_d=0$					
Xb	5.102	5.102	5.102	5.102	5.102
Xb*	8.041	10.041	12.041	14.041	16.041
$X\hat{b}:\alpha=0.01$	7.743	8.745	9.279	9.424	9.278
$X\hat{b}:\alpha=0.05$	6.995	7.308	7.280	7.061	6.761
$X\hat{b}:\alpha=0.30$	5.632	5.560	5.450	5.348	5.269
$X\hat{b}:\alpha=0.75$	5.189	5.164	5.141	5.125	5.115
$\nu=\infty, \lambda_d=0$					
Xb	5.000	5.000	5.000	5.000	5.000
Xb*	8.000	10.000	12.000	14.000	16.000
$X\hat{b}:\alpha=0.01$	7.708	8.707	9.225	9.337	9.149
$X\hat{b}:\alpha=0.05$	6.935	7.225	7.168	6.916	6.583
$X\hat{b}:\alpha=0.30$	5.530	5.445	5.327	5.222	5.142
$X\hat{b}:\alpha=0.75$	5.085	5.058	5.034	5.021	5.011
$\nu=5, \lambda_d=1$					
Xb	10.333	10.333	10.333	10.333	10.333
Xb*	11.333	13.333	15.333	17.333	19.333
$X\hat{b}:\alpha=0.01$	11.118	12.284	13.120	13.697	14.080
$X\hat{b}:\alpha=0.05$	10.876	11.542	11.944	12.172	12.293
$X\hat{b}:\alpha=0.30$	10.495	10.642	10.707	10.730	10.735
$X\hat{b}:\alpha=0.75$	10.363	10.386	10.395	10.396	10.396
$\nu=10, \lambda_d=1$					
Xb	8.250	8.250	8.250	8.250	8.250
Xb*	10.500	12.500	14.500	16.500	18.500
$X\hat{b}:\alpha=0.01$	10.268	11.404	12.152	12.582	12.770
$X\hat{b}:\alpha=0.05$	9.770	10.307	10.537	10.572	10.496
$X\hat{b}:\alpha=0.30$	8.752	8.791	8.762	8.705	8.645
$X\hat{b}:\alpha=0.75$	8.348	8.342	8.330	8.315	8.303
$\nu=100, \lambda_d=1$					
Xb	7.102	7.102	7.102	7.102	7.102
Xb*	10.041	12.041	14.041	16.041	18.041
$X\hat{b}:\alpha=0.01$	9.824	10.943	11.597	11.845	11.771
$X\hat{b}:\alpha=0.05$	9.127	9.521	9.543	9.338	9.029
$X\hat{b}:\alpha=0.30$	7.701	7.629	7.508	7.394	7.303
$X\hat{b}:\alpha=0.75$	7.204	7.174	7.149	7.132	7.119
$\nu=\infty, \lambda_d=1$					
Xb	7.000	7.000	7.000	7.000	7.000
Xb*	10.000	12.000	14.000	16.000	18.000
$X\hat{b}:\alpha=0.01$	9.786	10.905	11.547	11.770	11.659
$X\hat{b}:\alpha=0.05$	9.068	9.444	9.437	9.202	8.861
$X\hat{b}:\alpha=0.30$	7.600	7.515	7.386	7.266	7.175
$X\hat{b}:\alpha=0.75$	7.098	7.068	7.042	7.024	7.014
$\nu=5, \lambda_d=3$					
Xb	14.333	14.333	14.333	14.333	14.333
Xb*	15.333	17.333	19.333	21.333	23.333
$X\hat{b}:\alpha=0.01$	15.287	16.624	17.607	18.296	18.748
$X\hat{b}:\alpha=0.05$	15.074	15.842	16.300	16.555	16.675
$X\hat{b}:\alpha=0.30$	14.585	14.739	14.802	14.817	14.810
$X\hat{b}:\alpha=0.75$	14.383	14.404	14.410	14.409	14.406
$\nu=10, \lambda_d=3$					
Xb	12.250	12.250	12.250	12.250	12.250
Xb*	14.500	16.500	18.500	20.500	22.500
$X\hat{b}:\alpha=0.01$	14.415	15.745	16.681	17.266	17.559
$X\hat{b}:\alpha=0.05$	13.987	14.656	14.965	15.035	14.960
$X\hat{b}:\alpha=0.30$	12.867	12.912	12.874	12.804	12.727
$X\hat{b}:\alpha=0.75$	12.372	12.363	12.346	12.329	12.314
$\nu=100, \lambda_d=3$					
Xb	11.102	11.102	11.102	11.102	11.102
Xb*	14.041	16.041	18.041	20.041	22.041
$X\hat{b}:\alpha=0.01$	13.943	15.271	16.163	16.630	16.733
$X\hat{b}:\alpha=0.05$	13.361	13.926	14.065	13.919	13.607
$X\hat{b}:\alpha=0.30$	11.843	11.777	11.638	11.498	11.380
$X\hat{b}:\alpha=0.75$	11.233	11.197	11.163	11.139	11.126
$\nu=\infty, \lambda_d=3$					
Xb	11.000	11.000	11.000	11.000	11.000
Xb*	14.000	16.000	18.000	20.000	22.000
$X\hat{b}:\alpha=0.01$	13.902	15.231	16.117	16.569	16.646
$X\hat{b}:\alpha=0.05$	13.303	13.856	13.973	13.798	13.457
$X\hat{b}:\alpha=0.30$	11.747	11.667	11.518	11.370	11.249
$X\hat{b}:\alpha=0.75$	11.130	11.091	11.056	11.033	11.020
$\nu=5, \lambda_d=10$					
Xb	28.333	28.333	28.333	28.333	28.333
Xb*	29.333	31.333	33.333	35.333	37.333
$X\hat{b}:\alpha=0.01$	29.495	31.312	32.872	34.117	35.034
$X\hat{b}:\alpha=0.05$	29.564	30.803	31.618	32.084	32.298
$X\hat{b}:\alpha=0.30$	28.958	29.170	29.223	29.206	29.156
$X\hat{b}:\alpha=0.75$	28.472	28.486	28.479	28.467	28.454
$\nu=10, \lambda_d=10$					
Xb	26.250	26.250	26.250	26.250	26.250
Xb*	28.500	30.500	32.500	34.500	36.500
$X\hat{b}:\alpha=0.01$	28.593	30.397	31.945	33.172	34.058
$X\hat{b}:\alpha=0.05$	28.491	29.688	30.437	30.798	30.867
$X\hat{b}:\alpha=0.30$	27.317	27.434	27.380	27.256	27.116
$X\hat{b}:\alpha=0.75$	26.485	26.460	26.424	26.387	26.359
$\nu=100, \lambda_d=10$					
Xb	25.102	25.102	25.102	25.102	25.102
Xb*	30.082	32.082	32.041	34.041	36.041
$X\hat{b}:\alpha=0.01$	29.492	30.448	31.408	32.616	33.458
$X\hat{b}:\alpha=0.05$	27.977	27.984	29.715	29.964	29.876
$X\hat{b}:\alpha=0.30$	25.573	25.427	26.230	26.005	25.785
$X\hat{b}:\alpha=0.75$	25.114	25.108	25.244	25.193	25.158
$\nu=\infty, \lambda_d=10$					
Xb	25.000	25.000	25.000	25.000	25.000
Xb*	30.000	32.000	32.000	34.000	36.000
$X\hat{b}:\alpha=0.01$	29.403	30.355	31.359	32.564	33.400
$X\hat{b}:\alpha=0.05$	27.876	27.869	29.646	29.882	29.774
$X\hat{b}:\alpha=0.30$	25.462	25.312	26.120	25.881	25.654
$X\hat{b}:\alpha=0.75$	25.123	25.072	25.135	25.084	25.050

Estimator	8.0	9.0	10.0	12.0	14.0
$\nu=5, \lambda_d=0$					
Xb	8.333	8.333	8.333	8.333	8.333
Xb*	19.333	21.333	23.333	27.333	31.333
$X\hat{b}:\alpha=0.01$	11.986	12.117	12.182	12.172	12.060
$X\hat{b}:\alpha=0.05$	10.174	10.183	10.168	10.088	9.981
$X\hat{b}:\alpha=0.30$	8.698	8.687	8.672	8.640	8.612
$X\hat{b}:\alpha=0.75$	8.389	8.385	8.382	8.377	8.372
$\nu=10, \lambda_d=0$					
Xb	6.250	6.250	6.250	6.250	6.250
Xb*	18.500	20.500	22.500	26.500	30.500
$X\hat{b}:\alpha=0.01$	10.376	10.268	10.096	9.658	9.194
$X\hat{b}:\alpha=0.05$	8.156	8.006	7.851	7.556	7.304
$X\hat{b}:\alpha=0.30$	6.559	6.514	6.476	6.414	6.371
$X\hat{b}:\alpha=0.75$	6.291	6.282	6.276	6.270	6.260
$\nu=100, \lambda_d=0$					
Xb	5.102	5.102	5.102	5.102	5.102
Xb*	18.041	20.041	22.041	26.041	30.041
$X\hat{b}:\alpha=0.01$	8.942	8.504	8.026	7.113	6.395
$X\hat{b}:\alpha=0.05$	6.443	6.155	5.907	5.541	5.333
$X\hat{b}:\alpha=0.30$	5.211	5.172	5.147	5.119	5.110
$X\hat{b}:\alpha=0.75$	5.109	5.109	5.106	5.102	5.102
$\nu=\infty, \lambda_d=0$					
Xb	5.000	5.000	5.000	5.000	5.000
Xb*	18.000	20.000	22.000	26.000	30.000
$X\hat{b}:\alpha=0.01$	8.763	8.275	7.755	6.793	6.066
$X\hat{b}:\alpha=0.05$	6.245	5.943	5.692	5.345	5.159
$X\hat{b}:\alpha=0.30$	5.088	5.053	5.029	5.011	5.005
$X\hat{b}:\alpha=0.75$	5.006	5.003	5.000	5.000	5.000
$\nu=5, \lambda_d=1$					
Xb	10.333	10.333	10.333	10.333	10.333
Xb*	21.333	23.333	25.333	29.333	33.333
$X\hat{b}:\alpha=0.01$	14.318	14.448	14.506	14.476	14.338
$X\hat{b}:\alpha=0.05$	12.341	12.343	12.318	12.218	12.094
$X\hat{b}:\alpha=0.30$	10.728	10.714	10.694	10.662	10.625
$X\hat{b}:\alpha=0.75$	10.392	10.389	10.388	10.378	10.376
$\nu=10, \lambda_d=1$					
Xb	8.250	8.250	8.250	8.250	8.250
Xb*	20.500	22.500	24.500	28.500	32.500
$X\hat{b}:\alpha=0.01$	12.782	12.676	12.497	12.019	11.505
$X\hat{b}:\alpha=0.05$	10.358	10.192	10.017	9.688	9.411
$X\hat{b}:\alpha=0.30$	8.589	8.540	8.495	8.428	8.383
$X\hat{b}:\alpha=0.75$	8.294	8.285	8.278	8.270	8.265
$\nu=100, \lambda_d=1$					
Xb	7.102	7.102	7.102	7.102	7.102
Xb*	20.041	22.041	24.041	28.041	32.041
$X\hat{b}:\alpha=0.01$	11.475	11.041	10.541	9.545	8.716
$X\hat{b}:\alpha=0.05$	8.686	8.364	8.079	7.654	7.397
$X\hat{b}:\alpha=0.30$	7.236	7.188	7.158	7.124	7.110
$X\hat{b}:\alpha=0.75$	7.112	7.109	7.106	7.102	7.102
$\nu=\infty, \lambda_d=1$					
Xb	7.000	7.000	7.000	7.000	7.000
Xb*	20.000	22.000	24.000	28.000	32.000
$X\hat{b}:\alpha=0.01$	11.316	10.832	10.289	9.225	8.379
$X\hat{b}:\alpha=0.05$	8.495	8.153	7.863	7.445	7.213
$X\hat{b}:\alpha=0.30$	7.110	7.066	7.038	7.016	7.005
$X\hat{b}:\alpha=0.75$	7.009	7.004	7.003	7.000	7.000
$\nu=5, \lambda_d=3$					
Xb	14.333	14.333	14.333	14.333	14.333
Xb*	25.333	27.333	29.333	33.333	37.333
$X\hat{b}:\alpha=0.01$	19.025	19.168	19.220	19.151	18.950
$X\hat{b}:\alpha=0.05$	16.713	16.698	16.647	16.501	16.336
$X\hat{b}:\alpha=0.30$	14.792	14.770	14.747	14.704	14.661
$X\hat{b}:\alpha=0.75$	14.403	14.397	14.395	14.385	14.381
$\nu=10, \lambda_d=3$					
Xb	12.250	12.250	12.250	12.250	12.250
Xb*	24.500	26.500	28.500	32.500	36.500
$X\hat{b}:\alpha=0.01$	17.634	17.555	17.367	16.823	16.209
$X\hat{b}:\alpha=0.05$	14.800	14.606	14.394	13.988	13.642
$X\hat{b}:\alpha=0.30$	12.658	12.596	12.539	12.457	12.404
$X\hat{b}:\alpha=0.75$	12.300	12.292	12.284	12.271	12.265
$\nu=100, \lambda_d=3$					
Xb	11.102	11.102	11.102	11.102	11.102
Xb*	24.041	26.041	28.041	32.041	36.041
$X\hat{b}:\alpha=0.01$	16.549	16.162	15.658	14.525	13.479
$X\hat{b}:\alpha=0.05$	13.228	12.843	12.409	11.922	11.561
$X\hat{b}:\alpha=0.30$	11.291	11.228	11.183	11.135	11.115
$X\hat{b}:\alpha=0.75$	11.115	11.110	11.105	11.102	11.102
$\nu=\infty, \lambda_d=3$					
Xb	11.000	11.000	11.000	11.000	11.000
Xb*	24.000	26.000	28.000	32.000	36.000
$X\hat{b}:\alpha=0.01$	16.425	15.996	15.445	14.233	13.147
$X\hat{b}:\alpha=0.05$	13.050	12.642	12.272	11.705	11.358
$X\hat{b}:\alpha=0.30$	11.162	11.102	11.062	11.022	11.008
$X\hat{b}:\alpha=0.75$	11.010	11.007	11.004	11.000	11.000
$\nu=5, \lambda_d=10$					
Xb	28.333	28.333	28.333	28.333	28.333
Xb*	39.333	41.333	43.333	47.333	51.333
$X\hat{b}:\alpha=0.01$	35.653	36.021	36.189	36.104	35.724
$X\hat{b}:\alpha=0.05$	32.344	32.280	32.150	31.811	31.445
$X\hat{b}:\alpha=0.30$	29.097	29.036	28.980	28.880	28.801
$X\hat{b}:\alpha=0.75$	28.442	28.433	28.422	28.405	28.396
$\nu=10, \lambda_d=10$					
Xb	26.250	26.250	26.250	26.250	26.250
Xb*	38.500	40.500	42.500	46.500	50.500
$X\hat{b}:\alpha=0.01$	34.609	34.866	34.873	34.365	33.491
$X\hat{b}:\alpha=0.05$	30.734	30.475	30.150	29.451	28.801
$X\hat{b}:\alpha=0.30$	26.979	26.862	26.762	26.607	26.503
$X\hat{b}:\alpha=0.75$	26.337	26.318	26.307	26.285	26.272
$\nu=100, \lambda_d=10$					
Xb	25.102	25.102	25.102	25.102	25.102
Xb*	38.041	40.041	42.041	46.041	50.041
$X\hat{b}:\alpha=0.01$	33.932	34.059	33.891	32.894	31.435
$X\hat{b}:\alpha=0.05$	29.545	29.075	28.533	27.475	26.619
$X\hat{b}:\alpha=0.30$	25.597	25.449	25.341	25.207	25.146
$X\hat{b}:\alpha=0.75$	25.136	25.121	25.115	25.107	25.102
$\nu=\infty, \lambda_d=10$					
Xb	25.000	25.000	25.000	25.000	25.000
Xb*	38.000	40.000	42.000	46.000	50.000
$X\hat{b}:\alpha=0.01$	33.866	33.975	33.781	32.719	31.183
$X\hat{b}:\alpha=0.05$	29.422	28.923	28.357	27.254	26.384
$X\hat{b}:\alpha=0.30$	25.463	25.314	25.209	25.086	25.034
$X\hat{b}:\alpha=0.75$	25.030	25.016	25.009	25.004	25.000

TABLE A4.2.3: Relative Risks of Xb, Xb* and $X\hat{b}$

$v = 16,\ k = 4,\ m = 1.$

Estimator							λ_n								
	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$\nu=5, \lambda_d=0$															
Xb	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667
Xb*	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
$X\hat{b}:\alpha=0.01$	5.118	6.180	7.117	7.911	8.564	9.505	10.072	10.386	10.531	10.570	10.537	10.461	10.361	10.121	9.863
$X\hat{b}:\alpha=0.05$	5.422	6.399	7.112	7.616	7.966	8.360	8.512	8.541	8.504	8.435	8.352	8.267	8.179	8.017	7.874
$X\hat{b}:\alpha=0.30$	6.278	6.641	6.829	6.926	6.976	7.003	6.993	6.970	6.947	6.921	6.899	6.880	6.862	6.835	6.809
$X\hat{b}:\alpha=0.75$	6.652	6.667	6.673	6.676	6.678	6.677	6.677	6.676	6.674	6.674	6.674	6.673	6.673	6.671	6.671
$\nu=10, \lambda_d=0$															
Xb	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
Xb*	3.750	4.750	5.750	6.750	7.750	9.750	11.750	13.750	15.750	17.750	19.750	21.750	23.750	27.750	31.750
$X\hat{b}:\alpha=0.01$	3.838	4.901	5.843	6.639	7.289	8.166	8.607	8.742	8.686	8.516	8.287	8.030	7.767	7.278	6.859
$X\hat{b}:\alpha=0.05$	4.066	5.042	5.745	6.224	6.529	6.791	6.787	6.661	6.490	6.309	6.140	5.989	5.854	5.645	5.489
$X\hat{b}:\alpha=0.30$	4.709	5.068	5.240	5.312	5.333	5.308	5.255	5.206	5.165	5.133	5.105	5.085	5.069	5.049	5.034
$X\hat{b}:\alpha=0.75$	4.989	5.003	5.009	5.011	5.011	5.009	5.007	5.005	5.004	5.003	5.003	5.003	5.000	5.000	5.000
$\nu=100, \lambda_d=0$															
Xb	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082
Xb*	3.061	4.061	5.061	6.061	7.061	9.061	11.061	13.061	15.061	17.061	19.061	21.061	23.061	27.061	31.061
$X\hat{b}:\alpha=0.01$	3.133	4.198	5.144	5.945	6.589	7.412	7.706	7.621	7.303	6.870	6.405	5.964	5.572	4.966	4.579
$X\hat{b}:\alpha=0.05$	3.319	4.295	4.989	5.441	5.700	5.815	5.630	5.339	5.039	4.783	4.581	4.429	4.319	4.186	4.131
$X\hat{b}:\alpha=0.30$	3.844	4.199	4.354	4.401	4.391	4.314	4.232	4.173	4.135	4.114	4.099	4.090	4.085	4.082	4.082
$X\hat{b}:\alpha=0.75$	4.072	4.087	4.092	4.092	4.091	4.087	4.085	4.084	4.082	4.082	4.082	4.082	4.082	4.082	4.082
$\nu=\infty, \lambda_d=0$															
Xb	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000
Xb*	3.000	4.000	5.000	6.000	7.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	27.000	31.000
$X\hat{b}:\alpha=0.01$	3.071	4.135	5.082	5.884	6.527	7.344	7.620	7.504	7.150	6.680	6.188	5.724	5.323	4.724	4.371
$X\hat{b}:\alpha=0.05$	3.253	4.228	4.921	5.371	5.624	5.720	5.512	5.199	4.888	4.626	4.427	4.281	4.182	4.070	4.027
$X\hat{b}:\alpha=0.30$	3.767	4.122	4.275	4.319	4.306	4.221	4.138	4.079	4.044	4.022	4.011	4.007	4.004	4.000	4.000
$X\hat{b}:\alpha=0.75$	3.991	4.005	4.010	4.010	4.009	4.006	4.003	4.002	4.002	4.000	4.000	4.000	4.000	4.000	4.000
$\nu=5, \lambda_d=1$															
Xb	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667
Xb*	7.000	8.000	9.000	10.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	31.000	35.000
$X\hat{b}:\alpha=0.01$	7.099	8.161	9.134	9.987	10.709	11.780	12.442	12.810	12.981	13.022	12.977	12.885	12.759	12.470	12.170
$X\hat{b}:\alpha=0.05$	7.378	8.392	9.162	9.721	10.115	10.560	10.728	10.751	10.702	10.615	10.518	10.413	10.313	10.121	9.962
$X\hat{b}:\alpha=0.30$	8.246	8.652	8.860	8.966	9.017	9.042	9.027	8.998	8.970	8.939	8.917	8.893	8.874	8.841	8.819
$X\hat{b}:\alpha=0.75$	8.650	8.667	8.675	8.677	8.679	8.678	8.677	8.677	8.676	8.676	8.674	8.672	8.671	8.670	8.670
$\nu=10, \lambda_d=1$															
Xb	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000
Xb*	5.750	6.750	7.750	8.750	9.750	11.750	13.750	15.750	17.750	19.750	21.750	23.750	25.750	29.750	33.750
$X\hat{b}:\alpha=0.01$	5.819	6.881	7.854	8.707	9.424	10.456	11.020	11.236	11.218	11.055	10.807	10.522	10.222	9.656	9.163
$X\hat{b}:\alpha=0.05$	6.023	7.034	7.795	8.334	8.691	9.019	9.036	8.904	8.711	8.505	8.311	8.133	7.980	7.733	7.552
$X\hat{b}:\alpha=0.30$	6.676	7.080	7.275	7.357	7.381	7.350	7.290	7.232	7.184	7.146	7.116	7.093	7.075	7.051	7.036
$X\hat{b}:\alpha=0.75$	6.987	7.004	7.011	7.012	7.012	7.010	7.008	7.005	7.004	7.003	7.003	7.003	7.000	7.000	7.000
$\nu=100, \lambda_d=1$															
Xb	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082
Xb*	5.061	6.061	7.061	8.061	9.061	11.061	13.061	15.061	17.061	19.061	21.061	23.061	25.061	29.061	33.061
$X\hat{b}:\alpha=0.01$	5.114	6.175	7.148	8.001	8.716	9.711	10.171	10.208	9.955	9.535	9.043	8.548	8.082	7.327	6.816
$X\hat{b}:\alpha=0.05$	5.276	6.285	7.038	7.556	7.877	8.079	7.927	7.624	7.291	6.990	6.741	6.548	6.409	6.233	6.150
$X\hat{b}:\alpha=0.30$	5.811	6.213	6.395	6.453	6.445	6.359	6.265	6.195	6.149	6.120	6.103	6.094	6.089	6.082	6.082
$X\hat{b}:\alpha=0.75$	6.071	6.088	6.094	6.094	6.093	6.088	6.086	6.084	6.082	6.082	6.082	6.082	6.082	6.082	6.082
$\nu=\infty, \lambda_d=1$															
Xb	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
Xb*	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
$X\hat{b}:\alpha=0.01$	5.051	6.112	7.086	7.939	8.654	9.872	10.335	10.363	10.085	9.634	9.111	8.580	8.090	7.313	6.811
$X\hat{b}:\alpha=0.05$	5.210	6.219	6.970	7.486	7.802	7.988	7.815	7.492	7.142	6.833	6.581	6.397	6.261	6.110	6.044
$X\hat{b}:\alpha=0.30$	5.734	6.136	6.316	6.371	6.361	6.268	6.170	6.099	6.055	6.028	6.017	6.011	6.004	6.000	6.000
$X\hat{b}:\alpha=0.75$	5.989	6.007	6.012	6.012	6.010	6.007	6.003	6.002	6.002	6.000	6.000	6.000	6.000	6.000	6.000
$\nu=5, \lambda_d=3$															
Xb	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667
Xb*	11.000	12.000	13.000	14.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	29.000	31.000	35.000	39.000
$X\hat{b}:\alpha=0.01$	11.072	12.126	13.137	14.073	14.909	16.242	17.137	17.667	17.929	18.004	17.961	17.841	17.677	17.282	16.873
$X\hat{b}:\alpha=0.05$	11.310	12.365	13.229	13.897	14.387	14.962	15.182	15.208	15.136	15.018	14.883	14.745	14.609	14.362	14.153
$X\hat{b}:\alpha=0.30$	12.182	12.671	12.923	13.048	13.105	13.124	13.095	13.055	13.016	12.982	12.951	12.920	12.897	12.858	12.832
$X\hat{b}:\alpha=0.75$	12.646	12.669	12.677	12.681	12.682	12.682	12.680	12.677	12.676	12.676	12.674	12.672	12.671	12.670	12.670
$\nu=10, \lambda_d=3$															
Xb	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000
Xb*	9.750	10.750	11.750	12.750	13.750	15.750	17.750	19.750	21.750	23.750	25.750	27.750	29.750	33.750	37.750
$X\hat{b}:\alpha=0.01$	9.794	10.845	11.852	12.782	13.613	14.925	15.765	16.195	16.309	16.205	15.960	15.635	15.272	14.542	13.884
$X\hat{b}:\alpha=0.05$	9.956	11.008	11.863	12.514	12.981	13.468	13.557	13.432	13.209	12.956	12.704	12.473	12.271	11.939	11.701
$X\hat{b}:\alpha=0.30$	10.612	11.102	11.346	11.451	11.480	11.440	11.362	11.288	11.226	11.178	11.141	11.113	11.092	11.060	11.041
$X\hat{b}:\alpha=0.75$	10.983	11.006	11.014	11.016	11.015	11.013	11.010	11.007	11.004	11.003	11.003	11.003	11.000	11.000	11.000
$\nu=100, \lambda_d=3$															
Xb	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082
Xb*	9.061	10.061	11.061	12.061	13.061	15.061	17.061	19.061	21.061	23.061	25.061	27.061	29.061	33.061	37.061
$X\hat{b}:\alpha=0.01$	9.090	10.138	11.141	12.066	12.892	14.182	14.967	15.287	15.233	14.914	14.435	13.876	13.300	12.259	11.460
$X\hat{b}:\alpha=0.05$	9.210	10.260	11.105	11.742	12.182	12.580	12.530	12.239	11.860	11.476	11.139	10.862	10.644	10.361	10.215
$X\hat{b}:\alpha=0.30$	9.747	10.239	10.473	10.557	10.558	10.458	10.339	10.244	10.179	10.138	10.114	10.101	10.091	10.086	10.082
$X\hat{b}:\alpha=0.75$	10.067	10.090	10.098	10.098	10.096	10.090	10.086	10.083	10.083	10.082	10.082	10.082	10.082	10.082	10.082
$\nu=\infty, \lambda_d=3$															
Xb	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000
Xb*	9.000	10.000	11.000	12.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	29.000	33.000	37.000
$X\hat{b}:\alpha=0.01$	9.027	10.076	11.077	12.002	12.826	14.114	14.892	15.198	15.126	14.778	14.268	13.674	13.076	12.002	11.202
$X\hat{b}:\alpha=0.05$	9.143	10.193	11.038	11.672	12.109	12.497	12.430	12.120	11.719	11.324	10.979	10.700	10.485	10.222	10.090
$X\hat{b}:\alpha=0.30$	9.670	10.162	10.396	10.477	10.474	10.368	10.243	10.146	10.085	10.047	10.025	10.012	10.008	10.000	10.000
$X\hat{b}:\alpha=0.75$	9.985	10.009	10.016	10.017	10.014	10.009	10.004	10.002	10.002	10.000	10.000	10.000	10.000	10.000	10.000
$\nu=5, \lambda_d=10$															
Xb	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667
Xb*	25.000	26.000	27.000	28.000	29.000	31.000	33.000	35.000	37.000	39.000	41.000	43.000	45.000	49.000	53.000
$X\hat{b}:\alpha=0.01$	25.034	26.058	27.079	28.090	29.082	30.964	32.624	33.980	34.972	35.624	35.940	36.009	35.893	35.276	34.476
$X\hat{b}:\alpha=0.05$	25.180	26.250	27.260	28.178	28.975	30.159	30.815	31.062	31.037	30.855	30.599	30.313	30.029	29.509	29.075
$X\hat{b}:\alpha=0.30$	25.987	26.707	27.127	27.348	27.448	27.465	27.393	27.303	27.219	27.149	27.087	27.036	26.996	26.930	26.888
$X\hat{b}:\alpha=0.75$	26.632	26.676	26.690	26.694	26.694	26.691	26.686	26.684	26.680	26.678	26.677	26.675	26.673	26.672	26.671
$\nu=10, \lambda_d=10$															
Xb	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000
Xb*	23.750	24.750	25.750	26.750	27.750	29.750	31.750	33.750	35.750	37.750	39.750	41.750	43.750	47.750	51.750
$X\hat{b}:\alpha=0.01$	23.762	24.781	25.797	26.804	27.790	29.649	31.276	32.586	33.529	34.106	34.346	34.302	34.022	33.059	31.868
$X\hat{b}:\alpha=0.05$	23.836	24.902	25.905	26.811	27.593	28.735	29.331	29.487	29.341	29.017	28.612	28.187	27.772	27.062	26.520
$X\hat{b}:\alpha=0.30$	24.417	25.150	25.571	25.782	25.860	25.816	25.681	25.539	25.419	25.325	25.252	25.199	25.155	25.100	25.068
$X\hat{b}:\alpha=0.75$	24.969	25.014	25.029	25.031	25.030	25.022	25.017	25.013	25.009	25.007	25.006	25.004	25.003	25.000	25.000
$\nu=100, \lambda_d=10$															
Xb	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082
Xb*	23.061	24.061	25.061	26.061	27.061	29.061	31.061	33.061	35.061	37.061	39.061	41.061	43.061	47.061	51.061
$X\hat{b}:\alpha=0.01$	23.065	24.078	25.090	26.092	27.072	28.913	30.512	31.784	32.676	33.179	33.318	33.136	32.701	31.342	29.736
$X\hat{b}:\alpha=0.05$	23.101	24.163	25.159	26.053	26.819	27.921	28.459	28.526	28.261	27.794	27.240	26.678	26.156	25.320	24.776
$X\hat{b}:\alpha=0.30$	23.552	24.299	24.720	24.919	24.976	24.871	24.672	24.485	24.343	24.246	24.183	24.141	24.118	24.093	24.087
$X\hat{b}:\alpha=0.75$	24.052	24.099	24.114	24.115	24.111	24.101	24.092	24.087	24.084	24.084	24.082	24.082	24.082	24.082	24.082
$\nu=\infty, \lambda_d=10$															
Xb	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000
Xb*	23.000	24.000	25.000	26.000	27.000	29.000	31.000	33.000	35.000	37.000	39.000	41.000	43.000	47.000	51.000
$X\hat{b}:\alpha=0.01$	23.003	24.015	25.027	26.029	27.008	28.847	30.442	31.710	32.597	33.095	33.221	33.023	32.568	31.165	29.511
$X\hat{b}:\alpha=0.05$	23.036	24.098	25.093	25.985	26.750	27.847	28.377	28.435	28.156	27.671	27.103	26.523	25.990	25.148	24.611
$X\hat{b}:\alpha=0.30$	23.475	24.224	24.644	24.843	24.896	24.784	24.577	24.387	24.243	24.146	24.086	24.050	24.027	24.007	24.000
$X\hat{b}:\alpha=0.75$	23.971	24.017	24.033	24.034	24.030	24.018	24.010	24.006	24.003	24.002	24.000	24.000	24.000	24.000	24.000

TABLE A4.2.4: Relative Risks of Xb, Xb* and Xb̂.
ν = 16, k = 4, m = 3.

Estimator	λ_n 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$\nu=5, \lambda_d=0$															
Xb	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667	6.667
Xb*	1.667	2.667	3.667	4.667	5.667	7.667	9.667	11.667	13.667	15.667	17.667	19.667	21.667	25.667	29.667
Xb̂:α=0.01	1.843	2.918	3.958	4.938	5.843	7.403	8.634	9.572	10.270	10.777	11.131	11.373	11.525	11.646	11.604
Xb̂:α=0.05	2.381	3.513	4.509	5.361	6.075	7.152	7.867	8.330	8.620	8.794	8.885	8.925	8.928	8.866	8.761
Xb̂:α=0.30	4.505	5.290	5.834	6.213	6.479	6.800	6.961	7.040	7.074	7.084	7.082	7.071	7.058	7.026	6.994
Xb̂:α=0.75	6.350	6.492	6.572	6.619	6.648	6.679	6.691	6.696	6.697	6.696	6.695	6.694	6.692	6.690	6.687
$\nu=10, \lambda_d=0$															
Xb	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000
Xb*	1.250	2.250	3.250	4.250	5.250	7.250	9.250	11.250	13.250	15.250	17.250	19.250	21.250	25.250	29.250
Xb̂:α=0.01	1.382	2.459	3.503	4.492	5.409	6.987	8.202	9.074	9.656	9.998	10.155	10.176	10.098	9.759	9.301
Xb̂:α=0.05	1.786	2.918	3.917	4.768	5.472	6.490	7.089	7.391	7.495	7.476	7.379	7.243	7.085	6.764	6.468
Xb̂:α=0.30	3.379	4.162	4.692	5.042	5.268	5.486	5.540	5.521	5.476	5.419	5.364	5.317	5.271	5.199	5.150
Xb̂:α=0.75	4.763	4.903	4.977	5.016	5.036	5.047	5.044	5.038	5.032	5.026	5.020	5.017	5.013	5.009	5.009
$\nu=100, \lambda_d=0$															
Xb	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082	4.082
Xb*	1.020	2.020	3.020	4.020	5.020	7.020	9.020	11.020	13.020	15.020	17.020	19.020	21.020	25.020	29.020
Xb̂:α=0.01	1.128	2.205	3.253	4.252	5.183	6.789	8.003	8.816	9.258	9.390	9.276	8.990	8.588	7.647	6.723
Xb̂:α=0.05	1.458	2.591	3.592	4.444	5.142	6.103	6.570	6.777	6.551	6.294	5.980	5.659	5.357	4.871	4.547
Xb̂:α=0.30	2.758	3.539	4.055	4.377	4.560	4.668	4.610	4.500	4.389	4.298	4.230	4.180	4.148	4.110	4.095
Xb̂:α=0.75	3.888	4.027	4.096	4.126	4.135	4.127	4.113	4.101	4.093	4.088	4.085	4.085	4.082	4.082	4.082
$\nu=\infty, \lambda_d=0$															
Xb	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000	4.000
Xb*	1.000	2.000	3.000	4.000	5.000	7.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	25.000	29.000
Xb̂:α=0.01	1.106	2.183	3.232	4.231	5.164	6.774	7.990	8.796	9.220	9.323	9.172	8.838	8.389	7.358	6.380
Xb̂:α=0.05	1.428	2.562	3.563	4.416	5.114	6.067	6.518	6.599	6.438	6.149	5.808	5.466	5.153	4.666	4.357
Xb̂:α=0.30	2.703	3.483	3.998	4.317	4.494	4.588	4.516	4.395	4.281	4.189	4.121	4.077	4.047	4.018	4.005
Xb̂:α=0.75	3.810	3.949	4.017	4.096	4.054	4.045	4.030	4.017	4.009	4.005	4.003	4.000	4.000	4.000	4.000
$\nu=5, \lambda_d=1$															
Xb	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667	8.667
Xb*	3.667	4.667	5.667	6.667	7.667	9.667	11.667	13.667	15.667	17.667	19.667	21.667	23.667	27.667	31.667
Xb̂:α=0.01	3.806	4.866	5.909	6.915	7.866	9.562	10.946	12.029	12.843	13.437	13.853	14.127	14.298	14.409	14.338
Xb̂:α=0.05	4.271	5.402	6.435	7.346	8.193	9.339	10.154	10.679	10.999	11.185	11.273	11.301	11.289	11.187	11.041
Xb̂:α=0.30	6.321	7.186	7.798	8.227	8.527	8.884	9.054	9.131	9.159	9.163	9.152	9.133	9.115	9.068	9.029
Xb̂:α=0.75	8.301	8.474	8.569	8.623	8.656	8.687	8.700	8.703	8.703	8.703	8.700	8.697	8.696	8.691	8.686
$\nu=10, \lambda_d=1$															
Xb	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000	7.000
Xb*	3.250	4.250	5.250	6.250	7.250	9.250	11.250	13.250	15.250	17.250	19.250	21.250	23.250	27.250	31.250
Xb̂:α=0.01	3.345	4.405	5.448	6.456	7.413	9.119	10.503	11.552	12.291	12.759	13.008	13.088	13.037	12.690	12.173
Xb̂:α=0.05	3.676	4.806	5.837	6.745	7.521	8.690	9.416	9.805	9.957	9.950	9.850	9.693	9.506	9.121	8.755
Xb̂:α=0.30	5.195	6.058	6.661	7.067	7.333	7.594	7.657	7.632	7.573	7.503	7.435	7.375	7.321	7.235	7.172
Xb̂:α=0.75	6.714	6.886	6.977	7.024	7.047	7.058	7.054	7.044	7.036	7.030	7.025	7.018	7.016	7.009	7.009
$\nu=100, \lambda_d=1$															
Xb	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082	6.082
Xb*	3.020	4.020	5.020	6.020	7.020	9.020	11.020	13.020	15.020	17.020	19.020	21.020	23.020	27.020	31.020
Xb̂:α=0.01	3.091	4.151	5.194	6.205	7.166	8.888	10.280	11.302	11.957	12.280	12.319	12.139	11.795	10.840	9.779
Xb̂:α=0.05	3.349	4.477	5.506	6.411	7.182	8.313	8.944	9.175	9.123	8.891	8.564	8.204	7.846	7.229	6.784
Xb̂:α=0.30	4.574	5.437	6.030	6.414	6.644	6.804	6.753	6.628	6.494	6.379	6.290	6.225	6.178	6.123	6.097
Xb̂:α=0.75	5.839	6.012	6.099	6.137	6.150	6.142	6.122	6.108	6.096	6.090	6.087	6.085	6.082	6.082	6.082
$\nu=\infty, \lambda_d=1$															
Xb	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000	6.000
Xb*	3.000	4.000	5.000	6.000	7.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	27.000	31.000
Xb̂:α=0.01	3.069	4.128	5.171	6.182	7.145	8.870	10.262	11.282	11.927	12.228	12.239	12.024	11.636	10.590	9.462
Xb̂:α=0.05	3.319	4.448	5.476	6.382	7.152	8.279	8.898	9.109	9.030	8.767	8.411	8.023	7.648	7.018	6.584
Xb̂:α=0.30	4.519	5.382	5.973	6.355	6.580	6.728	6.664	6.527	6.389	6.269	6.179	6.118	6.074	6.029	6.008
Xb̂:α=0.75	5.761	5.934	6.021	6.058	6.070	6.059	6.040	6.024	6.014	6.006	6.003	6.003	6.000	6.000	6.000
$\nu=5, \lambda_d=3$															
Xb	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667	12.667
Xb*	7.667	8.667	9.667	10.667	11.667	13.667	15.667	17.667	19.667	21.667	23.667	25.667	27.667	31.667	35.667
Xb̂:α=0.01	7.761	8.799	9.834	10.857	11.856	13.736	15.394	16.779	17.885	18.720	19.324	19.732	19.989	20.154	20.029
Xb̂:α=0.05	8.119	9.230	10.293	11.282	12.177	13.648	14.701	15.400	15.830	16.065	16.166	16.181	16.133	15.948	15.710
Xb̂:α=0.30	9.995	10.977	11.714	12.249	12.628	13.072	13.272	13.346	13.359	13.341	13.313	13.278	13.239	13.168	13.107
Xb̂:α=0.75	12.194	12.435	12.565	12.635	12.675	12.709	12.720	12.719	12.717	12.713	12.710	12.706	12.703	12.698	12.696
$\nu=10, \lambda_d=3$															
Xb	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000	11.000
Xb*	7.250	8.250	9.250	10.250	11.250	13.250	15.250	17.250	19.250	21.250	23.250	25.250	27.250	31.250	35.250
Xb̂:α=0.01	7.303	8.338	9.371	10.391	11.387	13.263	14.918	16.296	17.372	18.153	18.661	18.939	19.024	18.774	18.192
Xb̂:α=0.05	7.528	8.636	9.694	10.675	11.561	13.006	14.007	14.618	14.919	14.994	14.915	14.736	14.503	13.982	13.470
Xb̂:α=0.30	8.869	9.854	10.586	11.107	11.462	11.827	11.923	11.892	11.806	11.704	11.606	11.517	11.441	11.321	11.235
Xb̂:α=0.75	10.605	10.849	10.978	11.043	11.073	11.087	11.077	11.063	11.049	11.041	11.031	11.027	11.021	11.013	11.009
$\nu=100, \lambda_d=3$															
Xb	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082	10.082
Xb*	7.020	8.020	9.020	10.020	11.020	13.020	15.020	17.020	19.020	21.020	23.020	25.020	27.020	31.020	35.020
Xb̂:α=0.01	7.051	8.084	9.115	10.131	11.126	12.997	14.650	16.018	17.066	17.785	18.193	18.314	18.198	17.436	16.285
Xb̂:α=0.05	7.204	8.309	9.361	10.334	11.212	12.630	13.581	14.099	14.259	14.149	13.863	13.471	13.030	12.174	11.470
Xb̂:α=0.30	8.249	9.237	9.964	10.471	10.801	11.086	11.075	10.934	10.757	10.589	10.450	10.342	10.260	10.162	10.118
Xb̂:α=0.75	9.731	9.976	10.104	10.164	10.185	10.175	10.148	10.125	10.107	10.096	10.088	10.088	10.085	10.082	10.082
$\nu=\infty, \lambda_d=3$															
Xb	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000	10.000
Xb*	7.000	8.000	9.000	10.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	31.000	35.000
Xb̂:α=0.01	7.029	8.062	9.092	10.109	11.102	12.974	14.625	15.992	17.037	17.749	18.141	18.244	18.099	17.272	16.041
Xb̂:α=0.05	7.175	8.280	9.331	10.304	11.180	12.595	13.540	14.047	14.189	14.055	13.740	13.322	12.857	11.965	11.248
Xb̂:α=0.30	8.193	9.183	9.909	10.414	10.741	11.017	10.993	10.838	10.652	10.477	10.335	10.228	10.150	10.062	10.023
Xb̂:α=0.75	9.653	9.899	10.027	10.085	10.105	10.094	10.064	10.040	10.024	10.014	10.007	10.004	10.004	10.000	10.000
$\nu=5, \lambda_d=10$															
Xb	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667	26.667
Xb*	21.667	22.667	23.667	24.667	25.667	27.667	29.667	31.667	33.667	35.667	37.667	39.667	41.667	45.667	49.667
Xb̂:α=0.01	21.706	22.716	23.727	24.740	25.751	27.766	29.753	31.687	33.5[illegible]	35.224	36.745	38.058	39.131	40.500	41.093
Xb̂:α=0.05	21.885	22.933	23.980	25.017	26.039	27.994	29.760	31.257	32.443	33.272	33.801	34.070	34.126	33.817	33.255
Xb̂:α=0.30	23.216	24.338	25.335	26.181	26.869	27.796	28.252	28.406	28.395	28.302	28.175	28.040	27.913	27.689	27.516
Xb̂:α=0.75	25.769	26.263	26.546	26.700	26.778	26.829	26.825	26.807	26.788	26.772	26.758	26.747	26.737	26.722	26.711
$\nu=10, \lambda_d=10$															
Xb	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000	25.000
Xb*	21.250	22.250	23.250	24.250	25.250	27.250	29.250	31.250	33.250	35.250	37.250	39.250	41.250	45.250	49.250
Xb̂:α=0.01	21.261	22.268	23.275	24.284	25.292	27.299	29.278	31.199	33.029	34.727	36.251	37.568	38.646	40.020	40.453
Xb̂:α=0.05	21.330	22.369	23.406	24.436	25.449	27.381	29.115	30.578	31.713	32.511	32.979	33.161	33.099	32.478	31.527
Xb̂:α=0.30	22.112	23.248	24.247	25.088	25.763	26.654	27.050	27.120	27.007	26.805	26.572	26.345	26.143	25.812	25.571
Xb̂:α=0.75	24.173	24.680	24.972	25.130	25.205	25.234	25.204	25.162	25.125	25.094	25.074	25.059	25.043	25.026	25.021
$\nu=100, \lambda_d=10$															
Xb	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082	24.082
Xb*	21.020	22.020	23.020	24.020	25.020	27.020	29.020	31.020	33.020	35.020	37.020	39.020	41.020	45.020	49.020
Xb̂:α=0.01	21.022	22.025	23.030	24.035	25.040	27.039	29.009	30.914	32.727	34.400	35.898	37.190	38.250	39.597	39.933
Xb̂:α=0.05	21.044	22.072	23.101	24.122	25.126	27.035	28.740	30.163	31.258	31.999	32.398	32.486	32.310	31.386	30.058
Xb̂:α=0.30	21.515	22.665	23.666	24.500	25.163	26.017	26.354	26.341	26.126	25.816	25.486	25.181	24.920	24.538	24.316
Xb̂:α=0.75	23.289	23.810	24.112	24.273	24.344	24.350	24.290	24.226	24.175	24.140	24.118	24.104	24.094	24.086	24.082
$\nu=\infty, \lambda_d=10$															
Xb	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000	24.000
Xb*	21.000	22.000	23.000	24.000	25.000	27.000	29.000	31.000	33.000	35.000	37.000	39.000	41.000	45.000	49.000
Xb̂:α=0.01	21.001	22.004	23.008	24.013	25.017	27.016	28.984	30.889	32.697	34.368	35.862	37.152	38.200	39.541	39.854
Xb̂:α=0.05	21.019	22.046	23.075	24.095	25.097	27.004	28.705	30.125	31.213	31.950	32.338	32.418	32.224	31.263	29.895
Xb̂:α=0.30	21.463	22.615	23.616	24.448	25.111	25.959	26.290	26.268	26.039	25.717	25.378	25.065	24.794	24.417	24.197
Xb̂:α=0.75	23.211	23.733	24.036	24.197	24.268	24.271	24.208	24.140	24.090	24.054	24.031	24.016	24.009	24.004	24.000

TABLE A4.2.5: Relative Risks of Xb, Xb* and X$\hat{b}$

$v = 16$, $k = 4$, $m = 1$.

							λ_d								
Estimator	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$\nu=5, \lambda_n=0$															
Xb	6.667	7.667	8.667	9.667	10.667	12.667	14.667	16.667	18.667	20.667	22.667	24.667	26.667	30.667	34.667
Xb*	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
X$\hat{b}$: α=0.01	5.118	6.107	7.099	8.091	9.084	11.073	13.064	15.056	17.050	19.045	21.041	23.037	25.034	29.029	33.025
X$\hat{b}$: α=0.05	5.422	6.399	7.378	8.359	9.342	11.310	13.284	15.260	17.240	19.222	21.206	23.192	25.180	29.159	33.141
X$\hat{b}$: α=0.30	6.278	7.262	8.246	9.229	10.213	12.182	14.151	16.122	18.093	20.065	22.038	24.012	25.987	29.940	33.896
X$\hat{b}$: α=0.75	6.652	7.651	8.650	9.649	10.648	12.646	14.644	16.642	18.640	20.638	22.636	24.634	26.632	30.627	34.622
$\nu=10, \lambda_n=0$															
Xb	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
Xb*	3.750	4.750	5.750	6.750	7.750	9.750	11.750	13.750	15.750	17.750	19.750	21.750	23.750	27.750	31.750
X$\hat{b}$: α=0.01	3.838	4.828	5.819	6.811	7.805	9.794	11.786	13.779	15.774	17.770	19.767	21.764	23.762	27.759	31.757
X$\hat{b}$: α=0.05	4.066	5.044	6.023	7.004	7.987	9.956	11.930	13.908	15.889	17.872	19.858	21.846	23.836	27.819	31.806
X$\hat{b}$: α=0.30	4.709	5.692	6.676	7.660	8.644	10.612	12.582	14.552	16.523	18.495	20.468	22.443	24.418	28.371	32.327
X$\hat{b}$: α=0.75	4.072	5.071	6.071	7.070	8.069	10.067	12.065	14.063	16.061	18.059	20.057	22.055	24.055	28.048	32.043
$\nu=100, \lambda_n=0$															
Xb	4.082	5.082	6.082	7.082	8.082	10.082	12.082	14.082	16.082	18.082	20.082	22.082	24.082	28.082	32.082
Xb*	3.061	4.061	5.061	6.061	7.061	9.061	11.061	13.061	15.061	17.061	19.061	21.061	23.061	27.061	31.061
X$\hat{b}$: α=0.01	3.133	4.123	5.114	6.107	7.100	9.090	11.082	13.077	15.073	17.070	19.067	21.066	23.065	27.063	31.062
X$\hat{b}$: α=0.05	3.320	4.297	5.276	6.257	7.240	9.210	11.104	13.163	15.146	17.131	19.119	21.109	23.101	27.089	31.080
X$\hat{b}$: α=0.30	3.844	4.827	5.811	6.795	7.779	9.747	11.716	13.687	15.658	17.630	19.603	21.577	23.552	27.506	31.463
X$\hat{b}$: α=0.75	4.072	5.071	6.071	7.070	8.069	10.067	12.065	14.063	16.061	18.059	20.057	22.055	24.052	28.048	32.043
$\nu=\infty, \lambda_n=0$															
Xb	4.000	5.000	6.000	7.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	28.000	32.000
Xb*	3.000	4.000	5.000	6.000	7.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	27.000	31.000
X$\hat{b}$: α=0.01	3.071	4.060	5.051	6.044	7.037	9.027	11.020	13.014	15.010	17.007	19.005	21.004	23.003	27.001	31.001
X$\hat{b}$: α=0.05	3.253	4.230	5.210	6.191	7.174	9.143	11.118	13.097	15.080	17.066	19.054	21.044	23.036	27.024	31.016
X$\hat{b}$: α=0.30	3.767	4.751	5.734	6.718	7.702	9.670	11.640	13.610	15.581	17.553	19.526	21.500	23.476	27.429	31.386
X$\hat{b}$: α=0.75	3.991	4.990	5.989	6.988	7.987	9.986	11.984	13.982	15.980	17.977	19.975	21.973	23.971	27.966	31.961
$\nu=5, \lambda_n=1$															
Xb	6.667	7.667	8.667	9.667	10.667	12.667	14.667	16.667	18.667	20.667	22.667	24.667	26.667	30.667	34.667
Xb*	7.000	8.000	9.000	10.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	27.000	31.000	35.000
X$\hat{b}$: α=0.01	7.117	8.128	9.134	10.138	11.139	13.137	15.130	17.121	19.112	21.103	23.094	25.086	27.079	31.066	35.056
X$\hat{b}$: α=0.05	7.112	8.138	9.162	10.183	11.201	13.229	15.249	17.262	19.269	21.271	23.270	25.266	27.260	31.245	25.228
X$\hat{b}$: α=0.30	6.829	7.845	8.860	9.876	10.891	12.923	14.954	16.985	19.015	21.045	23.073	25.101	27.127	31.178	35.223
X$\hat{b}$: α=0.75	6.673	7.674	8.675	9.675	10.676	12.677	14.679	16.681	18.683	20.684	22.686	24.688	26.690	30.694	34.699
$\nu=10, \lambda_n=1$															
Xb	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
Xb*	5.750	6.750	7.750	8.750	9.750	11.750	13.750	15.750	17.750	19.750	21.750	23.750	25.750	29.750	33.750
X$\hat{b}$: α=0.01	5.843	6.850	7.854	8.856	9.856	11.852	13.844	15.836	17.827	19.819	21.811	23.803	25.797	29.786	33.778
X$\hat{b}$: α=0.05	5.745	6.772	7.795	8.816	9.834	11.863	13.884	15.898	17.906	19.910	21.911	23.909	25.905	29.895	33.881
X$\hat{b}$: α=0.30	5.240	6.257	7.275	8.293	9.311	11.346	13.380	15.414	17.448	19.480	21.512	23.542	25.571	29.626	33.676
X$\hat{b}$: α=0.75	5.009	6.010	7.011	8.012	9.013	11.014	13.016	15.018	17.020	19.022	21.024	23.026	25.029	29.033	33.038
$\nu=100, \lambda_n=1$															
Xb	4.082	5.082	6.082	7.082	8.082	10.082	12.082	14.082	16.082	18.082	20.082	22.082	24.082	28.082	32.082
Xb*	5.061	6.061	7.061	8.061	9.061	11.061	13.061	15.061	17.061	19.061	21.061	23.061	25.061	29.061	33.061
X$\hat{b}$: α=0.01	5.144	6.147	7.148	8.148	9.146	11.141	13.133	15.125	17.117	19.109	21.102	23.096	25.090	29.081	33.075
X$\hat{b}$: α=0.05	4.989	6.015	7.038	8.058	9.076	11.105	13.127	15.142	17.152	19.158	21.161	23.161	25.159	29.152	33.142
X$\hat{b}$: α=0.30	4.355	5.375	6.395	7.415	8.434	10.473	12.512	14.549	16.586	18.621	20.655	22.688	24.720	28.779	32.834
X$\hat{b}$: α=0.75	4.092	5.093	6.094	7.095	8.096	10.098	12.100	14,102	16.104	18.106	20.109	22.112	24.114	28.119	32.125
$\nu=\infty, \lambda_n=1$															
Xb	4.000	5.000	6.000	7.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	28.000	32.000
Xb*	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
X$\hat{b}$: α=0.01	5.082	6.085	7.086	8.085	9.083	11.077	13.070	15.062	17.054	19.046	21.039	23.033	25.027	29.018	32.811
X$\hat{b}$: α=0.05	4.921	5.947	6.970	7.991	9.009	11.038	13.059	15.075	17.085	19.091	21.094	23.094	25.093	29.086	33.076
X$\hat{b}$: α=0.30	4.275	5.296	6.316	7.336	8.357	10.396	12.435	14.473	16.509	18.545	20.579	22.613	24.645	28.408	32.760
X$\hat{b}$: α=0.75	4.010	5.011	6.012	7.013	8.014	10.016	12.018	14.020	16.023	18.025	20.028	22.030	24.033	28.038	32.043
$\nu=5, \lambda_n=5$															
Xb	6.667	7.667	8.667	9.667	10.667	12.667	14.667	16.667	18.667	20.667	22.667	24.667	26.667	30.667	34.667
Xb*	15.000	16.000	17.000	18.000	19.000	21.000	23.000	25.000	27.000	29.000	31.000	33.000	35.000	39.000	43.000
X$\hat{b}$: α=0.01	10.386	11.597	12.811	14.025	15.240	17.667	20.080	22.476	24.844	27.179	29.478	31.743	33.980	38.342	42.596
X$\hat{b}$: α=0.05	8.541	9.642	10.751	11.860	12.974	15.208	17.456	19.711	21.976	24.245	26.516	28.791	31.062	35.595	40.100
X$\hat{b}$: α=0.30	6.971	7.986	8.998	10.013	11.027	13.055	15.087	17.119	19.153	21.189	23.225	25.264	27.303	31.387	35.477
X$\hat{b}$: α=0.75	6.676	7.676	8.678	9.678	10.678	12.677	14.679	16.679	18.681	20.681	22.683	24.683	26.684	30.686	34.687
$\nu=10, \lambda_n=5$															
Xb	5.000	6.000	7.000	8.000	9.000	11.000	13.000	15.000	17.000	19.000	21.000	23.000	25.000	29.000	33.000
Xb*	13.750	14.750	15.750	16.750	17.750	19.750	21.750	23.750	25.750	27.750	29.750	31.750	33.750	37.750	41.750
X$\hat{b}$: α=0.01	8.742	9.990	11.236	12.482	13.724	16.195	18.640	21.056	23.436	25.779	28.084	30.353	32.586	36.957	41.221
X$\hat{b}$: α=0.05	6.661	7.780	8.904	10.030	11.161	13.432	15.714	18.002	20.297	22.597	24.896	27.194	29.487	34.057	38.590
X$\hat{b}$: α=0.30	5.206	6.218	7.232	8.245	9.259	11.288	13.318	15.351	17.384	19.421	21.460	23.498	25.539	29.627	33.723
X$\hat{b}$: α=0.75	5.005	6.005	7.005	8.005	9.007	11.007	13.007	15.009	17.009	19.009	21.011	23.011	25.013	29.012	33.014
$\nu=100, \lambda_n=5$															
Xb	4.082	5.082	6.082	7.082	8.082	10.082	12.082	14.082	16.082	18.082	20.082	22.082	24.082	28.082	32.082
Xb*	13.061	14.061	15.061	16.061	17.061	19.061	21.061	23.061	25.061	27.061	29.061	31.061	33.061	37.061	41.061
X$\hat{b}$: α=0.01	7.621	8.916	10.208	11.489	12.765	15.287	17.769	20.207	22.605	24.958	27.271	29.545	31.784	36.168	40.447
X$\hat{b}$: α=0.05	5.339	6.479	7.624	8.772	9.926	12.239	14.565	16.893	19.225	21.557	23.887	26.209	28.526	33.132	37.692
X$\hat{b}$: α=0.30	4.173	5.184	6.195	7.207	8.217	10.244	12.272	14.301	16.335	18.369	20.405	22.444	24.485	28.576	32.673
X$\hat{b}$: α=0.75	4.084	5.084	6.084	7.084	8.084	10.083	12.085	14.085	16.085	18.085	20.085	22.087	24.087	28.087	32.089
$\nu=\infty, \lambda_n=5$															
Xb	4.000	5.000	6.000	7.000	8.000	10.000	12.000	14.000	16.000	18.000	20.000	22.000	24.000	28.000	32.000
Xb*	13.000	14.000	15.000	16.000	17.000	19.000	21.000	23.000	25.000	27.000	29.000	31.000	33.000	37.000	41.000
X$\hat{b}$: α=0.01	7.504	8.807	10.103	11.391	12.670	15.199	17.685	20.129	22.527	24.882	27.195	29.469	31.710	36.094	40.376
X$\hat{b}$: α=0.05	5.199	6.344	7.492	8.818	9.798	12.120	14.449	16.784	19.121	21.455	23.789	26.117	28.435	33.046	37.610
X$\hat{b}$: α=0.30	4.079	5.090	6.099	7.110	8.123	10.146	12.174	14.203	16.235	18.271	20.307	22.347	24.387	28.477	32.575
X$\hat{b}$: α=0.75	4.002	5.002	6.002	7.002	8.002	10.002	12.004	14.004	16.004	18.004	20.004	22.006	24.006	28.006	32.008

CHAPTER FIVE

SMALL SAMPLE PROPERTIES OF THE MIS-SPECIFIED PRE-TEST LINEAR RESTRICTIONS ESTIMATOR OF THE ERROR VARIANCE

5.1 Introduction

In the last chapter we investigated various estimators of the prediction vector, after a pre-test for exact linear restrictions on the coefficient vector, when the process generating the data is mis-specified by the omission of relevant regressors and the regression disturbances are taken to be normal when in fact they are spherically symmetric. In this chapter we use the model framework established in Chapter Four to investigate the risk properties of several estimators of the error variance.

We derive and evaluate the bias and the risk functions for the general family of estimators considered by Clarke *et al.* (1987b) and Clarke (1986). Three special members of this family are the least squares (L), the maximum likelihood (ML) and the minimum mean squared error (M) unrestricted and restricted estimators of the error variance which we discussed in Chapter Two. We assume that the researcher considers model (4.2.3), which omits relevant variables and mis-specifies the error distribution. Consequently, the estimators of the error variance that he believes are the maximum likelihood and the minimum mean squared error estimators no longer possess these properties. The estimators of the error variance under the maximum likelihood or the minimum mean squared error principle depend on the specific form of the error distribution. This contrasts with the estimators of β, which are the least squares and the maximum likelihood estimators for all members of the spherically symmetric family. So, too, the L estimators of the error variance under a normality assumption are least squares for the wider assumption of $e \sim SSD_N$.

This chapter is presented in the following manner. In the next section we detail the derivations of the exact bias and the exact risk functions of the unrestricted, restricted and the pre-test estimators of the error variance. This work extends the current literature by deriving the risk function of the pre-test estimator when the model is possibly mis-specified in two ways. The bias and the risk functions of the aforementioned special members, of the general family of estimators that we consider, are given in Appendix 5.1 of this chapter. We illustrate the results, as we did in Chapter Four, for the special case of multivariate Student-t regression disturbances and when the component estimators of the pre-test estimator are the L, ML, and the M estimators. These numerical evaluations enable us to more easily analyse the impact of the mis-specifications.

In Section 5.3 we compare the risk functions assuming that the design matrix is properly specified and we then consider the more general results in Section 5.4. In these sections we include only some illustrative results of the numerical evaluations. Appendix 5.2 of this chapter gives further examples. Some concluding remarks are given in the final section, followed by two Appendices. Appendix 5.1 presents the bias and the risk functions of the L, ML, and the M estimators, while Appendix 5.2 gives a small sample of the numerical evaluations of the relative risk functions. Further detail regarding their content is given in the discussion and in the introduction to each Appendix.

5.2 The Bias and Risk Functions

Under the framework of Chapter Four, a pre-test estimator of σ_e^2 is

$$\hat{\sigma}^2 = \begin{cases} \tilde{\sigma}^2 \ ; & \text{if } u > c \\ \sigma^{*2} \ ; & \text{if } u \leq c \end{cases} , \tag{5.2.1}$$

where $\tilde{\sigma}^2$, the unrestricted estimator, and σ^{*2}, the estimator which incorporates the restrictions, are specified to be of the form

$$\tilde{\sigma}^2 = (y-Xb)'(y-Xb)/(T+g) \tag{5.2.2}$$

$$\sigma^{*2} = (y-Xb^*)'(y-Xb^*)/(T+h)\ . \tag{5.2.3}$$

The L, ML and M estimators we discussed in Chapter Two are commonly used members of this family. First, when g=-k and h=(-k+m) we generate $\tilde{\sigma}^2_L$ and σ^{*2}_L respectively; secondly, if g=h=0 then $\tilde{\sigma}^2=\tilde{\sigma}^2_{ML}$ and $\sigma^{*2}=\sigma^{*2}_{ML}$; finally, if g=(-k+2) and h=(-k+m+2) then $\tilde{\sigma}^2=\tilde{\sigma}^2_M$ and $\sigma^{*2}=\sigma^{*2}_M$. In this chapter we derive the bias and the risk functions in terms of the general family $\hat{\sigma}^2$, $\tilde{\sigma}^2$ and σ^{*2}.

Care should be taken with the notation being used here. In (5.2.1), for example, $\hat{\sigma}^2$ is an estimator of σ^2_e, not of σ^2 (except when the errors are normal) - this simplified notation is used for convenience to avoid clumsy terminology in Appendix 5.1.

We first consider the bias functions of the estimators. We define the bias of an estimator $\bar{\sigma}^2$ of σ^2_e as $\text{bias}(\bar{\sigma}^2)=E(\bar{\sigma}^2)-\sigma^2_e=E(\bar{\sigma}^2)-E(\tau^2)$.

<u>Theorem 5.2.1</u>

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $SSD_N(0,I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\text{bias}(\tilde{\sigma}^2) = \left[2\theta_d - (k+g)E(\tau^2)\right]/(T+g)\ , \tag{5.2.4}$$

$$\text{bias}(\sigma^{*2}) = \left[2(\theta_n+\theta_d) + (m-k-h)E(\tau^2)\right]/(T+h)\ , \tag{5.2.5}$$

$$\text{bias}(\hat{\sigma}^2) = \Big\{2\theta_d(T+h)-(T+h)(k+g)E(\tau^2)+(g-h)\left[v\int_0^\infty \tau^2 P^{d\tau}_{02} f(\tau)d\tau+\right.$$

$$\left.2\theta_d\int_0^\infty P^{d\tau}_{04} f(\tau)d\tau\right]+(T+g)\left[m\int_0^\infty \tau^2 P^{d\tau}_{20} f(\tau)d\tau+\right.$$

$$2\theta_n\int_0^\infty P_{40}^{d\tau}f(\tau)d\tau\Big]\Big\}\Big/\Big[(T+g)(T+h)\Big], \tag{5.2.6}$$

where $P_{ij}^{d\tau}$ is as defined in Chapter Four.

Proof.

To establish (5.2.4) we have

$$\tilde{\sigma}^2 = e_2'Me_2/(T+g) \tag{5.2.7}$$

and $E(\tilde{\sigma}^2)=\int_0^\infty E_N(\tilde{\sigma}^2)f(\tau)d\tau$ where $E_N(\tilde{\sigma}^2)=E_N\Big(e_2'Me_2/(T+g)\Big)$ when $e\sim N(0,\tau^2I_T)$. Under this assumption $e_2'Me_2/\tau^2\sim\chi^{2\prime}_{v;\lambda_{d\tau}}$, $\lambda_{d\tau}=\theta_d/\tau^2$ and $\theta_d=\gamma'Z'MZ\gamma/2$. $E_N(e_2'Me_2/\tau^2)=(v+2\lambda_{d\tau})$ and $E(\tilde{\sigma}^2)=\frac{1}{(T+g)}\int_0^\infty(v\tau^2+2\theta_d)f(\tau)d\tau=\frac{1}{(T+g)}\Big(2\theta_d+vE(\tau^2)\Big)$, from which (5.2.4) follows.

(5.2.5) follows in a similar manner as

$$\sigma^{*2} = (e_2'Me_2+e_1'Ce_1)/(T+h) . \tag{5.2.8}$$

If $e\sim N(0,\tau^2I_T)$ then $(e_2'Me_2+e_1'Ce_1)/\tau^2\sim\chi^{2\prime}_{m+v;\lambda_{n\tau}+\lambda_{d\tau}}$, and so, $E_N\Big[(e_2'Me_2+e_1'Ce_1)/\tau^2\Big]=(m+v)+2(\lambda_{n\tau}+\lambda_{d\tau})$. Integrating with respect to τ gives $E(\sigma^{*2})=\frac{1}{(T+h)}\Big[(v+m)E(\tau^2)+2(\theta_n+\theta_d)\Big]$ from which we obtain (5.2.5).

To derive (5.2.6), we write using (5.2.7) and (5.2.8),

$$\begin{aligned}\hat{\sigma}^2 &= \tilde{\sigma}^2+(\sigma^{*2}-\tilde{\sigma}^2)I_{[0,c]}(u)\\ &= \Big\{(e_1'Me_1)(T+h)+\Big[(g-h)e_1'Me_1+(T+g)e_1'Ce_1\Big]\\ &\quad .I_{[0,c]}(ve_1'Ce_1/me_1'Me_1)\Big\}\Big/\Big[(T+g)(T+h)\Big]\end{aligned} \tag{5.2.9}$$

as $e_2'Me_2=e_1'Me_1$. Now, $E(\hat{\sigma}^2)=\int_0^\infty E_N(\hat{\sigma}^2)f(\tau)d\tau$ and $E_N(e_1'Me_1/\tau^2)=(v+2\lambda_{d\tau})$.

Further, using Lemma 1 of Clarke *et al.* (1987a),

$$E_N\Big[(e_1'Me_1/\tau^2)I_{[0,c]}\Big((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\Big)\Big] = vP_{02}^{d\tau}+2\lambda_{d\tau}P_{04}^{d\tau}$$

and

$$E_N\Big[(e_1'Ce_1/\tau^2)I_{[0,c]}\Big((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\Big)\Big] = mP_{20}^{d\tau}+2\lambda_{n\tau}P_{40}^{d\tau} .$$

So,

$$E_N(\hat{\sigma}^2) = \left\{(v\tau^2+2\theta_d)(T+h)+(g-h)\left(v\tau^2P_{02}^{d\tau}+2\theta_dP_{04}^{d\tau}\right)\right.$$

$$\left.+(T+g)\left(m\tau^2P_{20}^{d\tau}+2\theta_nP_{40}^{d\tau}\right)\right\}\Big/\left[(T+g)(T+h)\right] , \tag{5.2.10}$$

from which we obtain (5.2.6). #

Following from Theorem 5.2.1 we now give three corollaries which derive the bias functions of $\tilde{\sigma}^2$, σ^{*2} and $\hat{\sigma}^2$ for three special cases. Corollary 5.2.1 considers the situation of no omitted regressors, while Corollaries 5.2.2 and 5.2.3 derive the bias functions of the estimators when the regression disturbances are Mt and normal.

Corollary 5.2.1

If there are no omitted regressors ($Z\gamma=0$) and the regression disturbances are $SSD_N(0,I_T)$ then

$$\text{bias}_0(\tilde{\sigma}^2) = \left[-(k+g)E(\tau^2)\right]/(T+g) , \tag{5.2.11}$$

$$\text{bias}_0(\sigma^{*2}) = \left[2\theta+(m-k-h)E(\tau^2)\right]/(T+h) , \tag{5.2.12}$$

$$\text{bias}_0(\hat{\sigma}^2) = \left\{-(T+h)(k+g)E(\tau^2)+(g-h)v\int_0^\infty \tau^2P_{02}^{\tau}f(\tau)d\tau\right.$$

$$\left.+(T+g)\left[m\int_0^\infty \tau^2P_{20}^{\tau}f(\tau)d\tau+2\theta\int_0^\infty P_{40}^{\tau}f(\tau)d\tau\right]\right\}\Big/\left[(T+g)(T+h)\right], \tag{5.2.13}$$

where P_{ij}^{τ} is as defined in Chapter Four.

Proof.

$Z\gamma=0$, so $\Lambda=0$ and (5.2.11), (5.2.12) and (5.2.13) follow directly from, respectively, (5.2.4), (5.2.5) and (5.2.6). #

Corollary 5.2.2

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $Mt(0,\sigma_e^2 I_T)$, $\sigma_e^2=\nu\sigma^2/(\nu-2)$, and the pre-test is of H_0 in (4.3.2), then for $\nu>2$

$$\mathrm{bias}_{Mt}(\tilde{\sigma}^2) = \sigma^2\Big[2\lambda_d(\nu-2)-\nu(k+g)\Big]/\Big[(\nu-2)(T+g)\Big] , \tag{5.2.14}$$

$$\mathrm{bias}_{Mt}(\sigma^{*2}) = \sigma^2\Big[2(\lambda_n+\lambda_d)(\nu-2)+\nu(m-k-h)\Big]/\Big[(\nu-2)(T+h)\Big] , \tag{5.2.15}$$

$$\mathrm{bias}_{Mt}(\hat{\sigma}^2) = \sigma^2\Big\{2\lambda_d(T+h)(\nu-2)-\nu(T+h)(k+g)+(g-h)\Big[v\nu P^d_{021}$$

$$+2\lambda_d(\nu-2)P^d_{042}\Big]+(T+g)\Big[m\nu P^d_{201}+2\lambda_n(\nu-2)P^d_{402}\Big]\Big\}/\Big[(\nu-2)(T+g)(T+h)\Big]. \tag{5.2.16}$$

If there are no omitted regressors, $Z\gamma=0$, then

$$\mathrm{bias}_{OMt}(\tilde{\sigma}^2) = -\nu\sigma^2(k+g)/\Big[(\nu-2)(T+g)\Big] , \tag{5.2.17}$$

$$\mathrm{bias}_{OMt}(\sigma^{*2}) = \sigma^2\Big[2\lambda(\nu-2)+\nu(m-k-h)\Big]/\Big[(\nu-2)(T+h)\Big] , \tag{5.2.18}$$

$$\mathrm{bias}_{OMt}(\hat{\sigma}^2)= \sigma^2\Big[-\nu(T+h)(k+g)+v\nu(g-h)P_{021}$$

$$+(T+g)\Big(m\nu P_{201}+2\lambda(\nu-2)P_{402}\Big)\Big]/\Big[(\nu-2)(T+g)(T+h)\Big] . \tag{5.2.19}$$

Proof.

(5.2.14) and (5.2.15) are easily derived from (5.2.4) and (5.2.5): when $e\sim Mt\left(0,\nu\sigma^2/(\nu-2)I_T\right)$, $E(\tau^2)=\nu\sigma^2/(\nu-2)$ as $\tau\sim IG$. We use this result again and (4.3.19) and (4.3.37) to obtain (5.2.16) from (5.2.6). When $Z\gamma=0$, $\Lambda=0$ and so $\lambda_d=0$, $\lambda_n=\lambda$, $P^d_{ijn}=P_{ijn}$. and then (5.2.14), (5.2.15) and (5.2.16) collapse to (5.2.17), (5.2.18) and (5.2.19). #

Corollary 5.2.3

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are normally distributed as $N(0,\sigma^2 I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\text{bias}_N(\tilde{\sigma}^2) = \sigma^2\left[2\lambda_d-(k+g)\right]/(T+g) \ , \tag{5.2.20}$$

$$\text{bias}_N(\sigma^{*2}) = \sigma^2\left[2(\lambda_n+\lambda_d)+(m-k-h)\right]/(T+h) \ , \tag{5.2.21}$$

$$\text{bias}_N(\hat{\sigma}^2) = \sigma^2\left[2\lambda_d(T+h)-(T+h)(k+g)+(g-h)\left(vP^d_{02}+2\lambda_d P^d_{04}\right)\right.$$

$$\left.+(T+g)\left(mP^d_{20}+2\lambda_n P^d_{40}\right)\right]/\left[(T+g)(T+h)\right] \ . \tag{5.2.22}$$

If there are no omitted regressors, $Z\gamma=0$, then

$$\text{bias}_{ON}(\tilde{\sigma}^2) = -\sigma^2(k+g)/(T+g) \ , \tag{5.2.23}$$

$$\text{bias}_{ON}(\sigma^{*2}) = \sigma^2\left(2\lambda+(m-k-h)\right)/(T+h) \ , \tag{5.2.24}$$

$$\text{bias}_{ON}(\hat{\sigma}^2) = \sigma^2\left[-(T+h)(k+g)+(g-h)vP_{02}\right.$$

$$\left.+(T+g)(mP_{20}+2\lambda P_{40})\right]/\left[(T+g)(T+h)\right] \ . \tag{5.2.25}$$

Proof.

These expressions are obtained from Corollary 5.2.2 as $e\sim N(0,\sigma^2 I_T)$ when $\nu=\infty$. Then, $\lim_{\nu\to\infty}\sigma_e^2=\sigma^2$, $\lim_{\nu\to\infty}P^d_{ijn}=\Pr.\left[F''_{(m+i,v+j;\lambda_n,\lambda_d)}\le\left(cm(v+j)\right)/\left(v(m+i)\right)\right]$ $=P^d_{ij}$ and $\lim_{\nu\to\infty}P_{ijn}=\Pr.\left[F'_{(m+i,v+j;\lambda)}\le\left(cm(v+j)\right)/\left(v(m+i)\right)\right]=P_{ij}$, $i,j=0,1,2,\ldots$.#

We have derived the bias functions for a family of estimators. Three members of this family are of particular interest: the L, ML, and the M component estimators. The bias functions for these special cases, using some of the above theorems and corollaries, are given in Appendix 5.1.

Given that our central interest lies with the risk functions of the estimators, we present here only a relatively short discussion of the features of the bias functions. To illustrate these features we have numerically evaluated the bias expressions given in the special cases of Corollaries 5.2.2 and 5.2.3 in Appendix 5.1 for various choices of ν, α, m, k, and v (and, hence, T) as functions of λ_n and λ_d. We recall that these special cases consider the L, ML, and the M unrestricted and restricted estimators and their corresponding pre-test estimators when the errors are Mt and normal.

It should be noted that the notation $\hat{\sigma}^2_L$, $\hat{\sigma}^2_{ML}$, and $\hat{\sigma}^2_M$ is not to be interpreted to imply that these are, respectively, the L, the ML, and the M pre-test estimators of σ^2_e. Rather we mean that these are the pre-test estimators of σ^2_e whose component estimators (the researcher believes) are the L, the ML, and the M estimators of σ^2_e.

A wide selection of values of the arguments was investigated: v=10, 16, 20, 30; k=4, 5; m=1, 3; α=0.01, 0.05, 0.30, 0.50, 0.75 and those values of α associated with a critical value of unity (c^*_L) and a critical value of v/(v+2) (c^*_M); ν=5, 10, 100, 1000, ∞; $\lambda_n \in [0,5(0.5);5,10(1.0);10,20(2.0)]$; and $\lambda_d \in [0,5(0.5);5,10(1.0);10,20(2.0)]$. We used Davies' (1980) algorithm and the subroutines GAMMLN and BETAI from Press *et al.* (1986) to assist with the evaluations of P^d_{ij} and P^d_{ijn}, respectively. These computer programs were executed on a VAX 6230 computer.

Figures 5.2.1 to 5.2.12 illustrate the results, full details of which are available upon request. The figures consider the case of T=20, k=4, m=1 and give the relative bias functions of the estimators as functions of λ_n for a given value of λ_d. We consider relative bias which, of an estimator $\bar{\sigma}^2$ of σ^2_e, is given by Rbias($\bar{\sigma}^2$)=bias($\bar{\sigma}^2$)/σ^2 , so as to eliminate the scale parameter σ^2. It is for this reason that we also utilise λ_n and λ_d rather than θ_n and θ_d in the diagrams.

We consider the relative bias functions of $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ in Figures 5.2.1 to 5.2.4; $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ in Figures 5.2.5 to 5.2.8; and, $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ in Figures 5.2.9 to 5.2.12. In each of these sets of figures we present the relative bias functions when $\lambda_d=0$ (that is, $\lambda_n=\lambda$,), and when $\lambda_d=5$ for $\nu=5$ and $\nu=\infty$. The scales on the diagrams are different to enable their features to be evident. Further, negative bias values are in parentheses and the legend associated with each of the estimators follows.

Legend for Figures 5.2.1 to 5.2.12

Rbias($\tilde{\sigma}^2_i$)

Rbias(σ^{*2}_i)

Rbias($\hat{\sigma}^2_i$)
$\alpha = 0.01$

Rbias($\hat{\sigma}^2_i$)
$\alpha = 0.05$

Rbias($\hat{\sigma}^2_i$)
$\alpha = 0.30$

Rbias($\hat{\sigma}^2_i$)
$\alpha = 0.75$

Rbias($\hat{\sigma}^2_i$)
$c = 1$ or $v/(v+2)$

As there are two bias functions with the same line type, we have distinguished the unrestricted estimator with the aid of an arrow and a label. We now give some characteristics of the bias functions. The features labelled (b), (c) and (d) apply for all members of the family of SSD_N errors, while the final two points are specific to the L, ML, and the M component estimators when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$. The figures, nevertheless, illustrate all of the discussed features.

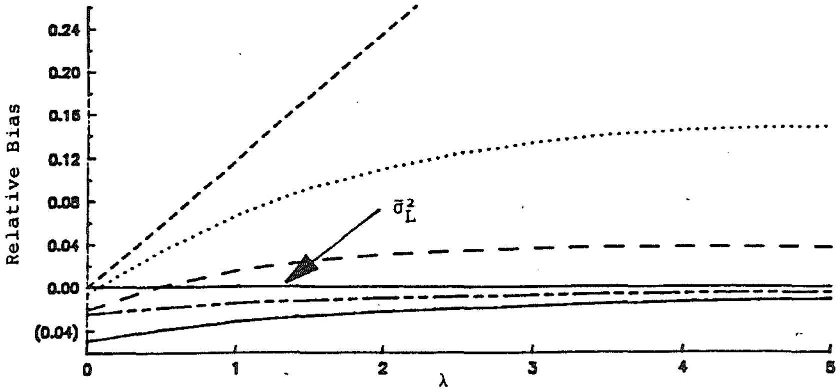

FIGURE 5.2.1: Relative bias functions for $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$.

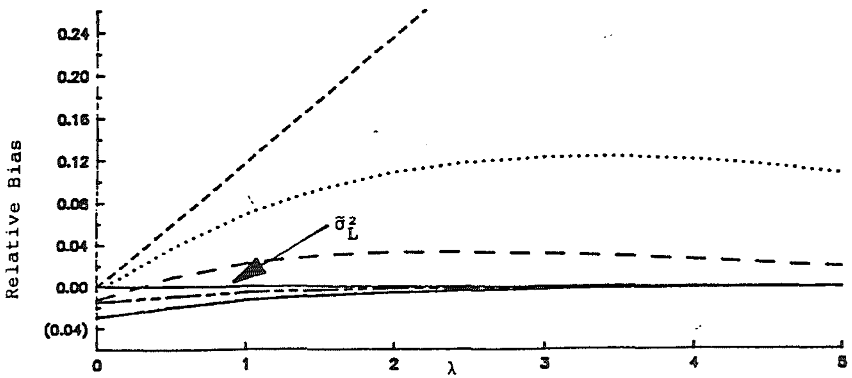

FIGURE 5.2.2: Relative bias functions for $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ when $e \sim N(0, \sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$.

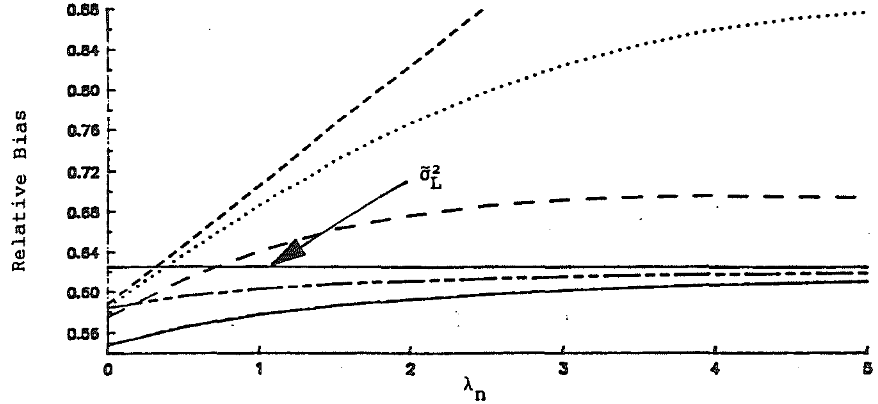

FIGURE 5.2.3: Relative bias functions for $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$, and $\lambda_d = 5$.

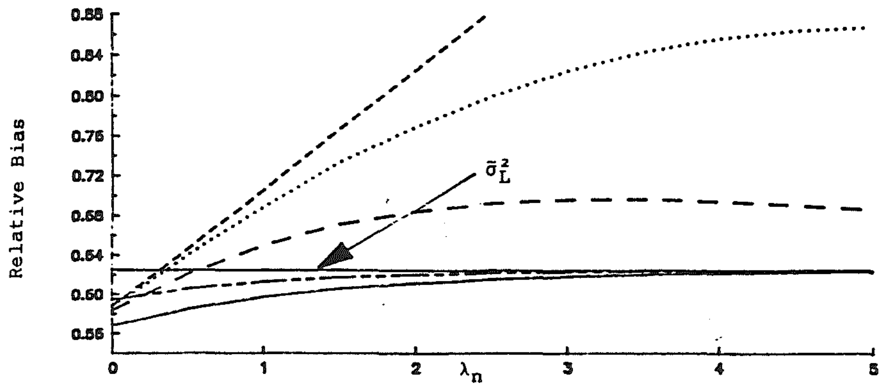

FIGURE 5.2.4: Relative bias functions for $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ when $e \sim N(0, \sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$, and $\lambda_d = 5$.

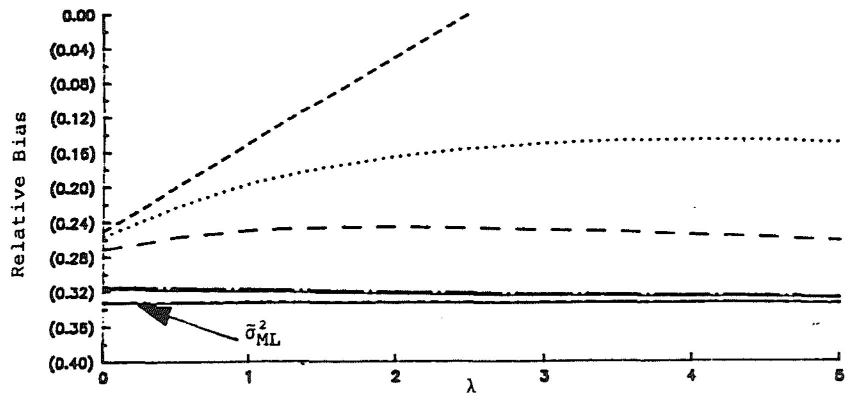

FIGURE 5.2.5: Relative bias functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$.

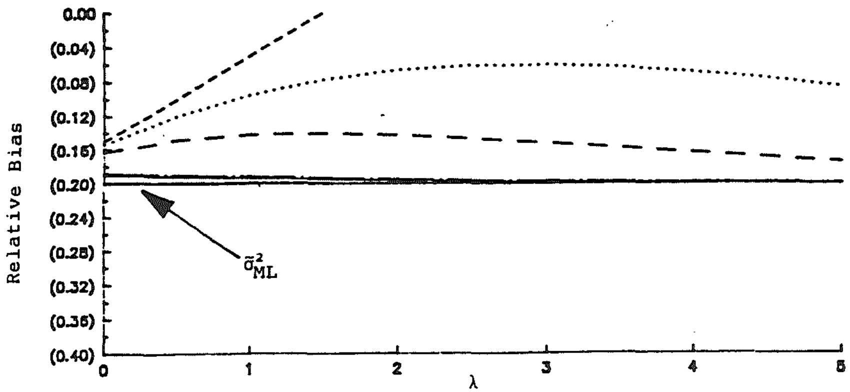

FIGURE 5.2.6: Relative bias functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim N(0,\sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$.

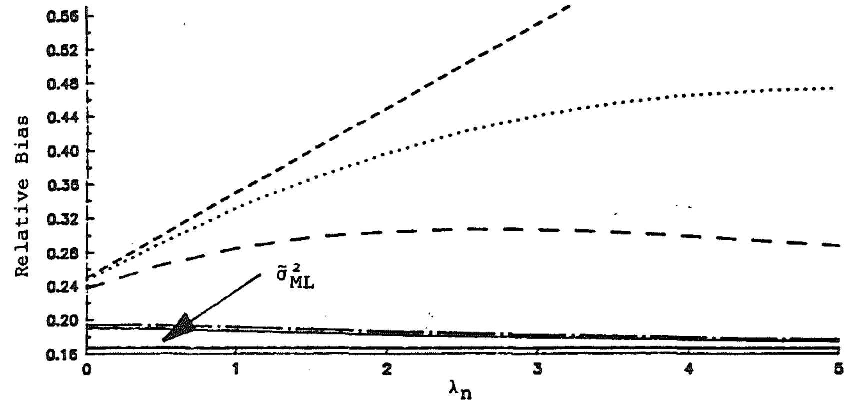

FIGURE 5.2.7: Relative bias functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$, and $\lambda_d = 5$.

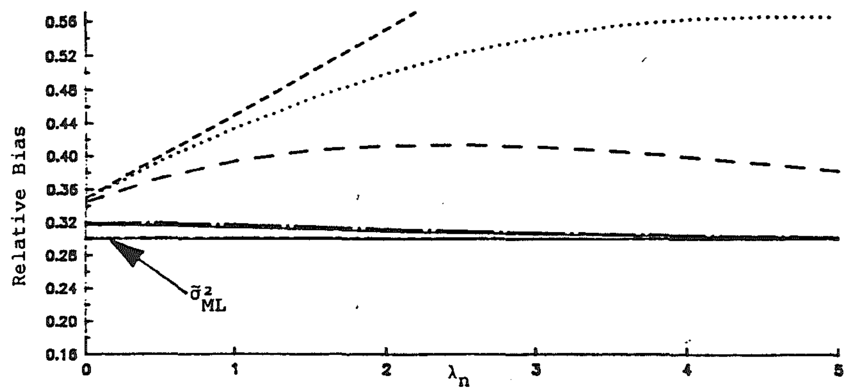

FIGURE 5.2.8: Relative bias functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim N(0,\sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$, and $\lambda_d = 5$.

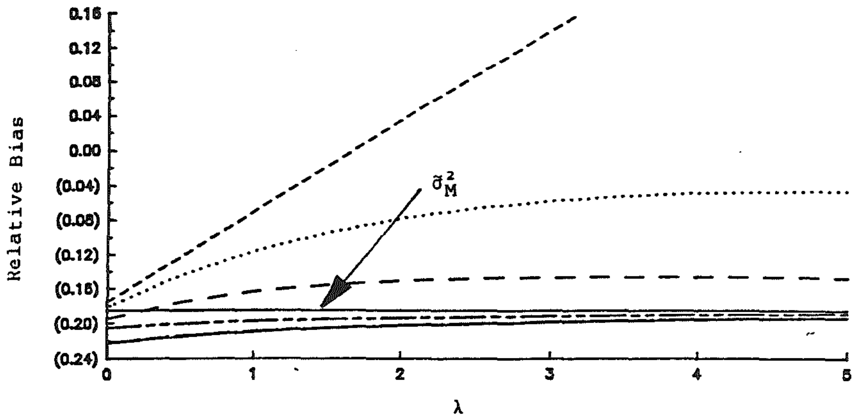

FIGURE 5.2.9: Relative bias functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$.

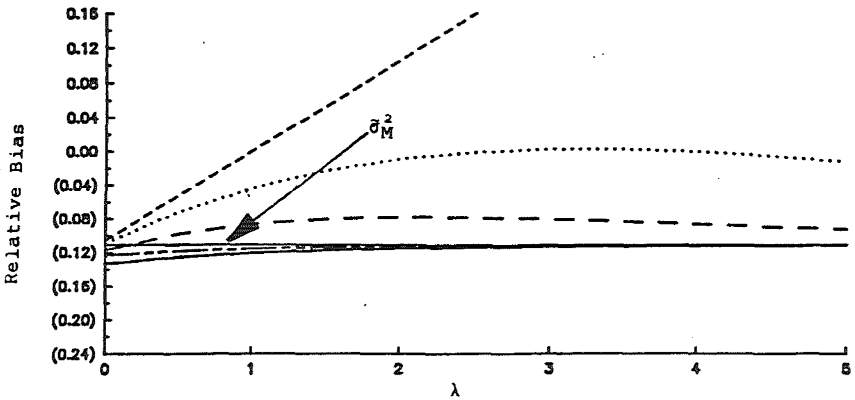

FIGURE 5.2.10: Relative bias functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim N(0, \sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$.

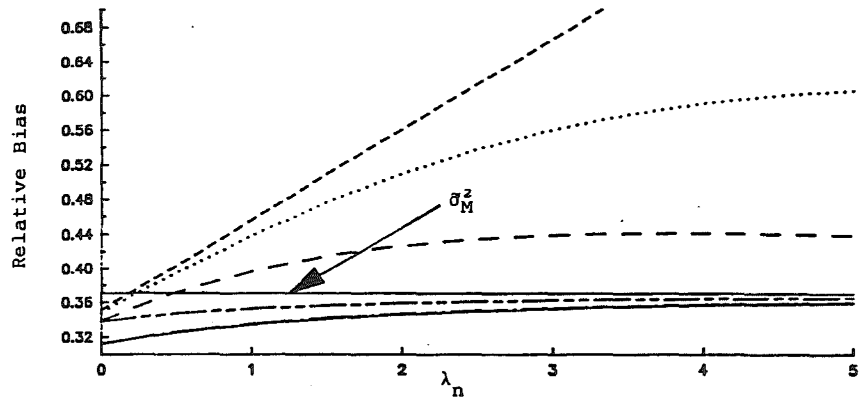

FIGURE 5.2.11: Relative bias functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$, and $\lambda_d = 5$.

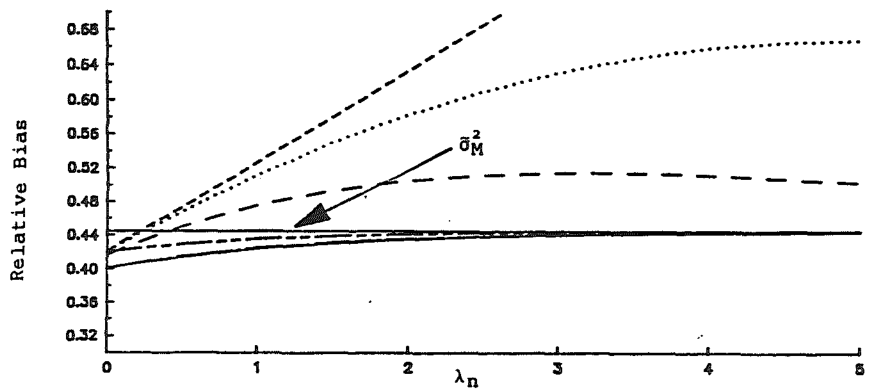

FIGURE 5.2.12: Relative bias functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim N(0, \sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$, and $\lambda_d = 5$.

(a) Equations (5.2.23), (5.2.24), and (5.2.25) are equivalent to the appropriate expressions derived by Clarke *et al.* (1987b).

(b) As $\alpha \to 1$, the value at which we always reject H_0, $c \to 0$, and hence $P_{ij}^{d\tau} \to 0$ for all i,j. So, the bias of the pre-test estimator approaches that of the unrestricted estimator. Conversely, as $\alpha \to 0$, the value at which we never reject H_0, $c \to \infty$, and $P_{ij}^{d\tau} \to 1$ for all i,j. Then, the bias of the pre-test estimator approaches that of the restricted estimator.

(c) For a given degree of mis-specification, as the hypothesis error increases, $\theta_n \to \infty$, $P_{ij}^{d\tau} \to 0$ and the bias of the pre-test estimator approaches that of the unrestricted estimator.

(d) The bias function of the unrestricted estimator is independent of θ_n, while it monotonically increases with θ_d. The bias function of the restricted estimator monotonically increases with both θ_n and θ_d, while that of the pre-test estimator is a second-order function in both θ_n and θ_d.

So, for a given value of θ_d, the bias functions of $\tilde{\sigma}^2$ and of $\hat{\sigma}^2$ are bounded, while that of σ^{*2} is unbounded, as $\theta_n \to \infty$. However, for a given value of θ_n, the bias functions of $\tilde{\sigma}^2$, σ^{*2} and $\hat{\sigma}^2$ are unbounded as $\theta_d \to \infty$. Further, the bias difference, bias($\tilde{\sigma}^2$)-bias(σ^{*2}),is unbounded as $\theta_d \to \infty$, for a given value of θ_n, for $g \neq h$. If $g=h$, as it does for the ML estimators, then $\text{bias}(\tilde{\sigma}^2)-\text{bias}(\sigma^{*2})=-\left(2\theta_n+mE(\tau^2)\right)/(T+g)$, which is bounded as $\theta_d \to \infty$, given θ_n. Similarly, the bias difference bias($\tilde{\sigma}^2$)-bias($\hat{\sigma}^2$) is unbounded for $g \neq h$. However, if $g=h$ then, $\text{bias}(\tilde{\sigma}^2)-\text{bias}(\hat{\sigma}^2)=-\left(mE(\tau^2)+2\theta_n\right)/(T+g)$ as $\theta_d \to \infty$, for a given value of θ_n. So, the bias difference bias(σ^{*2})-bias($\hat{\sigma}^2$) is unbounded for $g \neq h$, but is equal to zero when $g=h$, when $\theta_d=\infty$, given θ_n.

(e) The bias functions of $\tilde{\sigma}_L^2$ and σ_L^{*2} are independent of τ as is evident from equations (A5.1) and (A5.2). The bias($\tilde{\sigma}_L^2$) depends on θ_d but not on θ_n, and so $\tilde{\sigma}_L^2$ is unbiased when $\theta_d=0$. However, bias(σ_L^{*2}) is determined by both θ_d and θ_n. The bias ($\hat{\sigma}_L^2$) depends on all of the arguments, including τ, θ_d

and θ_n. When $e \sim Mt\left(0, \nu\sigma^2/(\nu-2)I_T\right)$ we find that $\text{bias}_{Mt}(\hat{\sigma}_L^2)$ shifts upwards as ν increases. If there are no omitted regressors then this results in $|\text{bias}_{ON}(\hat{\sigma}_L^2)|<|\text{bias}_{OMt}(\hat{\sigma}_L^2)|\nu<\infty|$. However, if there is sufficient mis-specification of the design matrix then $|\text{bias}_N(\hat{\sigma}_L^2)|>|\text{bias}_{Mt}(\hat{\sigma}_L^2)|\nu<\infty|$. Figures 5.2.1 to 5.2.4 illustrate these features for the case of Mt errors. (Recall that then λ_n replaces θ_n, etc.).

The bias functions of $\tilde{\sigma}_{ML}^2$, σ_{ML}^{*2}, and $\hat{\sigma}_{ML}^2$, and those of $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$, depend on τ. Given a value of θ_d, as ν increases the bias functions of these estimators shift upwards, as the figures illustrate. We find that the aforementioned inequalities given for the bias functions of $\hat{\sigma}_L^2$ also hold for $\hat{\sigma}_{ML}^2$ and for $\hat{\sigma}_M^2$.

For the ML component estimators $\text{bias}_{Mt}(\tilde{\sigma}_{ML}^2)\leq\text{bias}_{Mt}(\hat{\sigma}_{ML}^2)\leq\text{bias}_{Mt}(\sigma_{ML}^{*2})$ and for the L and the M component estimators (except for some $\lambda_n \in (0, \lambda_n^*; \lambda_n^*>0)$) a similar inequality occurs. For $\lambda_n \in (0, \lambda_n^*)$ the bias of the pre-test estimator is less than that of the unrestricted estimator. This range increases with α such that for some α, $\text{bias}_{Mt}(\hat{\sigma}_L^2)\leq\text{bias}_{Mt}(\tilde{\sigma}_L^2)$ and $\text{bias}_{Mt}(\hat{\sigma}_M^2)\leq\text{bias}_{Mt}(\tilde{\sigma}_M^2)$ for all $\lambda_n \geq 0$.

So, if the model is sufficiently mis-specified and the aim is to minimize relative bias using the ML component estimators, then it is preferable to always ignore the prior information, even if it is correct. Pre-testing in this case is never the optimal strategy. Conversely, if the L or the M component estimators are used then the strategy should be to always pre-test, even when the prior information is known to be valid.

The numerical evaluations suggest that the minimum pre-test bias for the L component estimators occurs when c=1; for the M component estimators it results when c=v/(v+2); and for the ML component estimators a critical value of c=0 results in the minimum pre-test bias. The proof of these results is given in the following proposition which, we note, holds for all members of the SSD_N family and whether or not we have excluded regressors.

Proposition 5.2.1

A sufficient condition for $\partial \text{bias}(\hat{\sigma}^2)/\partial c=0$ is for the critical value c to be equal to $c^*=\left(v(h-g)\right)/\left(m(T+g)\right)$.

Proof.

$$\text{bias}(\hat{\sigma}^2) = E\left(\hat{\sigma}^2-E(\tau^2)\right)$$

$$= E\left[\left(\tilde{\sigma}^2-E(\tau^2)\right)\left(1-I_{[0,c]}(u)\right)+\left(\sigma^{*2}-E(\tau^2)\right)I_{[0,c]}(u)\right]$$

$$= E\left[\left(e_1'Me_1/(T+g)-E(\tau^2)\right)+\left(e_1'Me_1(g-h)+e_1'Ce_1(T+g)\right)/\left((T+g)(T+h)\right)\right.$$

$$\left. \cdot I_{[0,c]}(ve_1'Ce_1/me_1'/Me_1)\right]$$

$$=\int_0^\infty E_N\left[\left(e_1'Me_1/(T+g)-E(\tau^2)\right)+\left(e_1'Me_1(g-h)+\right.\right.$$

$$\left.\left.e_1'Ce_1(T+g)\right)/\left((T+g)(T+h)\right)I(e_1'Ce_1\le mce_1'Me_1/v)\right]f(\tau)d\tau$$

$$=\int_0^\infty \tau^2E_N\left[\left((e_1'Me_1/\tau^2)/(T+g)-E(\tau^2)/\tau^2\right)+\left((e_1'Me_1/\tau^2)(g-h)\right.\right.$$

$$\left.\left.+(e_1'Ce_1/\tau^2)(T+g)\right)/\left((T+g)(T+h)\right)I\left((e_1'Ce_1/\tau^2)\le t\right)\right]f(\tau)d\tau \ ,$$

where $t=mc\left(e_1'Me_1/\tau^2\right)/v$, and $E_N[.]=E[.]$ when $e\sim N(0,\tau^2I_T)$. Under this normality assumption the quadratic forms $(e_1'Me_1/\tau^2)$ and $(e_1'Ce_1/\tau^2)$ are independent. Further, $(e_1'Me_1/\tau^2)\sim\chi^{2\prime}_{v;\lambda_{d\tau}}$ and $(e_1'Ce_1/\tau^2)\sim\chi^{2\prime}_{m;\lambda_{n\tau}}$. Then, if we let $E_N^1\{.\}$ be the $E\{.\}$ with respect to $(e_1'Me_1/\tau^2)$, $E_N^2[.]$ be the $E[.]$ with respect to $(e_1'Ce_1/\tau^2)$, and $f_N(e_1'Ce_1/\tau^2)$ be the density function of $(e_1'Ce_1/\tau^2)$ under the normality assumption, we have

$$\text{bias}(\hat{\sigma}^2) = \int_0^\infty \tau^2E_N^1\left\{E_N^2\left[\left((e_1'Me_1/\tau^2)/(T+g)-E(\tau^2)/\tau^2\right)+\left((e_1'Me_1/\tau^2)(g-h)+\right.\right.\right.$$

$$\left.\left.\left.(e_1'Ce_1/\tau^2)(T+g)\right)/\left((T+g)(T+h)\right)I\left((e_1'Ce_1/\tau^2)\le t\right)\right]\right\}f(\tau)d\tau$$

$$= \int_0^\infty \tau^2 E_N^1 \left\{ \left[(e_1' Me_1/\tau^2)/(T+g) - E(\tau^2)/\tau^2 \right] + \int_0^t \left[(e_1' Me_1/\tau^2)(g-h) \right.\right.$$

$$\left.\left. + (e_1' Ce_1/\tau^2)(T+g) \right] \Big/ \left[(T+g)(T+h) \right] f_N(e_1' Ce_1/\tau^2) d(e_1' Ce_1/\tau^2) \right\} f(\tau) d\tau .$$

So,

$$\frac{\partial \text{bias}(\hat{\sigma}^2)}{\partial c} = \int_0^\infty \tau^2 E_N^1 \left\{ \frac{\partial t}{\partial c} \cdot \frac{\partial}{\partial t} \int_0^t \left[(e_1' Me_1/\tau^2)(g-h) + (e_1' Ce_1/\tau^2)(T+g) \right] \right.$$

$$\left. \Big/ \left[(T+g)(T+h) \right] f_N(e_1' Ce_1/\tau^2) d(e_1' Ce_1/\tau^2) \right\} f(\tau) d\tau$$

$$= \int_0^\infty \tau^2 E_N^1 \left\{ m \left[e_1' Me_1/\tau^2 \right]^2 f \left[mc(e_1' Me_1/\tau^2)/v \right] \Big/ \left[v^2 (T+g)(T+h) \right] \right.$$

$$\left. \cdot \left[v(g-h) + (T+g)mc \right] \right\} f(\tau) d\tau .$$

To obtain this we have used the results that first, $\frac{\partial}{\partial t} \int_0^t F(x)dx = F(t)$ as $f_N(e_1' Ce_1/\tau^2)$ is continuous over the range of integration, and secondly, that $\frac{\partial t}{\partial c} = m(e_1' Me_1/\tau^2)/v$. So, a sufficient condition for $\partial \text{bias}(\hat{\sigma}^2)/\partial c$ to be zero is for $v(g-h)+(T+g)mc=0$, i.e. $c^* = \left[v(h-g) \right] / \left[m(T+g) \right]$. It is relatively straightforward to check that generally this solution results in a minimum of the bias function. #

Hence, for the L component estimators $c_L^*=1$; for the ML component estimators $c_{ML}^*=0$; while for the M component estimators $c_M^*=v/(v+2)$. So, if the model is sufficiently mis-specified these then would be the appropriate critical values. For the case illustrated in the figures a critical value of unity corresponds to a nominal size of 33.2% while a size of 36% results in a critical value of $\left[v/(v+2) \right]$.

(f) The numerical evaluations suggest, of the three component estimators considered, that it is preferable when there are no omitted regressors and the errors are normal, in terms of pre-test absolute relative bias, to use the L component estimators when $\alpha\geq0.05$,[1] and the M component estimators when $\alpha=0.01$. It is better to use the L components when ν is small. However, if the model is sufficiently mis-specified then it is preferable to use the ML component estimators and, more specifically, given the aforementioned discussion, to use $\tilde{\sigma}^2_{ML}$.

We now consider the risk functions of the estimators. We define the risk of an estimator $\bar{\sigma}^2$ of σ_e^2 as $\rho(\sigma_e^2,\bar{\sigma}^2)=E(\bar{\sigma}^2-\sigma_e^2)^2=E\left(\bar{\sigma}^2-E(\tau^2)\right)^2$.

Theorem 5.2.2

Under the assumptions of Theorem 5.2.1

$$\rho(\sigma_e^2,\tilde{\sigma}^2) = \left[v(v+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T+g)(T+g-2v)\right.$$
$$\left.-4E(\tau^2)\theta_d(k+g-2)+4\theta_d^2\right]/(T+g)^2\ , \tag{5.2.26}$$

$$\rho(\sigma_e^2,\sigma^{*2}) = \left[(v+m)(v+m+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T+h)\left(T+h-2(v+m)\right)\right.$$
$$\left.-4E(\tau^2)(\theta_n+\theta_d)(k+h-m-2)+4(\theta_n+\theta_d)^2\right]/(T+h)^2\ , \tag{5.2.27}$$

$$\rho(\sigma_e^2,\hat{\sigma}^2) = \left\{\int_0^{\infty}\left[(T+h)^2\left(v(v+2)\tau^4+4(v+2)\tau^2\theta_d+4\theta_d^2\right)+(T+g)^2(T+h)^2\right.\right.$$
$$.\left(E(\tau^2)\right)^2-2(T+h)^2(T+g)^2E(\tau^2)(v\tau^2+2\theta_d)+(g-h)(2T+g+h)$$
$$.\left(v(v+2)\tau^4P_{04}^{d\tau}+4(v+2)\theta_d\tau^2P_{06}^{d\tau}+4\theta_d^2P_{08}^{d\tau}\right)+(T+g)^2\left(m(m+2)\tau^4P_{40}^{d\tau}\right.$$
$$\left.+4(m+2)\theta_n\tau^2P_{60}^{d\tau}+4\theta_n^2P_{80}^{d\tau}\right)+2(T+g)^2\left(mv\tau^4P_{22}^{d\tau}+2m\theta_d\tau^2P_{24}^{d\tau}\right.$$
$$\left.+2v\theta_n\tau^2P_{42}^{d\tau}+4\theta_n\theta_dP_{44}^{d\tau}\right)-2(T+g)(T+h)(g-h)E(\tau^2)\left(v\tau^2P_{02}^{d\tau}+2\theta_dP_{04}^{d\tau}\right)$$

---

[1] This is not surprising, given that $\hat{\sigma}^2_L\to\tilde{\sigma}^2_L$ as $\alpha\to1$, and that $\tilde{\sigma}^2_L$ is unbiased for all λ under least squares estimation when there are no omitted regressors.

$$-2(T+g)^2(T+h)E(\tau^2)\left(m\tau^2 P_{20}^{d\tau}+2\theta_n P_{40}^{d\tau}\right)\Big] f(\tau)d\tau\Big\} \Big/ \left[(T+g)^2(T+h)^2\right] \quad . \qquad (5.2.28)$$

Proof.

$E_N(\tilde{\sigma}^2)=\tau^2(v+2\lambda_{d\tau})/(T+g)$ and $E_N(\tilde{\sigma}^4)=\tau^4\left(v(v+2)+4(v+2)\lambda_{d\tau}+4\lambda_{d\tau}^2\right)/(T+g)^2$ as $\tilde{\sigma}^2=\left(\tau^2/(T+g)\right)(e_2'Me_2/\tau^2)\sim\left(\tau^2/(T+g)\right)\chi^{2\prime}_{v;\lambda_{d\tau}}$ when $e\sim N(0,\tau^2 I_T)$. So,

$$\rho(\sigma_e^2,\tilde{\sigma}^2) = \int_0^\infty \left[E_N(\tilde{\sigma}^4)+\left(E(\tau^2)\right)^2-2E(\tau^2)E_N(\tilde{\sigma}^2)\right]f(\tau)d\tau$$

$$= \Big\{\int_0^\infty \Big[v(v+2)\tau^4+4(v+2)\theta_d\tau^2+4\theta_d^2+\left(E(\tau^2)\right)^2(T+g)^2$$

$$-2E(\tau^2)(T+g)(v\tau^2+2\theta_d)\Big]f(\tau)d\tau\Big\}/(T+g)^2$$

from which (5.2.26) follows. Similarly, $E_N(\sigma^{*2})=\tau^2\left[(v+m)+2(\lambda_{n\tau}+\lambda_{d\tau})\right]/(T+h)$ and $E_N(\sigma^{*4})=\tau^4\left[(v+m)(v+m+2)+4(m+v+2)(\lambda_{n\tau}+\lambda_{d\tau})+4(\lambda_{n\tau}+\lambda_{d\tau})^2\right]/(T+h)^2$ as $\sigma^{*2}=\left(\tau^2/(T+h)\right)\left[(e_2'Me_2+e_1'Ce_1)/\tau^2\right]\sim\left(\tau^2/(T+h)\right)\chi^{2\prime}_{m+v;\lambda_{n\tau}+\lambda_{d\tau}}$ when $e\sim N(0,\tau^2 I_T)$. So,

$$\rho(\sigma_e^2,\sigma^{*2}) = \int_0^\infty \left[E_N(\sigma^{*4})+\left(E(\tau^2)\right)^2-2E(\tau^2)E_N(\sigma^{*2})\right]f(\tau)d\tau$$

$$= \Big\{\int_0^\infty \Big[(v+m)(v+m+2)\tau^4+4(m+v+2)(\theta_n+\theta_d)\tau^2+4(\theta_n+\theta_d)^2$$

$$+\left(E(\tau^2)\right)^2(T+h)^2-2E(\tau^2)(T+h)\left((v+m)\tau^2+2(\theta_n+\theta_d)\right)f(\tau)d\tau\Big]/(T+h)^2\Big\}$$

from which (5.2.27) follows.

Finally, to establish (5.2.28) we write, using (5.2.9),

$$\rho(\sigma_e^2,\hat{\sigma}^2) = E\Big\{\tau^4\Big[(e_1'Me_1/\tau^2)^2(T+h)^2+(T+g)^2(T+h)^2\left(E(\tau^2)\right)^2/\tau^4$$

$$-2(T+h)^2(T+g)\left(E(\tau^2)/\tau^2\right)(e_1'Me_1/\tau^2)+\Big\{(g-h)(2T+g+h)(e_1'Me_1/\tau^2)^2$$

$$+(T+g)^2(e_1'Ce_1/\tau^2)^2+2(T+g)^2(e_1'Me_1/\tau^2)(e_1'Ce_1/\tau^2)-2(T+g)(T+h)(g-h)$$

$$\cdot\left(E(\tau^2)/\tau^2\right)(e_1'Me_1/\tau^2)-2(T+g)^2(T+h)\left(E(\tau^2)/\tau^2\right)(e_1'Ce_1/\tau^2)\Big\}$$

$$\cdot I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\Big]\Big\}\Big/\left[(T+g)^2(T+h)^2\right]$$

$$=\int_0^\infty E_N\left\{\cdot\right\}f(\tau)d\tau. \tag{5.2.29}$$

Using Lemma 1 of Clarke *et al.* (1987a),

$$E_N(e_1'Me_1/\tau^2) = v+2\lambda_{d\tau} ,$$

$$E_N(e_1'Me_1/\tau^2)^2 = v(v+2)+4(v+2)\lambda_{d\tau}+4\lambda_{d\tau}^2 ,$$

$$E_N\left[(e_1'Me_1/\tau^2)I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\right] = vP_{02}^{d\tau}+2\lambda_{d\tau}P_{04}^{d\tau},$$

$$E_N\left[(e_1'Ce_1/\tau^2)I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\right] = mP_{20}^{d\tau}+2\lambda_{n\tau}P_{40}^{d\tau},$$

$$E_N\left[(e_1'Me_1/\tau^2)^2I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\right] = v(v+2)P_{04}^{d\tau}$$

$$+4(v+2)\lambda_{d\tau}P_{06}^{d}+4\lambda_{d\tau}^2P_{08}^{d} ,$$

$$E_N\left[(e_1'Ce_1/\tau^2)^2I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\right] = m(m+2)P_{40}^{d\tau}$$

$$+4(m+2)\lambda_{n\tau}P_{60}^{d\tau}+4\lambda_{n\tau}^2P_{80}^{d\tau} ,$$

$$E_N\left[(e_1'Me_1/\tau^2)(e_1'Ce_1/\tau^2)I_{[0,c]}\left((ve_1'Ce_1/\tau^2)/(me_1'Me_1/\tau^2)\right)\right] = mvP_{22}^{d\tau}$$

$$+2m\lambda_{d\tau}P_{24}^{d\tau}+2v\lambda_{n\tau}P_{42}^{d\tau}+4\lambda_{n\tau}\lambda_{d\tau}P_{44}^{d\tau}.$$

Substituting these expectations into (5.2.29) and noting that $\lambda_{d\tau}=\theta_d/\tau^2$ and $\lambda_{n\tau}=\theta_n/\tau^2$ completes the proof. #

The hypothesis error, δ, enters the risk functions via θ_n, while the specification error is reflected via both θ_n and θ_d. If there are no omitted regressors then $\theta_d=0$ and $\theta_n=\theta$. The risk functions for this particular case are given in Corollary 5.2.4 . The risk functions for two special members of the SSD_N family are given in Corollaries 5.2.5 and 5.2.6 In Corollary 5.2.5 we consider the case of Mt errors while Corollary 5.2.6 presents the risk functions under normal errors. We follow these corollaries with a discussion of the risk functions.

Corollary 5.2.4

If there are no omitted regressors ($Z\gamma=0$) and the regression disturbances are $SSD_N(0,I_T)$ then

$$\rho_0(\sigma_e^2,\tilde{\sigma}^2) = \left[v(v+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T+g)(T+g-2v)\right]/(T+g)^2 , \tag{5.2.30}$$

$$\rho_0(\sigma_e^2,\sigma^{*2}) = \left[(v+m)(v+m+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T+h)\left(T+h-2(v+m)\right)\right.$$

$$\left.-4E(\tau^2)\theta(k+h-m-2)+4\theta^2\right]/(T+h)^2 , \tag{5.2.31}$$

$$\rho_0(\sigma_e^2,\hat{\sigma}^2) = \left\{v(v+2)(T+h)^2E(\tau^4)+\left(E(\tau^2)\right)^2(T+g)(T+h)^2(T+g-2v)\right.$$

$$+\int_0^\infty\left((g-h)(2T+g+h)v(v+2)\tau^4P_{04}^\tau-2(T+g)(T+h)(g-h)E(\tau^2)v\tau^2P_{02}^\tau\right.$$

$$+(T+g)^2\left[m(m+2)\tau^4P_{40}^\tau+4(m+2)\theta\tau^2P_{60}^\tau+4\theta^2P_{80}^\tau\right]+2(T+g)^2\tau^2v(m\tau^2P_{22}^\tau+2\theta P_{42}^\tau)$$

$$\left.\left.-2(T+g)^2(T+h)E(\tau^2)(m\tau^2P_{20}^\tau+2\theta P_{40}^\tau)\right)f(\tau)d\tau\right\}/\left[(T+g)^2(T+h)^2\right]. \tag{5.2.32}$$

Proof.

If $Z\gamma=0$, $\Lambda=0$, $\theta_d=0$, $\theta_n=\theta$, and $P_{ij}^{d\tau}=P_{ij}^\tau$. So, (5.2.30), (5.2.31) and (5.2.32) follow from Theorem 5.2.2. #

Corollary 5.2.5

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $Mt(0,\sigma_e^2 I_T)$, $\sigma_e^2=\nu\sigma^2/(\nu-2)$ and the pre-test is of H_0 in (4.3.2), then for $\nu>4$

$$\rho_{Mt}(\sigma_e^2,\tilde{\sigma}^2) = \sigma^4\Big[2\nu^2 v(v+\nu-2)+\nu^2(\nu-4)(k+g)^2-4\lambda_d\nu(\nu-2)(\nu-4)(k+g-2)$$

$$+4\lambda_d^2(\nu-2)^2(\nu-4)\Big]\Big/\Big[(\nu-2)^2(\nu-4)(T+g)^2\Big] , \tag{5.2.33}$$

$$\rho_{Mt}(\sigma_e^2,\sigma^{*2}) = \sigma^4\Big[\nu^2(m-k-h)^2(\nu-4)+2\nu^2(v+m)(v+m+\nu-2)$$

$$+4(\lambda_n+\lambda_d)\nu(\nu-2)(\nu-4)(m-k-h+2)+4(\lambda_n+\lambda_d)^2(\nu-2)^2(\nu-4)\Big]\Big/$$

$$\Big[(\nu-2)^2(\nu-4)(T+h)^2\Big] , \tag{5.2.34}$$

$$\rho_{Mt}(\sigma_e^2,\hat{\sigma}^2) = \sigma^4\Big\{\nu^2(T+h)^2\Big((\nu-4)(k+g)^2+2v(v+\nu-2)\Big)+4\lambda_d(\nu-2)(\nu-4)(T+h)^2$$

$$.\Big((\nu-2)\lambda d+\nu(2-k-g)\Big)-2(T+g)(T+h)\nu(\nu-4)\Big(v\nu(g-h)P^d_{021}+m\nu(T+g)P^d_{201}$$

$$+2\lambda_d(\nu-2)(g-h)P^d_{042}+2\lambda_n(\nu-2)(T+g)P^d_{402}\Big)+(g-h)(2T+g+h)(\nu-2)$$

$$.\Big(v(v+2)\nu^2P^d_{040}+4(v+2)\lambda_d\nu(\nu-4)P^d_{061}+4\lambda_d^2(\nu-2)(\nu-4)P^d_{082}\Big)+$$

$$(T+g)^2(\nu-2)\Big(m(m+2)\nu^2P^d_{400}+4(m+2)\lambda_n\nu(\nu-4)P^d_{601}+4\lambda_n^2(\nu-2)(\nu-4)P^d_{802}\Big)$$

$$+2(T+g)^2(\nu-2)\Big(mv\nu^2P^d_{220}+2m\lambda_d\nu(\nu-4)P^d_{241}+2v\lambda_n\nu(\nu-4)P^d_{421}$$

$$+4\lambda_n\lambda_d(\nu-2)(\nu-4)P^d_{442}\Big)\Big\}\Big/\Big[(\nu-2)^2(\nu-4)(T+g)^2(T+h)^2\Big] . \tag{5.2.35}$$

If there are no omitted regressors, $Z\gamma=0$, then,

$$\rho_{OMt}(\sigma_e^2,\tilde{\sigma}^2) = \nu^2\sigma^4\left[2v(v+\nu-2)+(\nu-4)(k+g)^2\right]/\left[(\nu-2)^2(\nu-4)(T+g)^2\right], \qquad (5.2.36)$$

$$\rho_{OMt}(\sigma_e^2,\sigma^{*2}) = \sigma^4\left[\nu^2(m-k-h)^2(\nu-4)+2\nu^2(v+m)(v+m+\nu-2)+4\lambda\nu(\nu-2)(\nu-4)\right.$$
$$\left..(m-k-h+2)+4\lambda^2(\nu-2)^2(\nu-4)\right]/\left[(\nu-2)^2(\nu-4)(T+h)^2\right], \qquad (5.2.37)$$

$$\rho_{OMt}(\sigma_e^2,\hat{\sigma}^2) = \sigma^4\left[\nu^2(T+h)^2\left((\nu-4)(k+g)^2+2v(v+\nu-2)\right)-2(T+g)(T+h)\nu\right.$$
$$.(\nu-4)\left(v\nu(g-h)P_{021}+m\nu(T+g)P_{201}+2\lambda(T+g)(\nu-2)P_{402}\right)$$
$$v(v+2)(g-h)(2T+g+h)\nu^2(\nu-2)P_{040}+(T+g)^2(\nu-2)\left(m(m+2)\nu^2P_{400}\right.$$
$$\left.+4(m+2)\lambda\nu(\nu-4)P_{601}+4\lambda^2(\nu-2)(\nu-4)P_{802}\right)+2v(T+g)^2\nu(\nu-2)$$
$$\left..\left(m\nu P_{220}+2\lambda(\nu-4)P_{421}\right)\right]/\left[(\nu-2)^2(\nu-4)(T+g)^2(T+h)^2\right] . \qquad (5.2.38)$$

<u>Proof</u>.

If τ~IG with scale parameter σ^2 and degrees of freedom parameter ν then $E(\tau^2)=\nu\sigma^2/(\nu-2)$ and $E(\tau^4)=\nu^2\sigma^4/\left((\nu-2)(\nu-4)\right)$, and as, $\lambda_d=\theta_d/\sigma^2$, $\lambda_n=\theta_n/\sigma^2$, $\sigma_e^2=\nu\sigma^2/(\nu-2)$, (5.2.33) and (5.2.34) follow directly from Theorem 1.

Now, using equations (4.3.20) and (4.3.38) we have that $\int_0^\infty P_{ij}^{d\tau}f(\tau)d\tau=P_{ij2}^d$, $\int_0^\infty \tau^2P_{ij}^{d\tau}f(\tau)d\tau=\nu\sigma^2P_{ij1}^d/(\nu-2)$,and using the same procedure, it is straightforward to show that

$$\int_0^\infty \tau^4P_{ij}^{d\tau}f(\tau)d\tau = \nu^2\sigma^4P_{ij0}^d/\left[(\nu-2)(\nu-4)\right] . \qquad (5.2.39)$$

Substituting these into (5.2.28) and rearranging terms completes the proof of (5.2.35). (5.2.36), (5.2.37) and (5.2.38) follow directly from (5.2.33), (5.2.34) and (5.2.35) respectively, as $\lambda_d=0$, $\lambda_n=\lambda$ and $P_{ijn}^d=P_{ijn}$ when $Z\gamma=0$. #

Corollary 5.2.6

If the mis-specified model (4.2.3) is used to represent the true generating process (4.2.1), when the regression disturbances are distributed as $N(0,\sigma^2 I_T)$, and the pre-test is of H_0 in (4.3.2), then

$$\rho_N(\sigma^2,\tilde{\sigma}^2) = \sigma^4\left[2(v+4\lambda_d)+(k+g-2\lambda_d)^2\right]/(T+g)^2 , \tag{5.2.40}$$

$$\rho_N(\sigma^2,\sigma^{*2}) = \sigma^4\left[2\left(v+m+4(\lambda_n+\lambda_d)\right)+\left(m-k-h+2(\lambda_n+\lambda_d)\right)^2\right]/(T+h)^2, \tag{5.2.41}$$

$$\begin{aligned}\rho_N(\sigma^2,\hat{\sigma}^2) = \sigma^4\Big[&(T+h)^2\left(2(v+4\lambda_d)+(k+g-2\lambda_d)^2\right)-2(T+g)(T+h)\Big(v(g-h)P^d_{02}\\ &+m(T+g)P^d_{20}+2\lambda_d(g-h)P^d_{04}+2\lambda_n(T+g)P^d_{40}\Big)+(g-h)(2T+g+h)\Big(v(v+2)P^d_{04}\\ &+4(v+2)\lambda_d P^d_{06}+4\lambda_d^2 P^d_{08}\Big)+(T+g)^2\Big(m(m+2)P^d_{40}+4(m+2)\lambda_n P^d_{60}+4\lambda_n^2 P^d_{80}\Big)\\ &+2(T+g)^2\Big(mvP^d_{22}+2m\lambda_d P^d_{24}+2v\lambda_n P^d_{42}+4\lambda_n\lambda_d P^d_{44}\Big)\Big]/\left[(T+g)^2(T+h)^2\right].\end{aligned} \tag{5.2.42}$$

If there are no omitted regressors, $Z\gamma=0$, then

$$\rho_{ON}(\sigma^2,\tilde{\sigma}^2) = \sigma^4\left(2v+(k+g)^2\right)/(T+g)^2 , \tag{5.2.43}$$

$$\rho_{ON}(\sigma^2,\sigma^{*2}) = \sigma^4\left[2(v+m+4\lambda)+(m-k-h+2\lambda)^2\right]/(T+h)^2 , \tag{5.2.44}$$

$$\begin{aligned}\rho_{ON}(\sigma^2,\hat{\sigma}^2) = \sigma^4\Big[&(T+h)^2\left(2v+(k+g)^2\right)+(g-h)(2T+g+h)v(v+2)P_{04}\\ &-2(T+g)(T+h)\Big(v(g-h)P_{02}+m(T+g)P_{20}+2\lambda(T+g)P_{40}\Big)+(T+g)^2\Big(m(m+2)P_{40}\\ &+4(m+2)\lambda P_{60}+4\lambda^2 P_{80}\Big)+2(T+g)^2 v\Big(mP_{22}+2\lambda P_{42}\Big)\Big]/\left[(T+g)^2(T+h)^2\right] .\end{aligned} \tag{5.2.45}$$

Proof.

These expressions are obtained from Corollary 5.2.5 as $e\sim N(0,\sigma^2 I_T)$ when $\nu=\infty$. Then, $\lim_{\nu\to\infty}\sigma_e^2=\sigma^2$, $\lim_{\nu\to\infty}P^d_{ijn}=\Pr.\left[F''_{(m+i,v+j;\lambda_n\lambda_d)}\le\left(cm(v+j)\right)/\left(v(m+i)\right)\right]$

$=P^{d}_{ij}$ and $\lim_{\nu\to\infty} P_{ijn} = \Pr.\left[F'_{(m+i,v+j;\lambda}\le\left(cm(v+j)\right)/\left(v(m+i)\right)\right]=P_{ij}$. #

Remarks.

(i) $\rho_{ON}(\sigma^2,\tilde{\sigma}^2)$, $\rho_{ON}(\sigma^2,\sigma^{*2})$ and $\rho_{ON}(\sigma^2,\hat{\sigma}^2)$ equal the functions derived by Clarke *et al.* (1987b), while $\rho_N(\sigma^2,\tilde{\sigma}^2_{ML})$, $\rho_N(\sigma^2,\sigma^{*2}_{ML})$, and $\rho_N(\sigma^2,\hat{\sigma}^2_{ML})$ are those given by Giles and Clarke (1989). From these last expressions we can easily derive the results of Clarke *et al.* (1987a).

(ii) When $\alpha=1$, $c=0$, $P^{d\tau}_{ij}=0$, we reject H_0 and $\rho(\sigma^2_e,\hat{\sigma}^2)=\rho(\sigma^2_e,\tilde{\sigma}^2)$. Conversely, when $\alpha=0$, $c=\infty$, $P^{d\tau}_{ij}=1$, we accept H_0 and $\rho(\sigma^2_e,\hat{\sigma}^2)=\rho(\sigma^2_e,\sigma^{*2})$.

We have derived the risk functions for a family of estimators. Using some of the above results Appendix 5.1 gives the risk functions for the L, the ML, and the M estimators. In the next two sections we discuss the risk functions given here and those contained in Appendix 5.1 .

5.3 Comparisons of the Risk Functions when the Regressors are Correctly Specified

In this section we compare the risk functions of $\tilde{\sigma}^2$, σ^{*2}, and $\hat{\sigma}^2$ - the unrestricted, the restricted, and the pre-test estimators of the error variance, σ^2_e - when there are no omitted regressors. To aid this discussion, we have numerically evaluated the risk expressions, assuming Mt errors, of the L, the ML, and the M estimators, given in the special cases of Corollaries 5.2.5 and 5.2.6 in Appendix 5.1 for various choices of ν, α, m, k, and v (and hence, T) as functions of λ.[2]

We numerically evaluated the risk expressions for the same values of the arguments that we used to evaluate the bias functions. As it is impossible to include all of the results, a sample of them is given in Tables A5.2.1 to A5.2.6 of Appendix 5.2 of this chapter.[3] Figures 5.3.1 to 5.3.12 graph some of the esults of Tables A5.2.2, A5.2.4, and A5.2.6; these tables relate to the case of v=30, k=5 and m=3. We consider the relative risk functions of $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ in Figures 5.3.1 to 5.3.4; $\tilde{\sigma}_{ML}^2$, σ_{ML}^{*2}, and $\hat{\sigma}_{ML}^2$ in Figures 5.3.5 to 5.3.8; and $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ in Figures 5.3.9 to 5.3.12. For each of these sets of diagrams we give the relative risk functions for ν=5, 10, 100 and ∞.[4] A legend of the line types associated with each of the estimators follows. The relative risk functions of $\tilde{\sigma}_i^2$ and $\hat{\sigma}_i^2$ (c=1 or c=v/(v+2)) in the diagrams have the same line type, so we have identified the risk function of $\tilde{\sigma}_i^2$ by an arrow and a label, i=L,ML,M.

Though we have only included a subset of the results, the following comments are based on the full details. The remarks (a) to (c) pertain to all members of the SSD_N family, while the final comments are specific to the Mt numerical evaluations of the relative risk functions.

(a) As the hypothesis error grows and θ approaches infinity, the risk of the restricted estimator is unbounded while the risk of the pre-test

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[2] The following discussion is in terms of the risks of the estimators. For the numerical evaluations, to eliminate the scale parameter σ^2, we consider risk relative to σ^4 and parameterise with respect to λ, rather than with respect to θ. The relative risk of an estimator $\bar{\sigma}^2$ of σ_e^2 is $R(\sigma_e^2,\bar{\sigma}^2)=\rho(\sigma_e^2,\bar{\sigma}^2)/\sigma^4$.

[3] The tables in Appendix 5.2 are given in terms of the case of omitted regressors. The relevant results for this discussion are those for which λ_d=0: then, $\lambda_n=\lambda$.

[4] To enable the features of the risk functions to be discernable, the scales on the diagrams are not identical.

estimator approaches that of the unrestricted estimator. Intuitively, when the prior information is so wrong that δ, and hence θ, is very large, then pre-testing will lead us to do the right thing: to ignore the restrictions.

Legend for Figures 5.3.1 to 5.3.12

$R_0(\sigma_e^2, \tilde{\sigma}_i^2)$	$R_0(\sigma_e^2, \sigma_i^{*2})$	$R_0(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.01$
$R_0(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.05$	$R_0(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.30$	$R_0(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.75$
	$R_0(\sigma_e^2, \hat{\sigma}_i^2)$ $c = 1$ or $v/(v+2)$	

(b) The risk function of $\tilde{\sigma}^2$ is independent of θ; that of the restricted estimator is a second order function in θ; while the risk function of $\hat{\sigma}^2$ is a third order function in θ.

(c) The pre-test risk function has a minimum when $c^* = \left(v(h-g)\right) / \left(m(T+g)\right)$.

The proof of this proposition follows along the same lines as that given for Proposition 5.2.1 . Using the notation introduced there, we have:[5]

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[5] We note that $\rho(\sigma_e^2, \hat{\sigma}^2)$ has a minimum when c=c* even in the presence of specification error in the model.

Proposition 5.3.1

The pre-test risk function has a minimum when $c^* = \left(v(h-g)\right) / \left(m(T+g)\right)$.

Proof.

$$\rho_0(\sigma_e^2,\hat{\sigma}^2) = \left(E(\hat{\sigma}^2)-E(\tau^2)\right)^2$$

$$= E\left[\left(\tilde{\sigma}^2-E(\tau^2)\right)^2\left(1-I_{[0,c]}(u)\right)+\left(\sigma^{*2}-E(\tau^2)\right)^2 I_{[o,c]}(u)\right]$$

$$= \int_0^\infty \tau^4 E_N\left\{\left((e_1'Me_1/\tau^2)/(T+g)-E(\tau^2)/\tau^2\right)^2+\left((e_1'Me_1/\tau^2)(g-h)\right.\right.$$

$$\left.+(e_1'Ce_1/\tau^2)(T+g)\right)/\left((T+g)(T+h)\right)\left[\left((e_1'Me_1/\tau^2)(g-h)+(e_1'Ce_1/\tau^2)(T+g)\right)\right.$$

$$\left.\left./\left((T+g)(T+h)\right)+2\left((e_1'Me_1/\tau^2)/(T+g)-E(\tau^2)/\tau^2\right)\right]I\left((e_1'Ce_1/\tau^2)\le t\right)\right\}f(\tau)d\tau \ ,$$

$$= \int_0^\infty \ \tau^4 E_N^1\left\{\left((e_1'Me_1/\tau^2)/(T+g)-E(\tau^2)/\tau^2\right)^2+\int_0^t \left((e_1'Me_1/\tau^2)(g-h)+(e_1'Ce_1\right.\right.$$

$$\left./\tau^2)(T+g)\right)/\left((T+g)(T+h)\right)\left[\left((e_1'Me_1/\tau^2)(g-h)+(e_1'Ce_1/\tau^2)(T+g)\right)/\left((T+g)\right.\right.$$

$$\left.\left.\left..(T+h)\right)+2\left((e_1'Me_1/\tau^2)(T+g)-E(\tau^2)/\tau^2\right)\right]f_N(e_1'Ce_1/\tau^2)d(e_1'Ce_1/\tau^2)\right\}f(\tau)d\tau.$$

So,

$$\frac{\partial\rho_0(\sigma_e^2,\hat{\sigma}^2)}{\partial c} = \int_0^\infty \tau^4 E_N^1\left\{m(e_1'Me_1/\tau^2)^2/v\right)f_N\left(m(e_1'Me_1/\tau^2)c/v\right)\left((g-h)+mc(T+g)/v\right)$$

$$/\left((T+g)(T+h)\right)\left[\left[\left((e_1'Me_1/\tau^2)(g-h)+mc(e_1'Me_1/\tau^2)(T+g)/v\right)/\left((T+g)(T+h)\right)\right.\right.$$

$$\left.\left.\left.+2\left((e_1'Me_1/\tau^2)(T+g)-E(\tau^2)/\tau^2\right)\right]\right]\right\}f(\tau)d\tau \ . \tag{5.3.1}$$

A sufficient condition for (5.3.1) to be zero is for $(g-h)+mc(T+g)/v=0$, that is, $c^* = \left(v(h-g)\right)/\left(m(T+g)\right)$.[6] #

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[6] It is relatively straightforward to check that this generally results in a minimum, not a maximum.

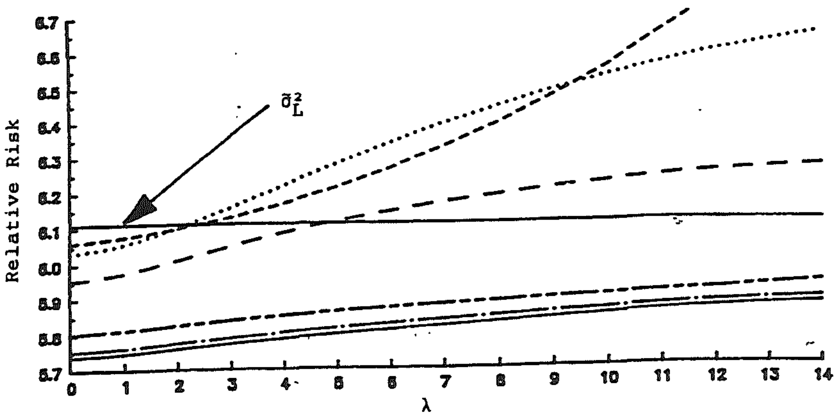

FIGURE 5.3.1: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 5.

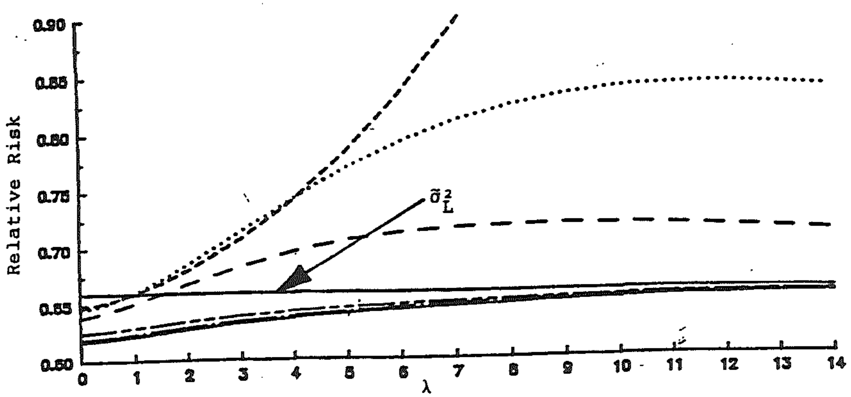

FIGURE 5.3.2: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 10.

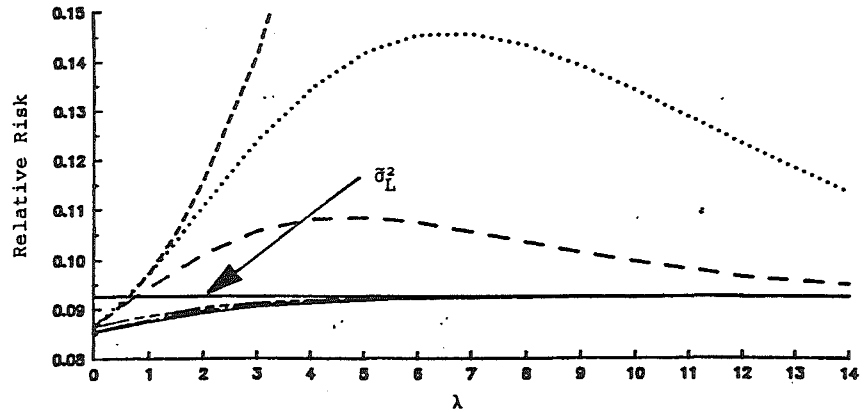

FIGURE 5.3.3: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 100.

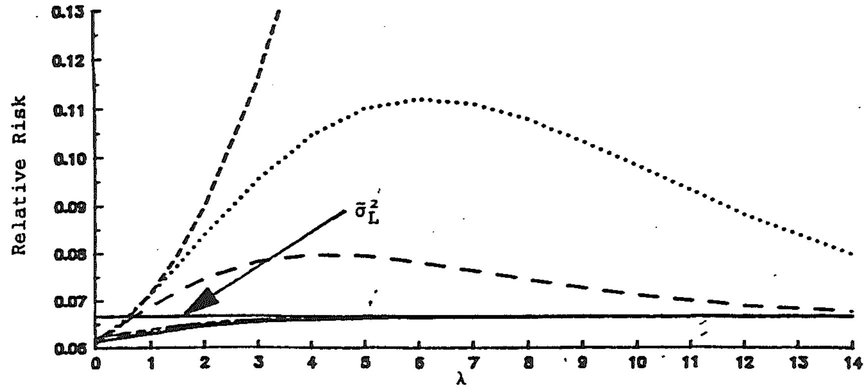

FIGURE 5.3.4: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim N(0, \sigma^2)I_T)$, T = 30, k = 5, m = 3, ν = ∞.

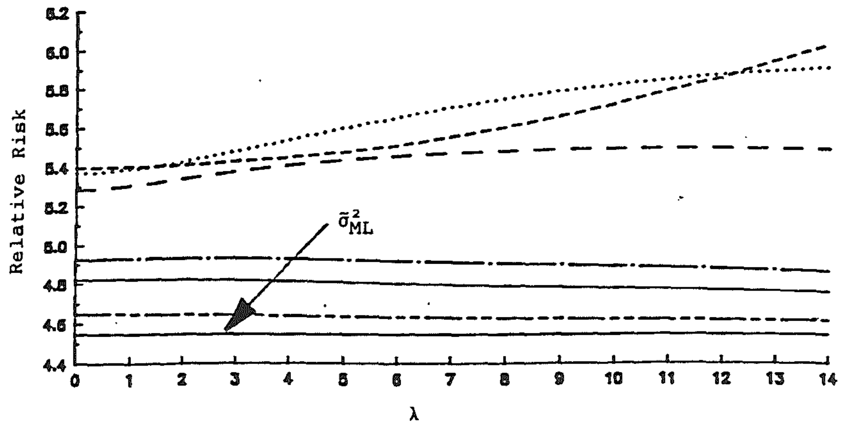

FIGURE 5.3.5: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 5$.

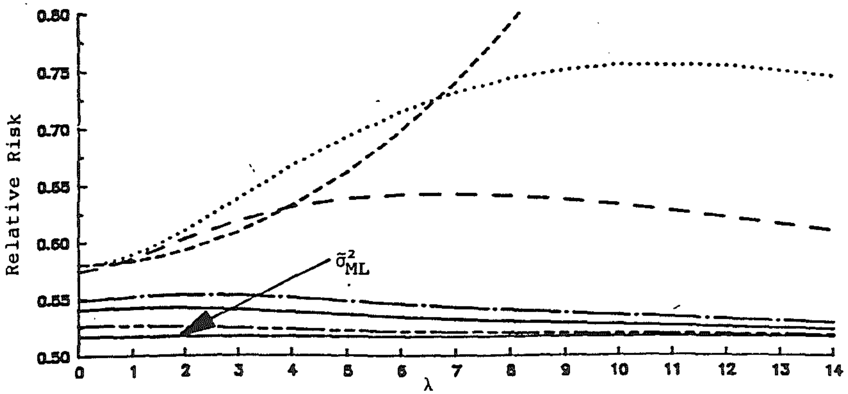

FIGURE 5.3.6: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 10$.

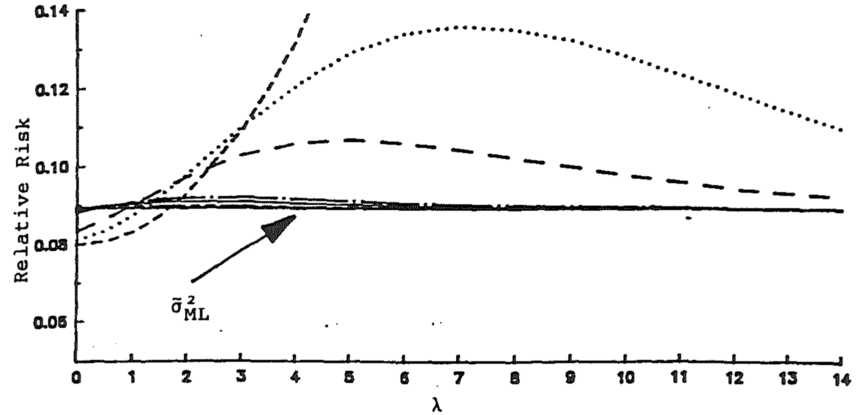

FIGURE 5.3.7: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 100$.

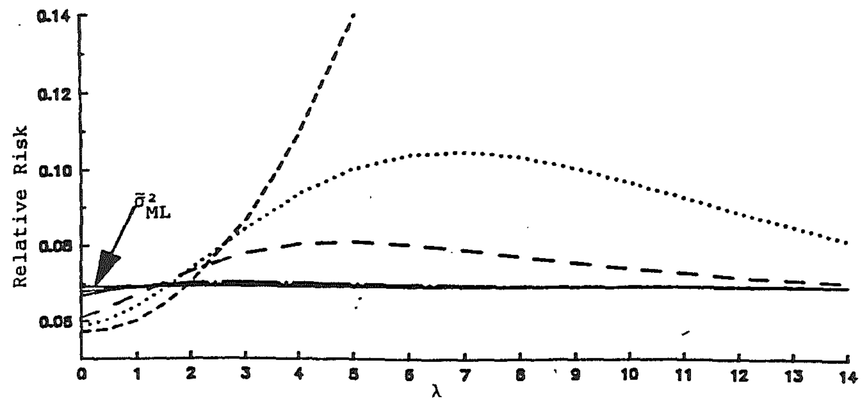

FIGURE 5.3.8: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim N(0,\sigma^2 I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = \infty$.

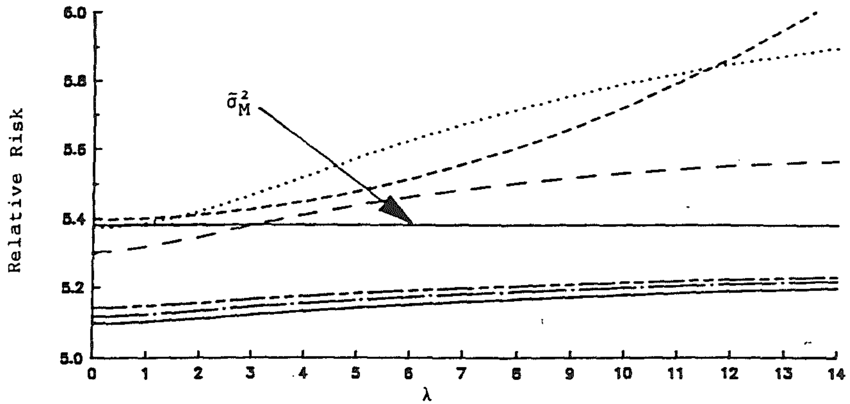

FIGURE 5.3.9: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 5$.

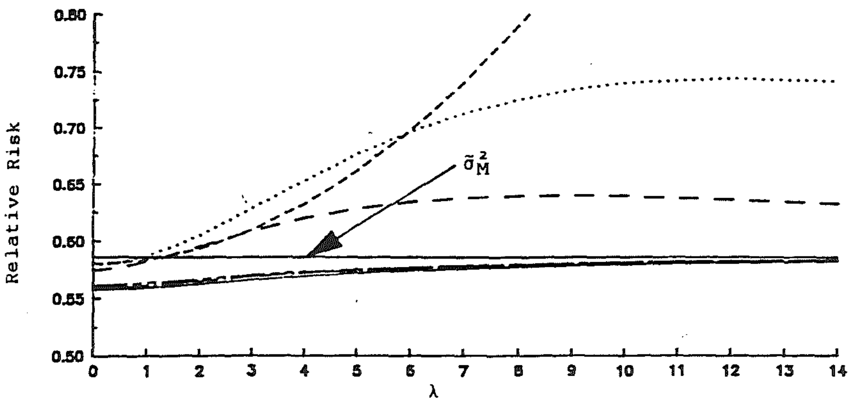

FIGURE 5.3.10: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 10$.

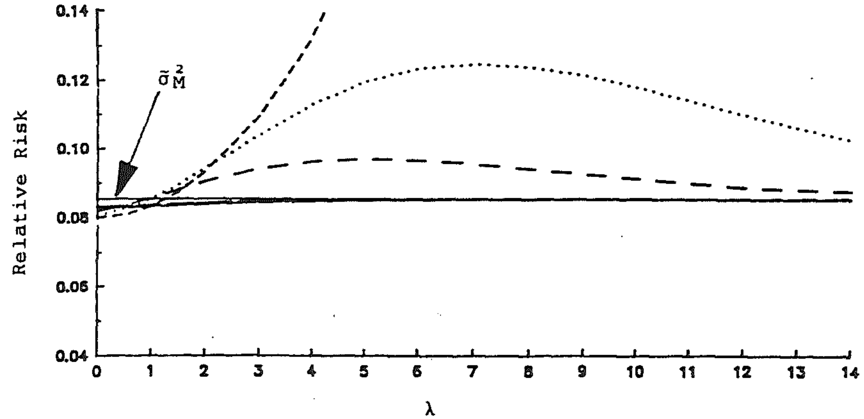

FIGURE 5.3.11: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 100$.

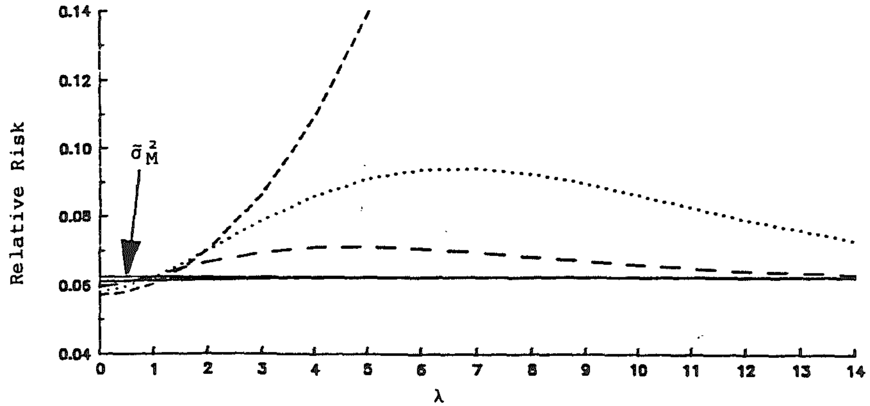

FIGURE 5.3.12: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim N(0, \sigma^2 I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = \infty$.

Note that this is the same critical value which results in a minimum of the pre-test bias function. And, as we noted in that discussion, $c^*_L=1$, $c^*_{ML}=0$, and $c^*_M=v/(v+2)$. So, $\hat{\sigma}^2_{ML}$ ($c\in(0,\infty)$) will never have smaller risk than that of either $\tilde{\sigma}^2_{ML}$ or σ^{*2}_{ML}.[7] When using the ML component estimators it is always preferable to either impose or ignore the restrictions.

However, the risk of the pre-test estimator can be smaller than that of both the unrestricted and the restricted estimators when the L or the M component estimators are used. That is, there will be some part of the θ-range over which pre-testing is preferable. Over this θ-range the smallest risk for the L component estimators is achieved by using $\hat{\sigma}^2_L$ with $c=1$, while for the M component estimators we obtain the smallest risk of $\hat{\sigma}^2_M$ when $c=v/(v+2)$.[8]

Given that $\rho(\sigma^2_e,\tilde{\sigma}^2_L)$ and $\rho(\sigma^2_e,\tilde{\sigma}^2_M)$ are independent of θ this result implies that $\tilde{\sigma}^2_L$ and $\tilde{\sigma}^2_M$ are strictly dominated, respectively, by $\hat{\sigma}^2_L|c=1$, and $\hat{\sigma}^2_M|c=v/(v+2)$. In fact, the family of pre-test estimators, $\hat{\sigma}^2_L$, with $c\in(0,1]$ strictly dominate $\tilde{\sigma}^2_L$, while those $\hat{\sigma}^2_M$ with $c\in(0,v/(v+2)]$ strictly dominate $\tilde{\sigma}^2_M$. The figures illustrate these features.

The result given here for the L component estimators has yet to be noted in the literature, even in the case of normal errors, while that for the M component estimators accords with the results of Ohtani (1988) and

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[7] (5.3.1) is also zero when $c=\infty$.

[8] That the risk of the pre-test estimator can dominate both of its components over any or all of the θ-range may seem counter-intuitive. We may believe, as the pre-test estimator is a weighted sum of its component estimators, that the pre-test estimators risk function should be enveloped by those of its components. This, however, confuses the distinction between a weighted sum of the moments of the component estimators and the moments of their weighted sum. The dominance of the pre-test estimator, for suitably chosen c, over the unrestricted estimator for all ϕ, and over the restricted estimator for some ϕ, also occurs when estimating the error variance after a pre-test for homogeneity. We recall that in this problem ϕ is the ratio of the error variances, $\phi=(\sigma^2_{e_1}/\sigma^2_{e_2})$.

Gelfand and Dey (1988a) under normality. We recall from our discussion in Chapter Two that $\hat{\sigma}^2_M | c=v/(v+2)$ is equivalent to the Stein (1964) estimator. We termed this estimator $\hat{\sigma}^2_S$ in Chapter Two. We have now shown that this result extends to all members of the SSD_N family.

(d) In some situations it is always better to use the unrestricted or the pre-test estimator, even if the restrictions are valid. We consider first the ML component estimators for which, we showed in point (c), it is never preferable to pre-test. So, if the risk of $\tilde{\sigma}^2_{ML}$ is less than that of σ^{*2}_{ML} under H_0 when $\delta=0$, then the optimal strategy is to ignore the restrictions. Using (A5.40) and (A5.41) of Appendix 5.1, this situation arises if

$$\rho\ (\sigma^2_e,\tilde{\sigma}^2_{ML})-\rho(\sigma^2_e,\sigma^{*2}_{ML}|\,\delta=0)$$

$$= \left\{-(m+2v+2)E(\tau^4)+2T\left[E(\tau^2)\right]^2\right\}/T^2 < 0\ , \tag{5.3.2}$$

which depends on m, T, k and, more importantly, $f(\tau)$. For instance, if the errors are normally distributed and k>2, then (5.3.2) is never satisfied: σ^{*2}_{ML} has the smallest risk when the restrictions are valid.

On the other hand, if we have Mt errors then (5.3.2) implies that it is preferable to ignore the prior information, even under H_0, if

$$\nu < 1+2T/(2k-2-m)\ . \tag{5.3.3}$$

For example, (5.3.3) is $\nu<15$ when T=35, k=5, and m=3. That is, we should always use $\tilde{\sigma}^2_{ML}$ if $\nu<15$, as is evident from Table A5.2.4 and from Figures 5.3.5 to 5.3.8. Basically, this result occurs because the increase in the error variance as ν decreases, increases the variance of σ^{*2}_{ML} relatively more than it does that of $\tilde{\sigma}^2_{ML}$.

So, when using the ML component estimators it is always better to either impose or to ignore the linear restrictions. Pre-testing is never the preferred strategy, and in fact, the pre-test estimator can have the highest risk of the estimators considered. These results accord with those found by Clarke *et al.* (1987a) for normally distributed regression disturbances. Then there is a λ-range over which σ^{*2}_{ML} has the smallest risk. We have shown that for small ν, however, we should always ignore the prior information, even if it is valid: $\tilde{\sigma}^2_{ML}$ has the smallest risk for all possible values of the hypothesis error.

Turning now to the L component estimators, we noted in point (c) that there exists a family of pre-test estimators which strictly dominate the unrestricted estimator, $\tilde{\sigma}^2_L$, for all θ, and which also dominate σ^{*2}_L over part of the θ range. We now consider the question of whether or not $\tilde{\sigma}^2_L$ or $\hat{\sigma}^2_L$ can have smaller risk than that of σ^{*2}_L when the prior information is correct; then, $\delta=\theta=\lambda=0$.

From (A5.37), (A5.38), and (A5.39) of Appendix 5.1

$$\rho_0(\sigma^2_e,\sigma^{*2}_L|\delta=0)-\rho_0(\sigma^2_e,\tilde{\sigma}_L) = -2mE(\tau^4)/\left(v(v+m)\right), \tag{5.3.4}$$

and $$\rho_0(\sigma^2_e,\hat{\sigma}^2_L|\delta=0)-\rho_0(\sigma^2_e,\sigma^{*2}_L|\delta=0) = m\Bigg\{2(v+m)E(\tau^4)\left[1-I_x\left(\frac{m}{2};\frac{v+4}{2}\right)\right]$$

$$+v(m+2)E(\tau^4)\left[I_x\left(\frac{m+4}{2};\frac{v}{2}\right)-I_x\left(\frac{m}{2};\frac{v+4}{2}\right)\right]+2mv^2E(\tau^4)\left[I_x\left(\frac{m+2}{2};\frac{v+2}{2}\right)-I_x\left(\frac{m}{2};\frac{v+4}{2}\right)\right]$$

$$+2mv(v+m)\left(E(\tau^2)\right)^2\left[I_x\left(\frac{m}{2};\frac{v+2}{2}\right)-I_x\left(\frac{m+2}{2};\frac{v}{2}\right)\right]\Bigg\}/\left(v(v+m)^2\right), \tag{5.3.5}$$

as $P^{\tau}_{ij}=I_x\left[\left(\frac{m+i}{2};\frac{v+j}{2}\right)\right]$ when $\delta=0$.

(5.3.4) is negative, so, σ^{*2}_L has smaller risk than $\tilde{\sigma}^2_L$ when the restrictions are valid. When using the L component estimators it is better to impose valid restrictions than to ignore them. However, it may still be better to pre-test. It is not possible to make a general statement about the sign of (5.3.5) as

$$I_x\left(\frac{m+4}{2};\frac{v}{2}\right)<I_x\left(\frac{m}{2};\frac{v+4}{2}\right),\ I_x\left(\frac{m+2}{2};\frac{v+2}{2}\right)<I_x\left(\frac{m}{2};\frac{v+4}{2}\right),\ I_x\left(\frac{m+4}{2};\frac{v}{2}\right)<I_x\left(\frac{m}{2};\frac{v+2}{2}\right). \tag{5.3.6}$$

Our numerical evaluations suggest that if the errors are Mt then $\hat{\sigma}^2_L$ can strictly dominate σ^{*2}_L. The diagrams, and tables, indicate that we can expect this for small values of ν but that it may also occur when $\nu=\infty$. (See, for example, the case of c=1 given in Table A5.2.1). The evaluations suggest that, if this event is to occur with normal errors then, we require a small value of m, say, m=1.

So, when using the L component estimators it is never preferable to ignore the restrictions. To minimize risk under quadratic loss the strategy should be to either impose the restrictions or to test their validity prior to estimation. If the errors are normally distributed and if m is greater than unity then our evaluations show that there exists a λ-range over which σ^{*2}_L has the smallest risk of the estimators considered.

For all other λ values the optimal strategy is to pre-test using a critical value of unity. However if m=1 (as it commonly does, for example, in t-tests), then the evaluations suggest that we should always pre-test using a critical value of unity even if the restriction is valid. These findings extend those of Clarke *et al.* (1987b).

The latter result carries over to the case of Mt errors with small ν, regardless of the values of v and m. For relatively small values of ν it is always better to pre-test using a critical value of unity if we wish to minimize risk under quadratic loss. As with the ML component estimators, the relatively higher increase in the variance of σ^{*2}_L as ν decreases is the dominating cause of this result.

We noted in point (c) that there also exists a family of pre-test estimators which strictly dominate $\tilde{\sigma}^2_M$ for all θ, and which dominate σ^{*2}_M for at least part of the θ-range. Of this family of pre-test estimators we have shown that $\hat{\sigma}^2_M | c=v/(v+2)$ has the smallest risk. We now consider whether or not $\tilde{\sigma}^2_M$ or $\hat{\sigma}^2_M$ can have smaller risk than that of σ^{*2}_M when $\delta=\theta=\lambda=0$.

From (A5.43), (A5.44), and (A5.45) of Appendix 5.1

$$\rho_0(\sigma_e^2,\sigma_M^{*2}|\delta=0)-\rho_0(\sigma_e^2,\tilde{\sigma}_M^2) = 2m\left[E(\tau^4)-2\left(E(\tau^2)\right)^2\right]\Big/\left((v+2)(v+m+2)\right) \tag{5.3.7}$$

$$\rho_0(\sigma_e^2,\hat{\sigma}_M^2|\delta=0)-\rho_0(\sigma_e^2,\sigma_M^{*2}|\delta=0) = \Bigg\{2m(v+m+2)\left[2\left(E(\tau^2)\right)^2-E(\tau^4)\right]-mv(2v+m+4)$$

$$.E(\tau^4)I_x\left(\frac{m}{2};\frac{v+4}{2}\right)-2mv(v+2)E(\tau^4)I_x\left(\frac{m+2}{2};\frac{v+2}{2}\right)+m(m+2)(v+2)^2E(\tau^4)I_x\left(\frac{m+4}{2};\frac{v}{2}\right)$$

$$+2m(v+m+2)\left(E(\tau^2)\right)^2\left[vI_x\left(\frac{m}{2};\frac{v+2}{2}\right)-(v+2)I_x\left(\frac{m+2}{2};\frac{v}{2}\right)\right]\Bigg\}\Big/\left((v+2)(v+m+2)^2\right). \tag{5.3.8}$$

(5.3.7) will be negative when

$$E(\tau^4) < 2\left(E(\tau^2)\right)^2 , \tag{5.3.9}$$

which depends on $f(\tau)$. If the errors are normally distributed then $E(\tau^4)=\left(E(\tau^2)\right)^2=\sigma^4$, and (5.3.9) is satisfied. Then it is preferable to impose valid restrictions than to ignore them. However, if the errors are Mt with finite ν then, (5.3.9) will not be satisfied for $\nu<6$. In such cases, one should never use the restrictions, even if they are valid. $\tilde{\sigma}_M^2$ and σ_M^{*2} have equal risk under H_0 when $\nu=6$, and for $\nu>6$ (5.3.9) is satisfied.

It does not seem possible to make a general statement about the sign of (5.3.8). Our numerical evaluations suggest, for a model with normal errors and valid prior information, that it is always better to impose the restrictions than to pre-test. So, when the regression disturbances are normal, there is a λ-range over which σ_M^{*2} has the smallest risk. For all other λ the strategy is to pre-test using c=v/(v+2). These results agree with the findings of Ohtani (1988) and Gelfand and Dey (1988a).

The evaluations also suggest that in most practical situations, when there are small to moderate degrees of freedom, the range of λ over which σ_M^{*2} dominates is relatively small. So, the practical prescription, given that δ (and hence, θ and λ) is unobservable, is to pre-test using a critical value of v/(v+2). This is certainly the optimal strategy when ν is small. Then, the pre-test estimator which utilizes a critical value of v/(v+2) strictly dominates all of the other considered estimators for all possible values of the hypothesis error, δ.

(e) We have already noted in this discussion various effects on the risk functions of changing ν. Two other major changes which result from a decrease in ν are first, a shift upwards of the estimator risk functions and secondly, a decrease in the rate at which the risk of the pre-test estimator approaches that of the unrestricted estimator.

(f) If we assume normality when in fact the distribution of the errors belongs to the wider class of SSD_N, then there is a range of δ (and hence of θ and of λ) over which we would choose to use the wrong estimator.

(g) Of the three component estimators we considered the numerical evaluations suggest that, if one adopted a pre-test strategy and a crude minimax risk criterion then, for normal errors the preferred estimator is $\hat{\sigma}^2_M$. This accords with the results of Clarke *et al.* (1987b). However, if ν is small then it is preferable to use the ML component estimators. So, given the aforementioned discussion, the optimal strategy for small ν is to always ignore the prior information regardless of its validity.

In this section we have compared the risk functions of $\tilde{\sigma}^2$, σ^{*2} and $\hat{\sigma}^2$ when the specified model does not exclude relevant regressors. In particular, we have considered three special cases - the L, the ML and the M component estimators. We should recall that the unrestricted and the restricted ML and M component estimators are the maximum likelihood and minimum mean squared error estimators associated with normally distributed regression disturbances. They do not possess these properties for the broader class of SSD_N disturbances.

Nevertheless, our results suggest that some of the features which we observe under normality do carry over. For instance, regardless of the specific error distribution, $\hat{\sigma}^2_{ML}$ has a minimum risk when c=0, c=1 for $\hat{\sigma}^2_L$, and c=v/(v+2) for $\hat{\sigma}^2_M$. However, our suggestions of which estimator to use for small ν as opposed to that for $\nu=\infty$ differ. We now extend this discussion by allowing for the possibility of omitted regressors.

5.4 Comparisons of the Risk Functions when Relevant Variables are Excluded

In this section we consider the risks of $\tilde{\sigma}^2$, σ^{*2}, and $\hat{\sigma}^2$ when the design matrix is missing variables. As in the previous section, we have undertaken numerical evaluations of the risk expressions given in the special cases of Corollaries 5.2.5 and 5.2.6 in Appendix 5.1 for various choices of ν, α, m, k, and v (and hence, T) as functions of λ_n and λ_d. These give the risk functions of the L, the ML, and the M estimators of σ^2_e when the errors are Mt and normal, respectively. Though the discussion in this section is in terms of the risks of the estimators, in the numerical evaluations we again investigate risk relative to σ^4 and parameterise with respect to λ_n and λ_d, rather than with respect to θ_n and θ_d.

We investigated the same values of ν, α, m, k, and v (and hence, T) as we considered for the evaluations of the bias functions. A sample of the results is given in Tables A5.2.1 to A5.2.9 of Appendix 5.2. Tables A5.2.1 and A5.2.2 give the relative risks of $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ as functions of λ_n for a given value of λ_d. The same information for the ML component estimators, $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$, and for the M component estimators, $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$, is given in Tables A5.2.3 and A5.2.4, and Tables A5.2.5 and A5.2.6, respectively. Figures 5.4.1 to 5.4.12 graph some of the results from Tables A5.2.2, A5.2.4 and A5.2.6.

We are also interested in the risk functions as λ_d varies. Accordingly, the relative risks of the L, the ML, and the M estimators as functions of λ_d for a given value of λ_n, are given in Tables A5.2.7, A5.2.8, and A5.2.9, respectively. Some of the results from these tables are graphed in Figures 5.4.13 to 5.4.24.

When comparing the figures we need to note that the same scales have not been used. Further, the legend associated with the figures follows.

Legend for Figures 5.4.1 to 5.4.24

$R(\sigma_e^2, \tilde{\sigma}_i^2)$	$R(\sigma_e^2, \sigma_i^{*2})$	$R(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.01$
$R(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.05$	$R(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.30$	$R(\sigma_e^2, \hat{\sigma}_i^2)$ $\alpha = 0.75$
	$R(\sigma_e^2, \hat{\sigma}_i^2)$ $c = 1$ or $v/(v+2)$	

As the relative risk functions of $\tilde{\sigma}_i^2$ and $\hat{\sigma}_i^2$ ($c=1$ or $c=v/(v+2)$) in the diagrams have the same line type, we have identified $\rho(\sigma_e^2, \tilde{\sigma}_i^2)$ by an arrow and a label, i=L,ML,M. Though we have only included some of the numerical evaluations, the following comments are based on the complete set:

(a) The risks of $\tilde{\sigma}^2$, σ^{*2}, and $\hat{\sigma}^2$ depend on the specification error, $Z\gamma$, through θ_d, and on the difference between the bias arising from the mis-specification, $\Lambda = RS^{-1}X'Z\gamma$, and the bias arising from the hypothesis error, $\delta = R\beta - r$, through θ_n. For any θ_d, $\rho(\sigma_e^2, \tilde{\sigma}^2)$ is independent of θ_n and hence, it is bounded as $\theta_n \to \infty$, but it is unbounded as $\theta_d \to \infty$. Similarly, $\rho(\sigma_e^2, \hat{\sigma}^2)$ is bounded (by $\rho(\sigma_e^2, \tilde{\sigma}^2)$) as $\theta_n \to \infty$ (given θ_d), but it is unbounded as $\theta_d \to \infty$ (given θ_n). In contrast, $\rho(\sigma_e^2, \sigma^{*2})$ is unbounded as either $\theta_n \to \infty$ or $\theta_d \to \infty$.

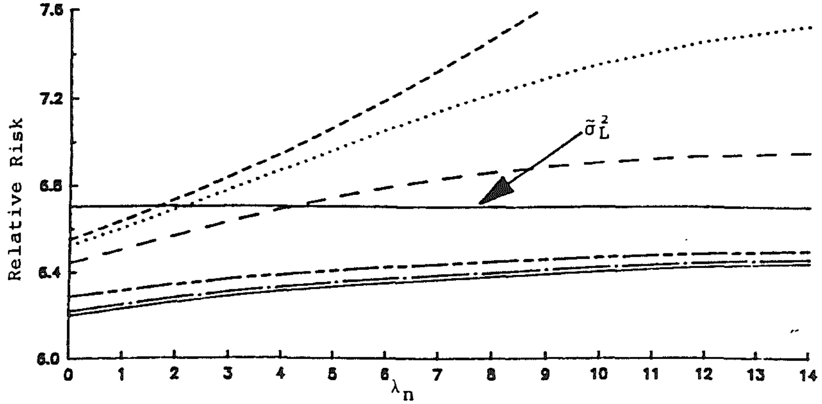

FIGURE 5.4.1: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 5$, and $\lambda_d = 10$.

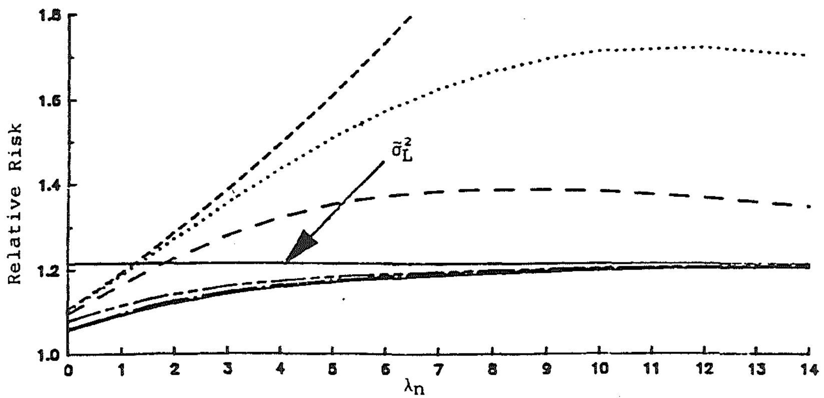

FIGURE 5.4.2: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 10$, and $\lambda_d = 10$.

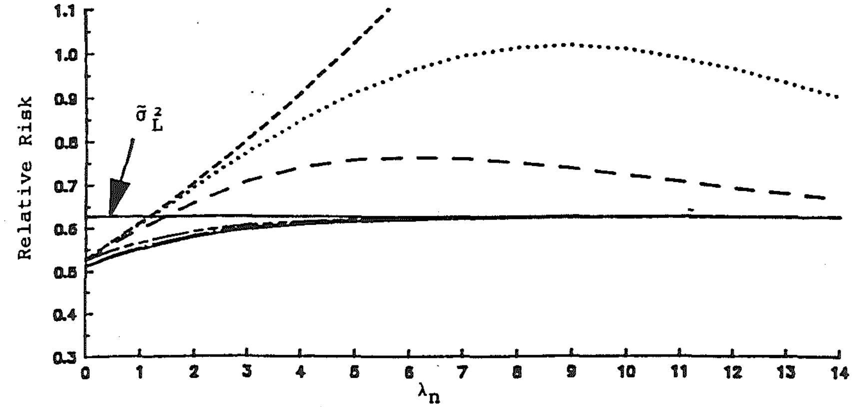

FIGURE 5.4.3: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 100$ and $\lambda_d = 10$.

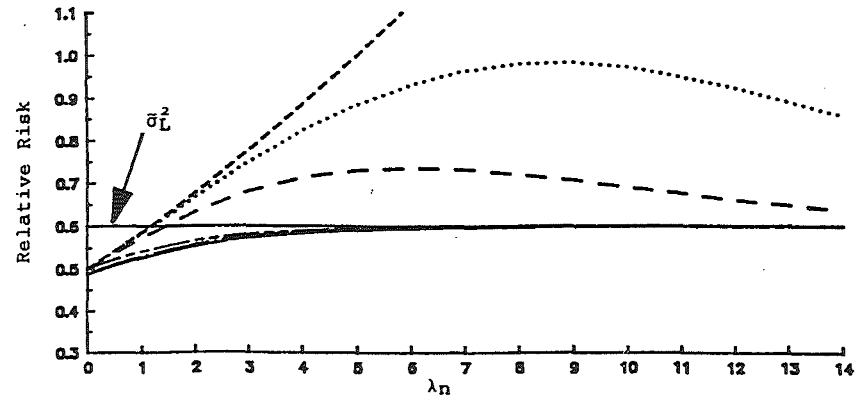

FIGURE 5.4.4: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim N(0, \sigma^2 I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = \infty$, and $\lambda_d = 10$.

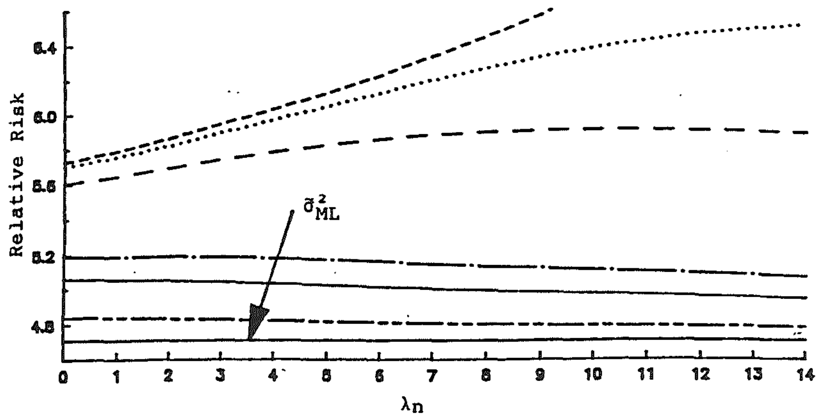

FIGURE 5.4.5: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 5, and λ_d = 10.

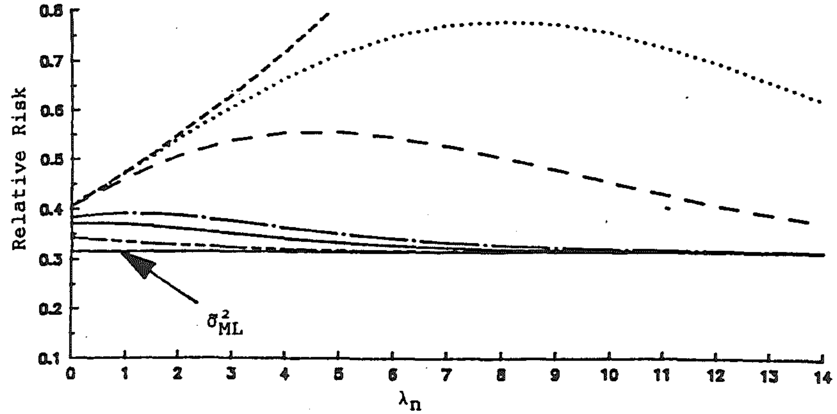

FIGURE 5.4.7: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 100, and λ_d = 10.

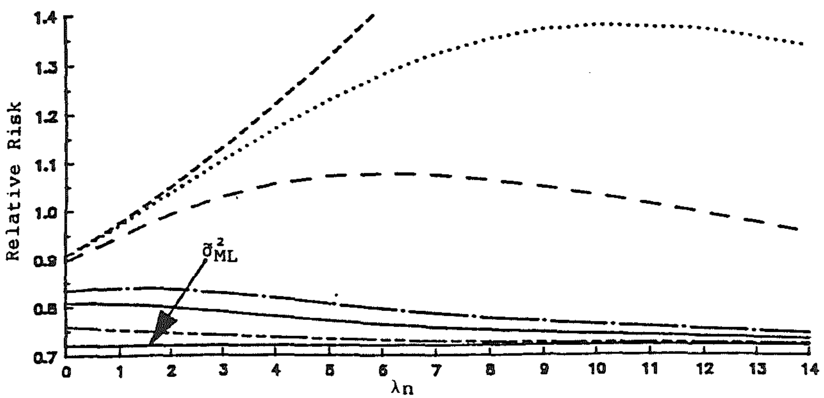

FIGURE 5.4.6: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 30, k = 5, m = 3, ν = 10, and λ_d = 10.

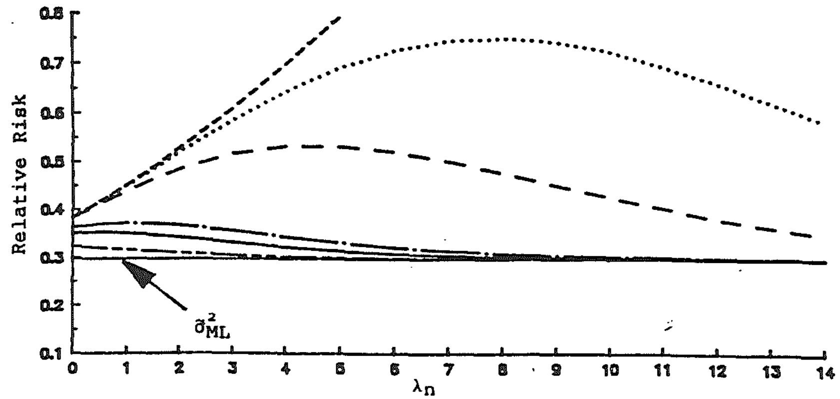

FIGURE 5.4.8: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim N(0, \sigma^2 I_T)$, T = 30, k = 5, m = 3, $\nu = \infty$, and λ_d = 10.

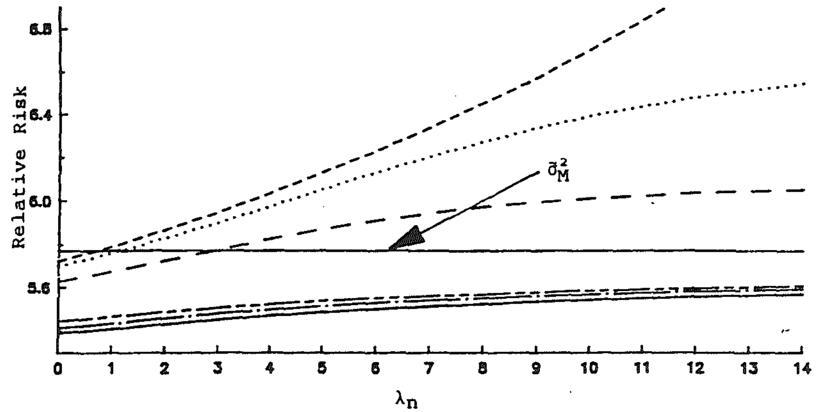

FIGURE 5.4.9: Relative risk functions for $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 5$, and $\lambda_d = 10$.

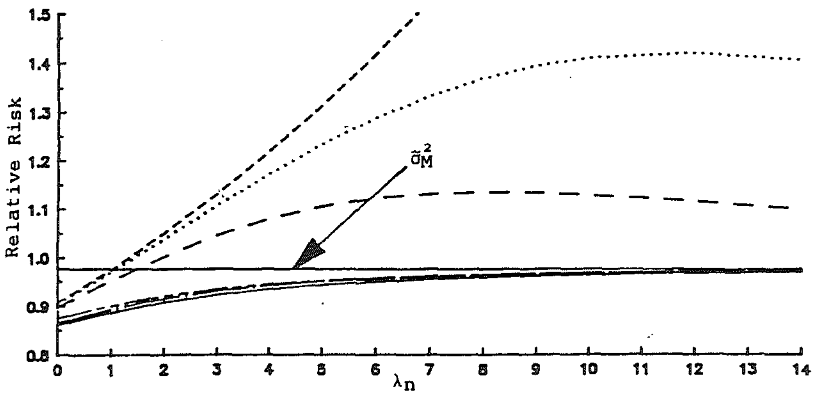

FIGURE 5.4.10: Relative risk functions for $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 10$, and $\lambda_d = 10$.

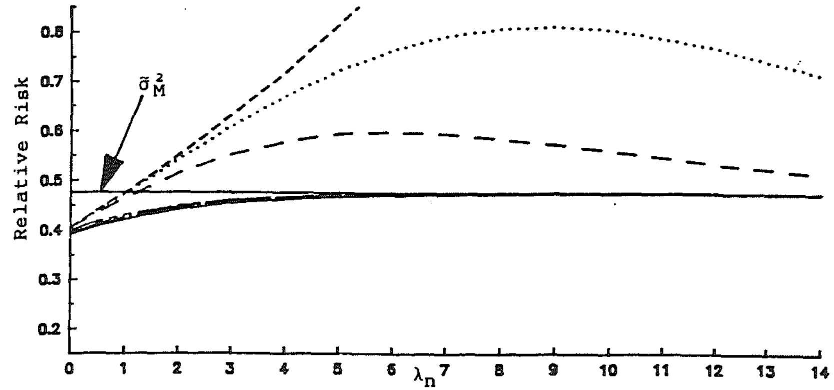

FIGURE 5.4.11: Relative risk functions for $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = 100$, and $\lambda_d = 10$.

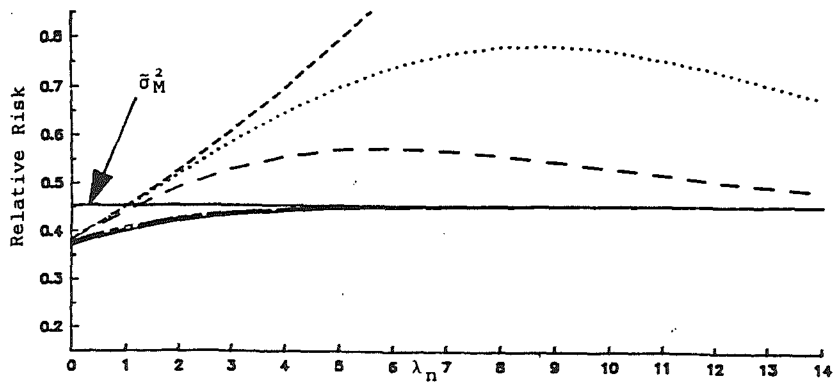

FIGURE 5.4.12: Relative risk functions for $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ when $e \sim N(0,\sigma^2 I_T)$, $T = 30$, $k = 5$, $m = 3$, $\nu = \infty$, and $\lambda_d = 10$.

The risk differences $\left(\rho(\sigma_e^2,\tilde{\sigma}^2)-\rho(\sigma_e^2,\sigma^{*2})\right)$, $\left(\rho(\sigma_e^2,\tilde{\sigma}^2)-\rho(\sigma_e^2,\hat{\sigma}^2)\right)$, $\left(\rho(\sigma_e^2,\sigma^{*2})-\rho(\sigma_e^2,\hat{\sigma}^2)\right)$ are unbounded as $\theta_d\to\infty$, given θ_n. (See Figures 5.4.13 to 5.4.24.) For a given value of θ_d, as $\theta_n\to\infty$ the differences are unbounded except for $\left(\rho(\sigma_e^2,\tilde{\sigma}^2)-\rho(\sigma_e^2,\hat{\sigma}^2)\right)$ which is bounded and is equal to zero. The results given here as $\theta_d\to\infty$, for a given value of θ_n, contrast with those we observed in Chapter Four when estimating $X\beta$. There we found that the risk differences $\left(\rho\left(E(y),Xb\right)-\rho\left(E(y),Xb^*\right)\right)$, $\left(\rho\left(E(y),Xb\right)-\rho\left(E(y),X\hat{b}\right)\right)$, and $\left(\rho\left(E(y),Xb^*\right)-\rho\left(E(y),X\hat{b}\right)\right)$ are all bounded as $\theta_d\to\infty$, given θ_n, and equal to $\left(mE(\tau^2)-2\theta_n\right)$, $\left(mE(\tau^2)-2\theta_n\right)$, and zero respectively.

(b) Proposition 5.3.1 applies to the mis-specified model as well as to the properly specified model. That is, $\rho(\sigma_e^2,\hat{\sigma}^2)$ has a minimum when $c^*=\left(v(h-g)\right)/\left(m(T+g)\right)$, and so, $c_L^*=1$, $c_{ML}^*=0$, $c_M^*=v/(v+2)$. So first, $\hat{\sigma}^2_{ML}$ $(c\in(0,\infty))$ can never dominate both $\tilde{\sigma}^2_{ML}$ and σ^{*2}_{ML} . Secondly, $\hat{\sigma}^2_L$ can dominate both $\tilde{\sigma}^2_L$ and σ^{*2}_L over some or all of the θ_n-range, and when this occurs the smallest risk occurs when c=1. Finally, $\hat{\sigma}^2_M$ can dominate both $\tilde{\sigma}^2_M$ and σ^{*2}_M over some or all of the range of θ_n, and the smallest pre-test risk under this situation occurs when $c=v/(v+2)$. These results hold for all θ_d and for all feasible members of the SSD_N family.

(c) In the last section we compared the risk functions of the estimators when the design matrix was properly specified. One feature we noted was that in some situations it is better to use the unrestricted or the pre-test estimator, even if the restrictions are valid. This characteristic carries over to the mis-specified model. The situation, however, becomes somewhat more complicated as θ_n is no longer zero when H_0 is true ($\delta=0$) unless θ_d is simultaneously zero or X and Z are orthogonal.

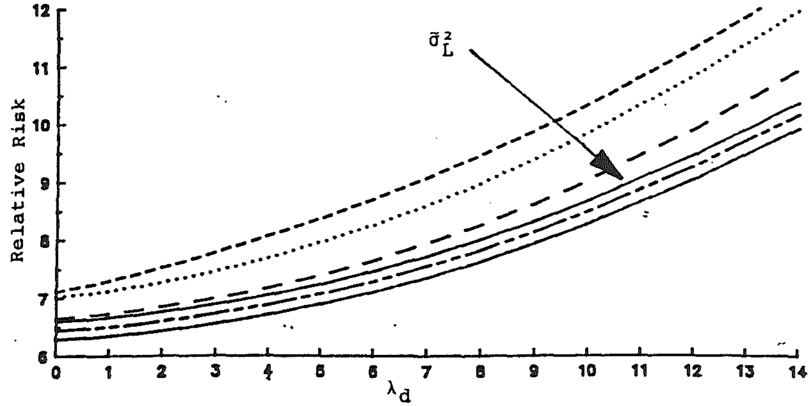

FIGURE 5.4.13: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 5, and λ_n = 5.

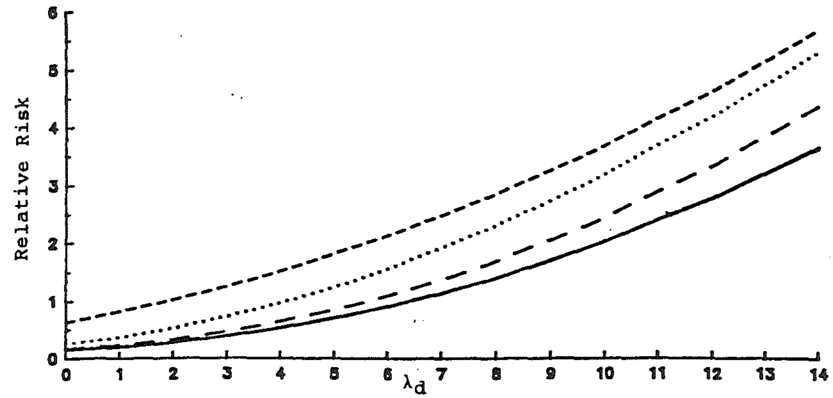

FIGURE 5.4.15: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 100, and λ_n = 5.

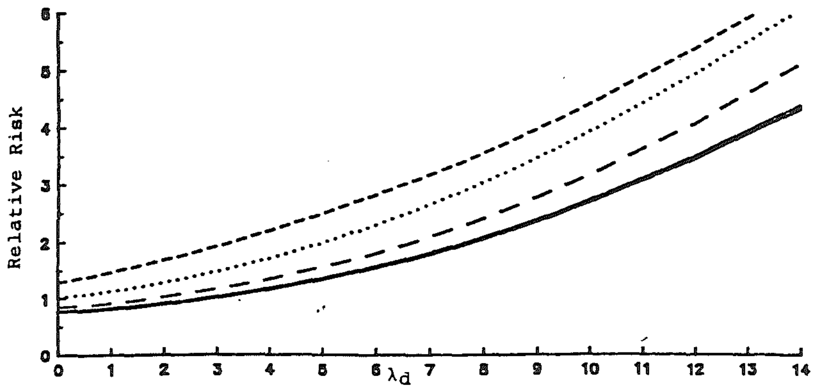

FIGURE 5.4.14: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, T = 20, k = 4, m = 1, ν = 10, and λ_n = 5.

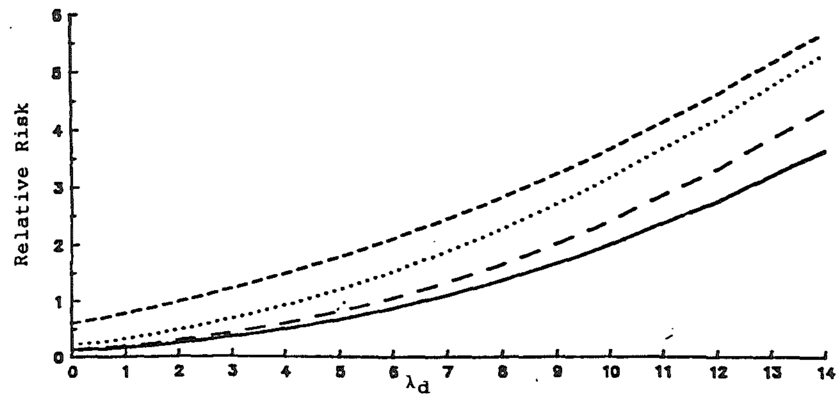

FIGURE 5.4.16: Relative risk functions for $\tilde{\sigma}^2_L$, σ^{*2}_L, and $\hat{\sigma}^2_L$ when $e \sim N(0, \sigma^2 I_T)$, T = 20, k = 4, m = 1, $\nu = \infty$, and λ_n = 5.

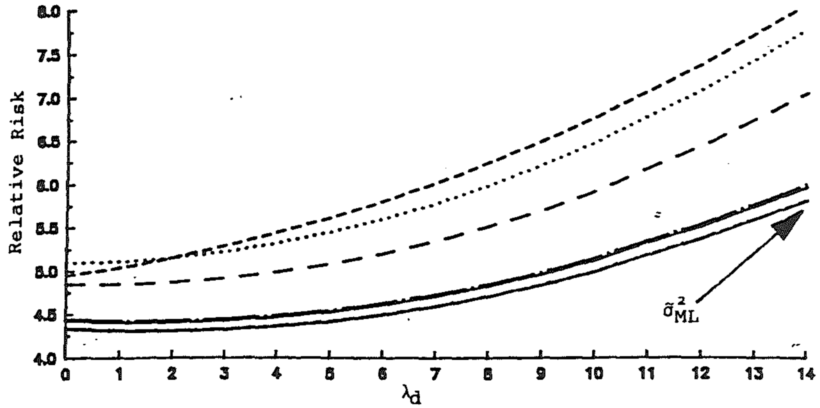

FIGURE 5.4.17: Relative risk functions for $\tilde{\sigma}^2_{ML}$, $\tilde{\sigma}^{*2}_{ML}$, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$, and $\lambda_n = 5$.

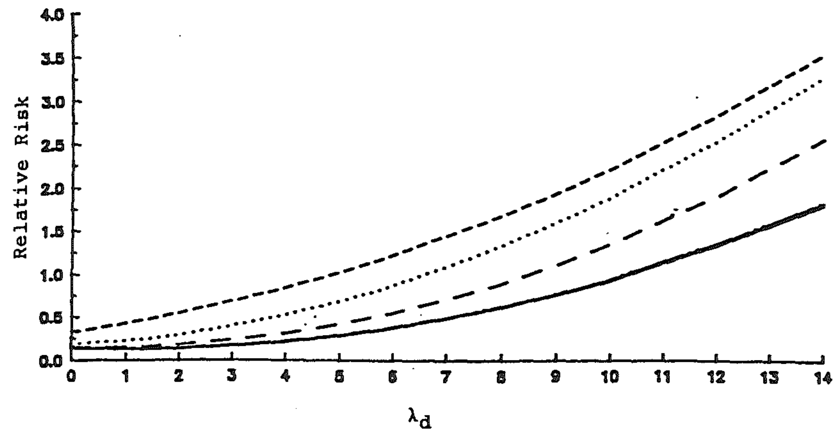

FIGURE 5.4.19: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 100$, and $\lambda_n = 5$.

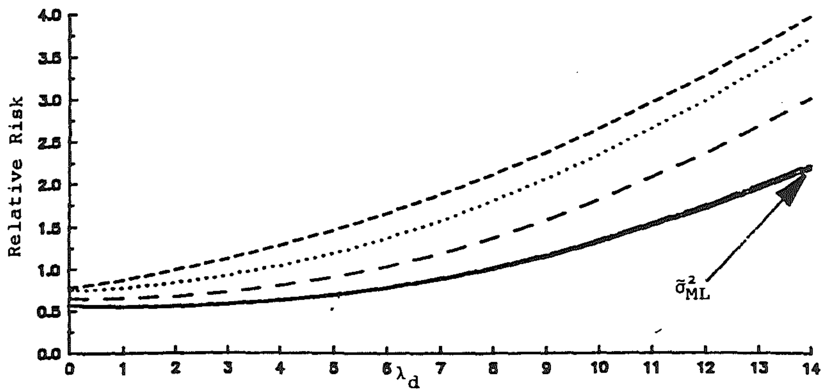

FIGURE 5.4.18: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim Mt(0,\nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 10$, and $\lambda_n = 5$.

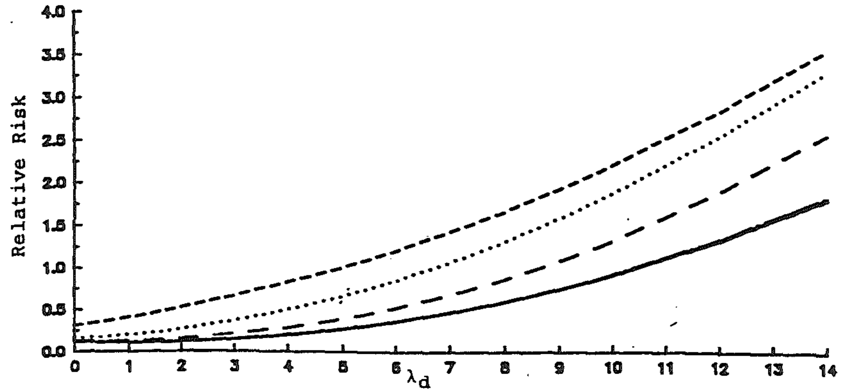

FIGURE 5.4.20: Relative risk functions for $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$ when $e \sim N(0,\sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$, and $\lambda_n = 5$.

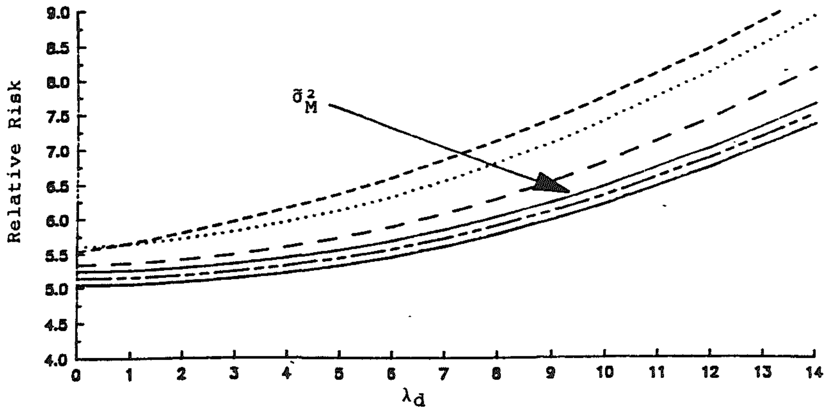

FIGURE 5.4.21: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 5$, and $\lambda_n = 5$.

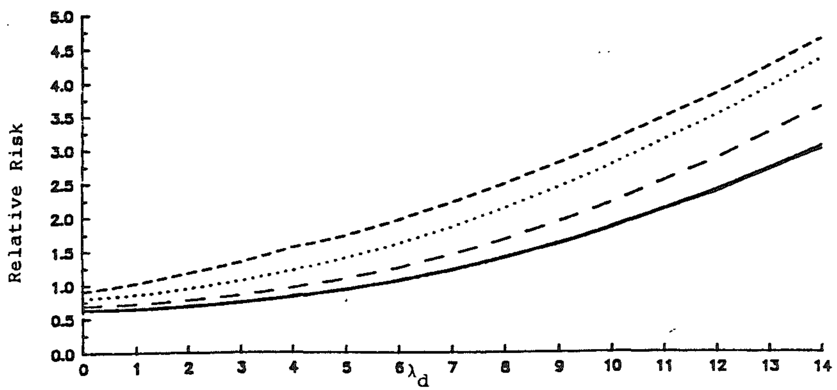

FIGURE 5.4.22: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 10$, and $\lambda_n = 5$.

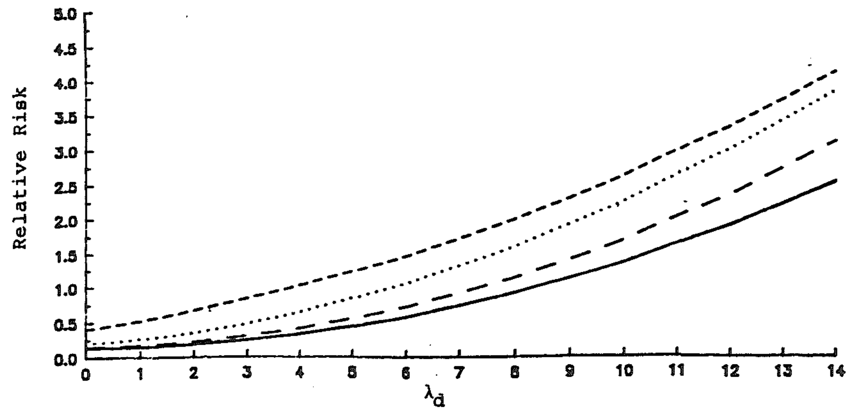

FIGURE 5.4.23: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim Mt(0, \nu\sigma^2/(\nu-2)I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = 100$, and $\lambda_n = 5$.

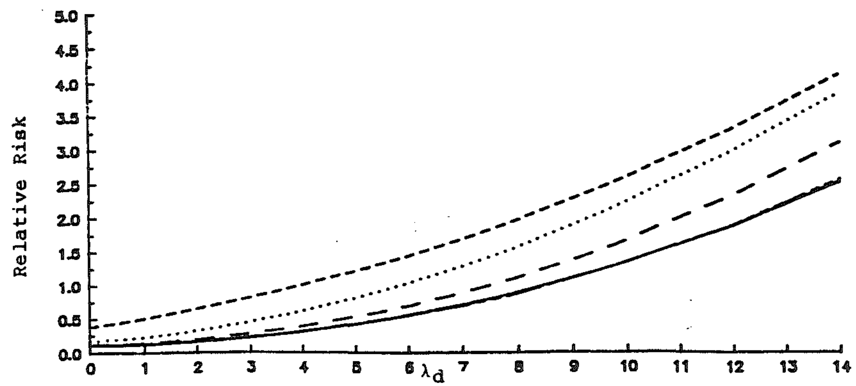

FIGURE 5.4.24: Relative risk functions for $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$ when $e \sim N(0, \sigma^2 I_T)$, $T = 20$, $k = 4$, $m = 1$, $\nu = \infty$, and $\lambda_n = 5$.

We consider, first, the ML component estimators. $\tilde{\sigma}^2_{ML}$ will have smaller risk than σ^{*2}_{ML} if, using (A5.31) and (A5.32),

$$-m(2v+m+2)E(\tau^4)+2Tm\left(E(\tau^2)\right)^2+4E(\tau^2)\left(\theta_n(k-m-2)-m\theta_d\right)-4\theta_n(\theta_n+2\theta_d)<0. \qquad (5.4.1)$$

When $e\sim Mt\left(0,\nu\sigma^2/(\nu-2)I_T\right)$ and $e\sim N(0,\sigma^2 I_T)$ (5.4.1) collapses respectively to

$$m\nu^2\left((2k-m-2)(\nu-4)-2(2v+m+2)\right)+4\nu(\nu-2)(\nu-4)\left(\lambda_n(k-m-2)-m\lambda_d\right)$$

$$-4(\nu-2)^2(\nu-4)\lambda_n(\lambda_n+2\lambda_d)<0 \ , \qquad (5.4.2)$$

and $$m(2k-m-2)+4\left(\lambda_n(k-m-2)-m\lambda_d\right)-4\lambda_n(\lambda_n+2\lambda_d) < 0 \ . \qquad (5.4.3)$$

(5.4.3) is the expression given by Giles and Clarke (1989) for the risk superiority of $\tilde{\sigma}^2_{ML}$ over σ^{*2}_{ML} for normal regression disturbances.

There are situations when (5.4.1), (5.4.2) and (5.4.3) will not be met even if the restrictions are valid. Then, $\theta_n=\theta_{n0}$, $\lambda_n=\lambda_{n0}=\theta_{n0}/\sigma^2$. Figures 5.4.5 to 5.4.8 illustrate this feature when $\lambda_d=10$. In each diagram $\tilde{\sigma}^2_{ML}$ strictly dominates all of the other considered estimators. Conversely, when $\lambda_d=1$ there still exists a range over which σ^{*2}_{ML} is risk superior to $\tilde{\sigma}^2_{ML}$ when $e\sim N(0,\sigma^2 I_T)$ (see Table A5.2.4). Of course, this range may not include the value of θ_{n0}, as θ_n depends on both δ and Λ and the two 'biases' may mitigate each other. So, if the model is suspected to be badly mis-specified through the omission of variables the best strategy is to always ignore the restrictions.

We turn now to the L component estimators. When the model has no omitted regressors we found that the best strategy is either to impose the restrictions or to test their validity prior to estimation using a critical value of unity, but never to totally ignore the prior information. Nevertheless, when comparing only $\tilde{\sigma}^2_L$ and σ^{*2}_L, we found that there exists a range of θ over which $\rho(\sigma^2_e,\tilde{\sigma}^2_L)>\rho(\sigma^2_e,\sigma^*_L)$ (around the neighbourhood of $\theta=\delta=0$) and a range over which the converse inequality holds. When there are omitted variables it is still a feature that neither $\tilde{\sigma}_L^2$ nor σ_L^{*2} can strictly dominate each other. To see this we have, using (A5.28) and (A5.29), that $\tilde{\sigma}_L^2$ is risk superior to σ_L^{*2} if

$$2mvE(\tau^4)+4\theta_d(v+m)^2\left(\theta_d+2E(\tau^2)\right)-4(\theta_n+\theta_d)v^2\left((\theta_n+\theta_d)+2E(\tau^2)\right)<0. \qquad (5.4.4)$$

$\rho(\sigma_e^2,\sigma_L^{*2})<\rho(\sigma_e^2,\tilde{\sigma}_L^2)$ if $\theta_n=0$, as then (5.4.4) equals $2mvE(\tau^4)+4\theta_d m(2v+m)\left(\theta_d+2E(\tau^2)\right)<0$, which is never satisfied. However, from (5.4.4) $\rho(\sigma_e^2,\tilde{\sigma}_L^2)<\rho(\sigma_e^2,\sigma_L^{*2})$ as $\theta_n\to\infty$, given θ_d. So, there is a range of θ_n over which we prefer σ_L^{*2} to $\tilde{\sigma}_L^2$ and a range over which the converse preference occurs. The range of θ_n for which $\rho(\sigma_e^2,\sigma_L^{*2})<\rho(\sigma_e^2,\tilde{\sigma}_L^2)$ may not include θ_{n0}. Then, $\tilde{\sigma}_L^2$ is risk superior to σ_L^{*2} even though the restrictions are valid.

It may still be better to pre-test. $\hat{\sigma}_L^2$ is risk superior to σ_L^{*2} if, using (A5.29) and (A5.30),

$$\begin{aligned}
&2mvE(\tau^4)+4\theta_d(v+m)^2(\theta_d+2E(\tau^2)\Big)-4(\theta_n+\theta_d)v^2\left((\theta_n+\theta_d)+2E(\tau^2)\right)\\
&+\int_0^\infty\Bigg[-m(2v+m)\left(v(v+2)\tau^4P_{04}^{d\tau}+4(v+2)\theta_d\tau^2P_{06}^{d\tau}+4\theta_d^2P_{08}^{d\tau}\right)+v^2\Big(m(m+2)\tau^4P_{40}^{d\tau}\\
&+4(m+2)\theta_n\tau^2P_{60}^{d\tau}+4\theta_n^2P_{80}^{d\tau}\Big)+2v^2\left(mv\tau^4P_{22}^{d\tau}+2m\theta_d\tau^2P_{24}^{d\tau}+2v\theta_n\tau^2P_{42}^{d\tau}+4\theta_n\theta_dP_{44}^{d\tau}\right)\\
&+2mv(v+m)E(\tau^2)\left(v\tau^2P_{02}^{d\tau}+2\theta_dP_{04}^{d\tau}\right)-2v^2(v+m)E(\tau^2)\\
&\cdot\left(m\tau^2P_{20}^{d\tau}+2\theta_nP_{40}^{d\tau}\right)\Bigg]f(\tau)d\tau<0\,. \qquad (5.4.5)
\end{aligned}$$

It does not seem possible to sign (5.4.5). Our numerical evaluations suggest that when $e\sim Mt\left(0,\nu\sigma^2/(\nu-2)I_T\right)$ it is better to pre-test using c=1 for all feasible ν and for $\lambda_d>0$. See, for example, Figures 5.4.1 to 5.4.4. So, we conclude that if we are using the L component estimators in a model with Mt regression disturbances which is mis-specified through the exclusion of regressors, then the optimal strategy is to pre-test using a critical value of unity, whether or not the restrictions are valid. This includes, of course, the case of normal errors.

$\hat{\sigma}^2_M$ is risk superior to σ^{*2}_M if

$$2(v+2)(v+m+2)\left[2(v+m)\left(E(\tau^2)\right)^2-mE(\tau^4)\right]+4\theta_d^2m(2v+m+4)-$$

$$4\theta_n(v+2)^2(\theta_n+2\theta_d)+\int_0^\infty\left[-m(2v+m+4)\left(v(v+2)\tau^4P_{04}^{d\tau}+4(v+2)\theta_d\tau^2P_{06}^{d\tau}+4\theta_d^2P_{08}^{d\tau}\right)\right.$$

$$+(v+2)^2\left(m(m+2)\tau^4P_{40}^{d\tau}+4(m+2)\theta_n\tau^2P_{60}^{d\tau}+4\theta_n^2P_{80}^{d\tau}\right)+2(v+2)^2\left(mv\tau^4P_{22}^{d\tau}\right.$$

$$\left.+2m\theta_d\tau^2P_{24}^{d\tau}+2v\theta_n\tau^2P_{42}^{d\tau}+4\theta_n\theta_dP_{44}^{d\tau}\right)+2m(v+2)(v+m+2)E(\tau^2)\left(v\tau^2P_{02}^{d\tau}+2\theta_dP_{04}^{d\tau}\right)$$

$$\left.-2(v+2)^2(v+m+2)E(\tau^2)\left(m\tau^2P_{20}^{d\tau}+2\theta_nP_{40}^{d\tau}\right)\right]f(\tau)d\tau < 0. \tag{5.4.6}$$

Signing (5.4.6) does not seem feasible. Our numerical evaluations suggest that for small values of ν there exist pre-test estimators which strictly dominate σ^{*2}_M for $\lambda_d>0$, and of these, we know that the pre-test estimator which uses $c=v/(v+2)$ has the smallest risk. See Figures 5.4.9 and 5.4.10.

When $e\sim N(0,\sigma^2I_T)$ the results suggest that for sufficiently serious mis-specification (say, $\lambda_d\geq3$), it is preferable to pre-test using $c=v/(v+2)$ than to impose the restrictions, even if they are valid. In some cases this appears to hold for all $\lambda_d>0$ (see, for example, Table A5.2.5),[9] while in other situations σ^{*2}_M has smaller risk than $\hat{\sigma}^2_M|c=v/(v+2)$ around the neighbourhood of $\theta_n=0$ (see, for example Table A5.2.6 when $\lambda_d=1$). Nevertheless, this is only for a small θ_n range and we find that the difference between the risk of σ^{*2}_M and that of $\hat{\sigma}^2_M|c=v/(v+2)$ is minimal.

Consequently, given that this range may not include the value of θ_n when H_0 is true, our advice if using the M components is to pre-test, using $c=v/(v+2)$, for all members of the Mt family if one believed that the model

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[9] The results suggest that this will occur for small m, say m=1.

could be mis-specified through the omission of relevant variables .

(d) Aside from the features already mentioned in this section, *ceteris paribus*, an increase in λ_d shifts the risk functions upwards;[10] increases the maximum regret of the risk function of $\hat{\sigma}_i^2$ from that of $\tilde{\sigma}_i^2$, i=L,ML,M; decreases the rate at which the risk of the pre-test estimator approaches that of the unrestricted estimator; and increases the λ_n range over which we prefer pre-testing (for all α's) to imposing the restrictions. When $\lambda_d=0$ and α is small, say 1%, there is a region over which pre-testing has the highest risk. Once the model is mis-specified this range decreases and in most cases even pre-testing with this test size is preferable to imposing the restrictions without testing their validity. This can be seen, for example, by comparing the cases of $\alpha=0.01$ in Figures 5.3.1 and 5.4.1, or Figures 5.3.9 and 5.4.9.

(e) Figures 5.4.13 to 5.4.24 illustrate the risk functions of the estimators for $\lambda_d \in [0,14]$ when $\lambda_n=5$. Then, when H_0 is true $\lambda_n=\lambda_{n0}\neq 0$. The diagrams illustrate some of the results given in Tables A5.2.7 to A5.2.9, and in particular, they illustrate that the risk functions, and their risk differences, are unbounded as $\theta_d \to \infty$, given θ_n. There is a risk penalty for mis-specifying the design matrix.

Generally, as λ_n increases, given λ_d, the risk functions of σ_i^{*2} and $\hat{\sigma}_i^2$ shift upwards, i=L,ML,M. Intuitively, there is a risk penalty in imposing false restrictions. $\rho(\sigma_e^2,\tilde{\sigma}^2)$ is independent of λ_n.[11]

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[10] Intuitively, there is a risk penalty for mis-specifying the model.

[11] When using the ML or the M component estimators there may be a decrease in the risk of the restricted estimator as λ_n increases marginally from zero. Over this small range of λ_n the bias of σ^{*2} decreases in absolute terms and so if this outweighs the increase in variance, risk decreases. See Table A5.2.14 when $\lambda_d=0$ and λ_n changes from 0 to 1.

(f) When the design matrix is properly specified and the regression disturbances are normal we suggested that $\hat{\sigma}^2_M$ is the preferred estimator under a crude minimax criterion, while for small values of ν it is preferable to use $\tilde{\sigma}^2_{ML}$ as it strictly dominates $\hat{\sigma}^2_{ML}$ and σ^{*2}_{ML}. When the model is mis-specified through the omission of relevant variables, our numerical evaluations suggest that these conclusions carry over if $\lambda_d<1$. However, if $\lambda_d>1$ then we find that for $\nu>4$ it is preferable to employ the ML component estimators for any given value of α. So, the optimal strategy is to ignore the prior information and to never pre-test.

In this section we have compared the risk functions of $\tilde{\sigma}^2$, σ^{*2} and $\hat{\sigma}^2$ when we may have omitted variables from the design matrix. We have seen that mis-specifying the model impacts, to some degree, more severely on the restricted estimator. There is usually a large range of λ_n over which this estimator has the highest risk, and furthermore, in many cases, it is inadmissible for all possible values of λ_n. So, imposing the prior information, even if it is valid, is not recommended.

5.5 Concluding Remarks

In this chapter we have investigated some finite sample properties of estimators of the error variance in a mis-specified linear regression model after a pre-test for exact linear restrictions. The researcher considers a model which may omit relevant regressors and for which the errors are assumed to be normally distributed when in fact they belong to the wider class of SSD_N. We concentrated our attention on the usual least squares, maximum likelihood, and minimum mean squared error component estimators of the error variance. The latter two, of course, only hold their stated properties for normal regression disturbances, and not for the wider family.

Our analysis has shown that mis-specifying the error distribution can have a substantial impact on the risk functions of the estimators. For instance, if the errors are Mt then, imposing the linear restrictions, even if they are valid, is rarely the optimal strategy, whether or not the design matrix is mis-specified. This contrasts with the results of Clarke *et al.* (1987a,b) who find that for normal regression disturbances there is (usually) always a range over which it is better to impose the prior information without first pre-testing.

The results presented here show that it is better to pre-test using a critical value of one when using the L estimators; to pre-test using a critical value of v/(v+2) when using the M estimators; and to simply ignore the prior information (that is, set c=0) when using the ML estimators, if the errors are Mt with a small degrees of freedom parameter. Moreover, we showed that the risk function of the pre-test estimator has a minimum for these critical values for all feasible members of the SSD_N family.

Rarely do these quoted critical values correspond to test sizes of one and five percent. Accordingly, we must question the *ad hoc* use of such test sizes if one pre-tests and is interested in minimizing the estimator's risk under quadratic loss when estimating the error variance.

In many of the cases investigated we showed that there exists an estimator which strictly dominates the others considered. Then the choice of the optimal test size is obvious. However, the problem of the choice of test size remains for those cases where we have no strictly dominating estimator. This remains for future research.

APPENDIX 5.1

BIAS AND RISK FUNCTIONS OF THE L, ML AND M ESTIMATORS

In this Appendix we present some special cases of the results given in Chapter Five. We consider the L, the ML, and the M unrestricted and restricted estimators and their corresponding pre-test estimators. We use some of the expressions given here for our numerical evaluations. The notation $\hat{\sigma}^2_L$, $\hat{\sigma}^2_{ML}$ and $\hat{\sigma}^2_M$ implies that these are the pre-test estimators of σ^2_e whose component estimators (the researcher believes) are the L, ML and M estimators of σ^2_e.

<u>Theorem 5.2.1</u> (Special Cases)

The following special cases are obtained from Theorem 5.2.1 by substituting in the appropriate values of g and h.

(i) Least Squares (L) components (g=-k, h=(-k+m)):

$$\text{bias}(\tilde{\sigma}^2_L) = 2\theta_d/v , \tag{A5.1}$$

$$\text{bias}(\sigma^{*2}_L) = 2(\theta_n+\theta_d)/(v+m) , \tag{A5.2}$$

$$\text{bias}(\hat{\sigma}^2_L) = \left[2\theta_d(v+m)+mv\int_0^\infty \tau^2\left(P^{d\tau}_{20}-P^{d\tau}_{02}\right)f(\tau)d\tau-2\theta_d m\int_0^\infty P^{d\tau}_{04}f(\tau)d\tau\right.$$
$$\left.+2\theta_n v\int_0^\infty P^{d\tau}_{40}f(\tau)d\tau\right]\Big/\Big(v(v+m)\Big) . \tag{A5.3}$$

(ii) Maximum Likelihood (ML) Components (g=h=0):

$$\text{bias}(\tilde{\sigma}^2_{ML}) = \left(2\theta_d-kE(\tau^2)\right)/T , \tag{A5.4}$$

$$\text{bias}(\sigma^{*2}_{ML}) = \left(2(\theta_n+\theta_d)+(m-k)E(\tau^2)\right)/T , \tag{A5.5}$$

$$\text{bias}(\hat{\sigma}^2_{ML}) = \left(2\theta_d - kE(\tau^2) + m\int_0^\infty \tau^2 P_{20}^{d\tau} f(\tau)d\tau + 2\theta_n \int_0^\infty P_{40}^{d\tau} f(\tau)d\tau\right)/T. \qquad (A5.6)$$

(iii) Minimum Mean Squared Error (M) Components (g=(-k+2), h=(-k+m+2)):

$$\text{bias}(\tilde{\sigma}^2_M) = 2\left(\theta_d - E(\tau^2)\right)/(v+2) \ , \qquad (A5.7)$$

$$\text{bias}(\sigma^{*2}_M) = 2\left((\theta_n+\theta_d) - E(\tau^2)\right)/(v+m+2) \ , \qquad (A5.8)$$

$$\text{bias}(\hat{\sigma}^2_M) = \left[2(v+m+2)\left(\theta_d - E(\tau^2)\right) + m\int_0^\infty \tau^2\left((v+2)P_{20}^{d\tau} - vP_{02}^{d\tau}\right)f(\tau)d\tau\right.$$

$$\left. -2m\theta_d \int_0^\infty P_{04}^{d\tau} f(\tau)d\tau + 2(v+2)\theta_n \int_0^\infty P_{40}^{d\tau} f(\tau)d\tau\right] / \left((v+2)(v+m+2)\right) . \qquad (A5.9)$$

<u>Corollary 5.2.2</u> (Special Cases)

The following special cases are obtained from Corollary 5.2.2 by substituting in the appropriate values of g and h.

(i) L Components (g=-k, h=(-k+m)):

$$\text{bias}_{Mt}(\tilde{\sigma}^2_L) = 2\sigma^2\lambda_d/v \ , \qquad (A5.10)$$

$$\text{bias}_{Mt}(\sigma^{*2}_L) = 2\sigma^2(\lambda_n+\lambda_d)/(v+m) \ , \qquad (A5.11)$$

$$\text{bias}_{Mt}(\hat{\sigma}^2_L) = \sigma^2\left[2(\nu-2)\left(\lambda_d(v+m) - m\lambda_d P^d_{042}\right.\right.$$

$$\left.\left. + v\lambda_n P^d_{402}\right) + mv\nu\left(P^d_{201} - P^d_{021}\right)\right] / \left((\nu-2)v(v+m)\right) . \qquad (A5.12)$$

(ii) ML Components (g=h=0):

$$\text{bias}_{Mt}(\tilde{\sigma}^2_{ML}) = \sigma^2\left(2\lambda_d(\nu-2) - \nu k\right)/\left((\nu-2)T\right), \qquad (A5.13)$$

$$\text{bias}_{Mt}(\sigma^{*2}_{ML}) = \sigma^2\left(2(\lambda_n+\lambda_d)(\nu-2) + \nu(m-k)\right)/\left((\nu-2)T\right) \ , \qquad (A5.14)$$

$$\text{bias}_{Mt}(\hat{\sigma}^2_{ML}) = \sigma^2\left[\nu\left(mP^d_{201} - k\right) + 2(\nu-2)\left(\lambda_d + \lambda_n P^d_{402}\right)\right] / \left((\nu-2)T\right) . \qquad (A5.15)$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\text{bias}_{Mt}(\tilde{\sigma}^2_M) = 2\sigma^2\left(\lambda_d(\nu-2) - \nu\right)/\left((\nu-2)(v+2)\right) \ , \qquad (A5.16)$$

$$\text{bias}_{Mt}(\sigma^{*2}_M) = 2\sigma^2\Big((\lambda_n+\lambda_d)(\nu-2)-\nu\Big)\Big/\Big((\nu-2)(v+m+2)\Big) \ , \tag{A5.17}$$

$$\text{bias}_{Mt}(\hat{\sigma}^2_M) = \sigma^2\Big[2\lambda_d(v+m+2)(\nu-2)-2\nu(v+m+2)-m\Big(v\nu P^d_{021}+2\lambda_d(\nu-2)P^d_{042}\Big)$$

$$+(v+2)\Big(m\nu P^d_{201}+2\lambda_n(\nu-2)P^d_{402}\Big)\Big]\Big/\Big((\nu-2)(v+2)(v+m+2)\Big) \ . \tag{A5.18}$$

<u>Corollary 5.2.3</u> (Special Cases)

The following special cases are obtained from Corollary 5.2.3 by substituting in the appropriate values of g and h.

(i) L Components (g=-k, h=(-k+m)):

$$\text{bias}_N(\tilde{\sigma}^2_L) = 2\sigma^2\lambda_d/v \ , \tag{A5.19}$$

$$\text{bias}_N(\sigma^{*2}_L) = 2\sigma^2(\lambda_n+\lambda_d)/(v+m) \ , \tag{A5.20}$$

$$\text{bias}_N(\hat{\sigma}^2_L) = \sigma^2\Big[2\lambda_d(v+m)+mv\Big(P^d_{20}-P^d_{02}\Big)-2m\lambda_d P^d_{04}+2v\lambda_n P^d_{40}\Big] \Big/\Big(v(v+m)\Big). \tag{A5.21}$$

(ii) ML Components (g=h=0):

$$\text{bias}_N(\tilde{\sigma}^2_{ML}) = \sigma^2(2\lambda_d-k)/T \ , \tag{A5.22}$$

$$\text{bias}_N(\sigma^{*2}_{ML}) = \sigma^2\Big(2(\lambda_n+\lambda_d)+(m-k)\Big)/T \ , \tag{A5.23}$$

$$\text{bias}_N(\hat{\sigma}^2_{ML}) = \sigma^2\Big(2\lambda_d-k+mP^d_{20}+2\lambda_n P^d_{40}\Big)/T \ . \tag{A5.24}$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\text{bias}_N(\tilde{\sigma}^2_M) = 2\sigma^2(\lambda_d-1)/(v+2) \ , \tag{A5.25}$$

$$\text{bias}_N(\sigma^{*2}_M) = 2\sigma^2(\lambda_n+\lambda_d-1)/(v+m+2) \ , \tag{A5.26}$$

$$\text{bias}_N(\hat{\sigma}^2_M) = \sigma^2\Big[2(\lambda_d-1)(v+m+2)-m\Big(vP^d_{02}+2\lambda_d P^d_{04}\Big)$$

$$+(v+2)\Big(mP^d_{20}+2\lambda_n P^d_{40}\Big)\Big]\Big/\Big((v+2)(v+m+2)\Big) \ . \tag{A5.27}$$

Theorem 5.2.2 (Special Cases)

The following special cases are obtained from Theorem 5.2.2 by substituting in the appropriate values of g and h.

(i) L Components (g=-k, h=(-k+m)):

$$\rho(\sigma_e^2,\tilde{\sigma}_L^2) = \left[v(v+2)E(\tau^4)-v^2\left(E(\tau^2)\right)^2+4\theta_d\left(\theta_d+2E(\tau^2)\right)\right]/v^2 \ , \qquad \text{(A5.28)}$$

$$\rho(\sigma_e^2,\sigma_L^{*2}) = \left[(v+m)(v+m+2)E(\tau^4)-(v+m)^2\left(E(\tau^2)\right)^2\right.$$
$$\left.+4(\theta_n+\theta_d)\left((\theta_n+\theta_d)+2E(\tau^2)\right)\right]/(v+m)^2 \ , \qquad \text{(A5.29)}$$

$$\rho(\sigma_e^2,\hat{\sigma}_L^2) = \left\{\int_0^\infty\left[(v+m)^2\left(v(v+2)\tau^4+4(v+2)\tau^2\theta_d+4\theta_d^2\right)\right.\right.$$
$$+v^2(v+m)^2\left(E(\tau^2)\right)^2-2v(v+m)^2E(\tau^2)(v\tau^2+2\theta_d)-m(2v+m)\left(v(v+2)\tau^4P^{d\tau}_{04}\right.$$
$$\left.+4(v+2)\theta_d\tau^2P^{d\tau}_{06}+4\theta_d^2P^{d\tau}_{08}\right)+v^2\left(m(m+2)\tau^4P^{d\tau}_{40}+4(m+2)\theta_n\tau^2P^{d\tau}_{60}+4\theta_n^2P^{d\tau}_{80}\right)$$
$$+2v^2\left(mv\tau^4P^{d\tau}_{22}+2m\theta_d\tau^2P^{d\tau}_{24}+2v\theta_n\tau^2P^{d\tau}_{42}+4\theta_n\theta_dP^{d\tau}_{44}\right)+2mv(v+m)E(\tau^2)\left(v\tau^2P^{d\tau}_{02}\right.$$
$$\left.\left.\left.+2\theta_dP^{d\tau}_{04}\right)-2v^2(v+m)E(\tau^2)\left(m\tau^2P^{d\tau}_{20}+2\theta_nP^{d\tau}_{40}\right)\right]f(\tau)d\tau\right\}/\left(v^2(v+m)^2\right). \qquad \text{(A5.30)}$$

(ii) ML Components (g=h=0):

$$\rho(\sigma_e^2,\tilde{\sigma}_{ML}^2) = \left[v(v+2)E(\tau^4)+\left(E(\tau^2)\right)^2T(T-2v)-4E(\tau^2)\theta_d(k-2)+4\theta_d^2\right]/T^2, \qquad \text{(A5.31)}$$

$$\rho(\sigma_e^2,\sigma_{ML}^{*2}) = \left[(v+m)(v+m+2)E(\tau^4)+\left(E(\tau^2)\right)^2T\left(T-2(v+m)\right)\right.$$
$$\left.-4E(\tau^2)(\theta_n+\theta_d)(k-m-2)+4(\theta_n+\theta_d)^2\right]/T^2 \ , \qquad \text{(A5.32)}$$

$$\rho(\sigma_e^2,\hat{\sigma}_{ML}^2) = \left\{\int_0^\infty\left[v(v+2)\tau^4+4(v+2)\tau^2\theta_d+4\theta_d^2+T^2\left(E(\tau^2)\right)^2-2E(\tau^2)T(v\tau^2+2\theta_d)\right.\right.$$
$$+m(m+2)\tau^4P^{d\tau}_{40}+4(m+2)\theta_n\tau^2P^{d\tau}_{60}+4\theta_n^2P^{d\tau}_{80}+2\left(mv\tau^4P^{d\tau}_{22}+2m\theta_d\tau^2P^{d\tau}_{24}+2v\theta_n\tau^2P^{d\tau}_{42}\right.$$

$$+4\theta_n\theta_d P_{44}^{d\tau}\Big) -2TE(\tau^2)\Big(m\tau^2 P_{20}^{d\tau}+2\theta_n P_{40}^{d\tau}\Big)\Big] f(\tau)d\tau\Big\}/T^2 \ . \qquad (A5.33)$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\rho(\sigma_e^2,\tilde{\sigma}_M^2) = \Big[v(v+2)E(\tau^4)-\Big(E(\tau^2)\Big)^2(v+2)(v-2)+4\theta_d^2\Big]/(v+2)^2 \ , \qquad (A5.34)$$

$$\rho(\sigma_e^2,\sigma_M^{*2}) = \Big[(v+m)(v+m+2)E(\tau^4)-\Big(E(\tau^2)\Big)(v+m+2)(v+m-2)$$

$$+4(\theta_n+\theta_d)^2\Big]/(v+m+2)^2 \ , \qquad (A5.35)$$

$$\rho(\sigma_e^2,\hat{\sigma}_M^2) = \Big\{(v+m+2)^2\Big(v(v+2)E(\tau^4)-(v+2)(v-2)\Big(E(\tau^2)\Big)^2+4\theta_d^2\Big)$$

$$+\int_0^\infty\Big[-m(2v+m+4)\Big(v(v+2)\tau^4 P_{04}^{d\tau}+4(v+2)\theta_d\tau^2 P_{06}^{d\tau}+4\theta_d^2 P_{08}^{d\tau}\Big)$$

$$+(v+2)^2\Big(m(m+2)\tau^4 P_{40}^{d\tau}+4(m+2)\theta_n\tau^2 P_{60}^{d\tau}+4\theta_n^2 P_{80}^{d\tau}\Big)+2(v+2)^2\Big(mv\tau^4 P_{22}^{d\tau}$$

$$+2m\theta_d\tau^2 P_{24}^{d\tau}+2v\theta_n\tau^2 P_{42}^{d\tau}+4\theta_n\theta_d P_{44}^{d\tau}\Big)+2m(v+2)(v+m+2)E(\tau^2)\Big(v\tau^2 P_{02}^{d\tau}$$

$$+2\theta_d P_{04}^{d\tau}\Big)-2(v+2)^2(v+m+2)E(\tau^2)\Big(m\tau^2 P_{20}^{d\tau}+2\theta_n P_{40}^{d\tau}\Big)\Big] f(\tau)d\tau\Big\}/$$

$$\Big((v+2)^2(v+m+2)^2\Big) \ . \qquad (A5.36)$$

<u>Corollary 5.2.4</u> (Special Cases)

The following special cases are obtained from Corollary 5.2.4 by substituting in the appropriate values of g and h.

(i) L Components (g=-k, h=(-k+m)):

$$\rho_0(\sigma_e^2,\tilde{\sigma}_L^2) = \Big[(v+2)E(\tau^4)-v\Big(E(\tau^2)\Big)^2\Big]/v \ , \qquad (A5.37)$$

$$\rho_0(\sigma_e^2,\sigma_L^{*2}) = \Big[(v+m)(v+m+2)E(\tau^4)-(v+m)^2\Big(E(\tau^2)\Big)^2$$

$$+4\theta\Big(\theta+2E(\tau^2)\Big)\Big]/(v+m)^2, \qquad (A5.38)$$

$$\rho_0(\sigma_e^2,\hat{\sigma}_L^2) = \Big\{(v+m)^2\Big[(v+2)E(\tau^4)-v\Big(E(\tau^2)\Big)^2\Big]+\int_0^\infty\Big[-m(v+2)(2v+m)\tau^4P_{04}^\tau$$

$$+2mv(v+m)E(\tau^2)\tau^2P_{02}^\tau+v\Big(m(m+2)\tau^4P_{40}^\tau+4(m+2)\theta\tau^2P_{60}^\tau+4\theta^2P_{80}^\tau\Big)$$

$$+2v^2\tau^2\Big(m\tau^2P_{22}^\tau+2\theta P_{42}^\tau\Big)-2v(v+m)E(\tau^2)\Big(m\tau^2P_{20}^\tau$$

$$+2\theta P_{40}^\tau\Big)\Big]f(\tau)d\tau\Big\}/\Big(v(v+m)^2\Big) \quad . \tag{A5.39}$$

(ii) ML Components (g=h=0):

$$\rho_0(\sigma_e^2,\tilde{\sigma}_{ML}^2) = \Big[v(v+2)E(\tau^4)+\Big(E(\tau^2)\Big)^2T(T-2v)\Big]/T^2 \quad , \tag{A5.40}$$

$$\rho_0(\sigma_e^2,\sigma_{ML}^{*2}) = \Big[(v+m)(v+m+2)E(\tau^4)+\Big(E(\tau^2)\Big)^2T\Big(T-2(v+m)\Big)$$

$$-4E(\tau^2)\theta(k-m-2)+4\theta^2\Big]/T^2 \quad , \tag{A5.41}$$

$$\rho_0(\sigma_e^2,\hat{\sigma}_{ML}^2) = \Big\{v(v+2)E(\tau^4)+\Big(E(\tau^2)\Big)^2T(T-2v)+\int_0^\infty\Big[m(m+2)\tau^4P_{40}^\tau$$

$$+4(m+2)\theta\tau^2P_{60}^\tau+4\theta^2P_{80}^\tau+2\tau^2v\Big(m\tau^2P_{22}^\tau+2\theta P_{42}^\tau\Big)$$

$$-2TE(\tau^2)\Big(m\tau^2P_{20}^\tau+2\theta P_{40}^\tau\Big)\Big]f(\tau)d\tau\Big\}/T^2 \quad . \tag{A5.42}$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\rho_0(\sigma_e^2,\tilde{\sigma}_M^2) = \Big[vE(\tau^4)-(v-2)\Big(E(\tau^2)\Big)^2\Big]/(v+2) \quad , \tag{A5.43}$$

$$\rho_0(\sigma_e^2,\sigma_M^{*2}) = \Big[(v+m)(v+m+2)E(\tau^4)-(v+m-2)(v+m+2)\Big(E(\tau^2)\Big)^2+4\theta^2\Big]$$

$$/(v+m+2)^2 \quad , \tag{A5.44}$$

$$\rho_0(\sigma_e^2,\hat{\sigma}_M) = \Big\{(v+2)(v+m+2)^2\Big[vE(\tau^4)-(v-2)\Big(E(\tau^2)\Big)^2\Big]$$

$$+\int_0^\infty\Big[-mv(v+2)(2v+m+4)\tau^4P^\tau_{04}+2mv(v+2)(v+m+2)E(\tau^2)\tau^2P^\tau_{02}$$

$$+(v+2)^2\left(m(m+2)\tau^4P^\tau_{40}+4(m+2)\theta\tau^2P^\tau_{60}+4\theta^2P^\tau_{80}\right)+2v(v+2)^2\tau^2\left(m\tau^2P^\tau_{22}+2\theta P^\tau_{42}\right)$$

$$-2(v+2)^2(v+m+2)E(\tau^2)\left(m\tau^2P^\tau_{20}+2\theta P^\tau_{40}\right)\Big]f(\tau)d\tau\Big\}\Big/\left((v+2)^2(v+m+2)^2\right). \qquad \text{(A5.45)}$$

Corollary 5.2.5 (Special Cases)

The following special cases are obtained from Corollary 5.2.5 by substituting in the appropriate values of g and h.

(i) L Components ($g=-k$, $h=(-k+m)$):

$$\rho_{Mt}(\sigma^2_e,\tilde{\sigma}^2_L) = \sigma^4\Big[2v\nu^2(v+\nu-2)+4\lambda_d(\nu-2)(\nu-4)\left(\lambda_d(\nu-2)+2\nu\right)\Big]\Big/$$

$$\left(v^2(\nu-2)^2(\nu-4)\right), \qquad \text{(A5.46)}$$

$$\rho_{Mt}(\sigma^2_e,\sigma^{*2}_L) = 2\sigma^4\Big[\nu^2(v+m)(v+m+\nu-2)+2(\lambda_n+\lambda_d)(\nu-2)(\nu-4)\Big((\lambda_n+\lambda_d)(\nu-2)$$

$$+2\nu\Big)\Big]\Big/\left((\nu-2)^2(\nu-4)(v+m)^2\right), \qquad \text{(A5.47)}$$

$$\rho_{Mt}(\sigma^2_e,\hat{\sigma}^2_L) = \sigma^4\Big\{2v(v+m)^2\nu^2(v+\nu-2)+2\lambda_d(\nu-2)(\nu-4)(v+m)^2\left((\nu-2)\lambda_d+2\nu\right)$$

$$-2v(v+m)\nu(\nu-4)\Big[mv\nu\left(P^d_{201}-P^d_{021}\right)-2m\lambda_d(\nu-2)P^d_{042}+2v\lambda_n(\nu-2)P^d_{402}\Big]$$

$$-m(m+2v)(\nu-2)\left(v(v+2)\nu^2P^d_{040}+4(v+2)\lambda_d\nu(\nu-4)P^d_{061}+4\lambda^2_d(\nu-2)(\nu-4)P^d_{082}\right)$$

$$+v^2(\nu-2)\left(m(m+2)\nu^2P^d_{400}+4(m+2)\lambda_n\nu(\nu-4)P^d_{601}+4\lambda^2_n(\nu-2)(\nu-4)P^d_{802}\right)$$

$$+2v^2(\nu-2)\Big(mv\nu^2P^d_{220}+2m\lambda_d\nu(\nu-4)P^d_{241}+2v\lambda_n\nu(\nu-4)P^d_{421}$$

$$+4\lambda_n\lambda_d(\nu-2)(\nu-4)P^d_{442}\Big)\Big\}\Big/\left((\nu-2)^2(\nu-4)v^2(v+m)^2\right). \qquad \text{(A5.48)}$$

(ii) ML Components (g=h=0):

$$\rho_{Mt}(\sigma_e^2,\tilde{\sigma}_{ML}^2) = \sigma^4\Big(2\nu^2 v(v+\nu-2)+\nu^2(\nu-4)k^2-4\lambda_d\nu(\nu-2)(\nu-4)(k-2)$$

$$+4\lambda_d^2(\nu-2)^2(\nu-4)\Big)\Big/\Big((\nu-2)^2(\nu-4)T^2\Big) \ , \qquad \text{(A5.49)}$$

$$\rho_{Mt}(\sigma_e^2,\sigma_{ML}^{*2}) = \sigma^4\Big(\nu^2(m-k)^2(\nu-4)+2\nu^2(v+m)(v+m+\nu-2)$$

$$+4(\lambda_n+\lambda_d)\nu(\nu-2)(\nu-4)(m-k+2)+4(\lambda_n+\lambda_d)^2(\nu-2)^2(\nu-4)\Big)$$

$$\Big/\Big((\nu-2)^2(\nu-4)T^2\Big) \ , \qquad \text{(A5.50)}$$

$$\rho_{Mt}(\sigma_e^2,\hat{\sigma}_{ML}^2) = \sigma^4\Big[\nu^2\Big((\nu-4)k^2+2v(v+\nu-2)\Big)+4\lambda_d(\nu-2)(\nu-4)$$

$$\cdot\Big((\nu-2)\lambda_d-\nu(k-2)\Big)-2\nu(\nu-4)T\Big(m\nu P_{201}^d+2\lambda_n(\nu-2)P_{402}^d\Big)+$$

$$(\nu-2)\Big(m(m+2)\nu^2P_{400}^d+4(m+2)\lambda_n\nu(\nu-4)P_{601}^d+4\lambda_n^2(\nu-2)(\nu-4)P_{802}^d\Big)$$

$$+2(\nu-2)\Big(mv\nu^2P_{220}^d+2m\lambda_d\nu(\nu-4)P_{241}^d+2v\lambda_n\nu(\nu-4)P_{421}^d$$

$$+4\lambda_n\lambda_d(\nu-2)(\nu-4)P_{442}^d\Big)\Big]\Big/\Big((\nu-2)^2(\nu-4)T^2\Big) \ . \qquad \text{(A5.51)}$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\rho_{Mt}(\sigma_e^2\ \tilde{\sigma}_M^2) = 2\sigma^4\Big(\nu^2 v(v+\nu-2)+2\nu^2(\nu-4)+2\lambda_d^2(\nu-2)^2(\nu-4)\Big)\Big/$$

$$\Big((\nu-2)^2(\nu-4)(v+2)^2\Big), \qquad \text{(A5.52)}$$

$$\rho_{Mt}(\sigma_e^2,\sigma_M^{*2}) = 2\sigma^4\Big(2\nu^2(\nu-4)+\nu^2(v+m)(v+m+\nu-2)+\ 2(\lambda_n+\lambda_d)^2(\nu-2)^2(\nu-4)\Big)\Big/$$

$$\Big((\nu-2)^2(\nu-4)(v+m+2)^2\Big) \ , \qquad \text{(A5.53)}$$

$$\rho_{Mt}(\sigma_e^2,\hat{\sigma}_M^2) = \sigma^4\Big[2\nu^2(v+m+2)^2\Big(2(\nu-4)+v(v+\nu-2)\Big]+4\lambda_d^2(\nu-2)^2$$

$$.(\nu-4)(v+m+2)^2-2(v+2)(v+m+2)\nu(\nu-4)\Big(-mv\nu P^d_{021}$$

$$+m\nu(v+2)P^d_{201}-2m\lambda_d(\nu-2)P^d_{042}+2\lambda_n(\nu-2)(v+2)P^d_{402}\Big)$$

$$-m(2v+m+4)(\nu-2)\Big(v(v+2)\nu^2P^d_{040}+4(v+2)\lambda_d\nu(\nu-4)P^d_{061}$$

$$+4\lambda_d^2(\nu-2)(\nu-4)P^d_{082}\Big)+(v+2)^2(\nu-2)\Big(m(m+2)\nu^2P^d_{400}+4(m+2)\lambda_n\nu$$

$$.(\nu-4)P^d_{601}+4\lambda_n^2(\nu-2)(\nu-4)P^d_{802}\Big)+2(v+2)^2(\nu-2)\Big(mv\nu^2P^d_{220}$$

$$+2m\lambda_d(\nu-4)P^d_{241}+2v\lambda_n\nu(\nu-4)P^d_{421}+4\lambda_n\lambda_d(\nu-2)(\nu-4)P^d_{442}\Big)\Big]\Big/$$

$$\Big((\nu-2)^2(\nu-4)(v+2)^2(v+m+2)^2\Big) \quad . \tag{A5.54}$$

<u>Corollary 5.2.6</u> (Special Cases)

The following special cases are obtained from Corollary 5.2.6 by substituting in the appropriate values of g and h.

(i) L components (g=-k, h=(-k+m)):

$$\rho_N(\sigma^2,\tilde{\sigma}^2_L) = 2\sigma^4\Big(v+2\lambda_d(\lambda_d+2)\Big)/v^2 \ , \tag{A5.55}$$

$$\rho_N(\sigma^2,\sigma^{*2}_L) = 2\sigma^4\Big(v+m+2(\lambda_n+\lambda_d)(\lambda_n+\lambda_d+2)\Big)/(v+m)^2 \ , \tag{A5.56}$$

$$\rho_N(\sigma^2,\hat{\sigma}^2_L) =\sigma^4\Big\{2(v+m)^2\Big(v+2\lambda_d(\lambda_d+2)\Big)-2v(v+m)\Big[mv\Big(P^d_{20}-P^d_{02}\Big)$$

$$-2\lambda_d mP^d_{04}+2\lambda_n vP^d_{40}\Big]-m(2v+m)\Big(v(v+2)P^d_{04}+4(v+2)\lambda_dP^d_{06}+4\lambda_d^2P^d_{08}\Big)$$

$$+v^2\Big(m(m+2)P^d_{40}+4(m+2)\lambda_nP^d_{60}+4\lambda_n^2P^d_{80}\Big)+2v^2\Big(mvP^d_{22}+2m\lambda_dP^d_{24}$$

$$+2v\lambda_nP^d_{42}+4\lambda_n\lambda_dP^d_{44}\Big)\Big\}\Big/\Big(v^2(v+m)^2\Big) \quad . \tag{A5.57}$$

(ii) ML Components (g=h=0).

$$\rho_N(\sigma^2,\tilde{\sigma}^2_{ML}) = \sigma^4\left(2(v+4\lambda_d)+(k-2\lambda_d)^2\right)/T^2 , \tag{A5.58}$$

$$\rho_N(\sigma^2,\sigma^{*2}_{ML}) = \sigma^4\left[2\left(v+m+4(\lambda_n+\lambda_d)\right)+\left(m-k+2(\lambda_n+\lambda_d)\right)^2\right]/T^2 , \tag{A5.59}$$

$$\rho_N(\sigma^2,\hat{\sigma}^2_{ML}) = \sigma^4\left[2(v+4\lambda_d)+(k-2\lambda_d)^2+2\left(mvP^d_{22}-mTP^d_{20}+2m\lambda_dP^d_{24}+2v\lambda_nP^d_{42}\right.\right.$$

$$\left.\left.-2\lambda_nTP^d_{40}+4\lambda_n\lambda_dP^d_{44}\right)+m(m+2)P^d_{40}+4(m+2)\lambda_nP^d_{60}+4\lambda_n^2P^d_{80}\right]/T^2 . \tag{A5.60}$$

(iii) M Components (g=(-k+2), h=(-k+m+2)):

$$\rho_N(\sigma^2,\tilde{\sigma}^2_M) = 2\sigma^4(v+2+2\lambda_d^2)/(v+2)^2 , \tag{A5.61}$$

$$\rho_N(\sigma^2,\sigma^{*2}_M) = 2\sigma^4\left(v+m+2+2(\lambda_n+\lambda_d)^2\right)/(v+m+2)^2 , \tag{A5.62}$$

$$\rho_N(\sigma^2,\hat{\sigma}^2_M) = \sigma^4\left[2(v+m+2)^2(v+2+2\lambda_d^2)-2(v+2)(v+m+2)\left(-mvP^d_{02}+m(v+2)P^d_{20}\right.\right.$$

$$\left.-2\lambda_dmP^d_{04}+2\lambda_n(v+2)P^d_{40}\right)-m(2v+m+4)\left(v(v+2)P^d_{04}+4(v+2)\lambda_dP^d_{06}+4\lambda_d^2P^d_{08}\right)$$

$$+(v+2)^2\left(m(m+2)P^d_{40}+4(m+2)\lambda_nP^d_{60}+4\lambda_n^2P^d_{80}\right)+2(v+2)^2\left(mvP^d_{22}+2m\lambda_dP^d_{24}\right.$$

$$\left.\left.+2v\lambda_nP^d_{42}+4\lambda_n\lambda_dP^d_{44}\right)\right]/\left((v+2)^2(v+m+2)^2\right) . \tag{A5.63}$$

APPENDIX 5.2

TABLES OF RELATIVE RISKS OF $\tilde{\sigma}_j^2$, σ_j^{*2}, AND $\hat{\sigma}_j^2$, j = L, ML, AND M

In this Appendix we give a small, though representative, sample of the numerical evaluations of the relative risks of $\tilde{\sigma}_j^2$, σ_j^{*2}, and $\hat{\sigma}_j^2$ (α=0.01, 0.05, 0.30, 0.75 and that value associated with a critical value of unity or of v/(v+2); j=L,ML,M. The relative risks of $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$ are given in Tables A5.2.1 and A5.2.2. Tables A5.2.3 and A5.2.4 present the relative risks of $\tilde{\sigma}_{ML}^2$, σ_{ML}^{*2}, and $\hat{\sigma}_{ML}^2$, while the relative risks of $\tilde{\sigma}_M^2$, σ_M^{*2}, and $\hat{\sigma}_M^2$ are given in Tables A5.2.5 and A5.2.6. In each case we consider risk as a function of λ_n for given values of λ_d. We recall that λ_d is a measure of the specification error, while λ_n depends on the specification error and on the hypothesis error. When λ_d=0 there are no omitted regressors, $\lambda_n=\lambda$ which will be zero when the prior information is valid. So, for nonzero λ_d, $\lambda_n>0$ even if H_0 is true.

In these tables we consider λ_d=0, 1, 3, 10; λ_n= [0,2(0.5);2,10(1.0);10,14(2.0)] and ν=5, 10, 100, ∞. For each of these values of λ_n, λ_d and ν, each table gives the relative risks of the estimators for different values of v, k, and m. Tables A5.2.1, A5.2.3, and A5.2.5 consider v=30, k=5, and m=1, while Tables A5.2.2, A5.2.4, and A5.2.6 consider v=30, k=5, and m=3.

Tables A5.2.7, A5.2.8, and A5.2.9 present the relative risks of $\tilde{\sigma}_j^2$, σ_j^{*2}, and $\hat{\sigma}_j^2$, j=L, ML, and M respectively, for given values of λ_n, as a function of λ_d. We consider v=16, k=4, m=1; ν=5, 10, 100, ∞; λ_n=0, 1, 5 and λ_d=[0,2(0.5);2,10(1.0);10,14(2.0)].

For the tables which consider the L and the ML component estimators (Tables A5.2.1 to A5.2.4, and Tables A5.2.7 and A5.2.9) the relative risks are presented in the following estimator order:

$$\tilde{\sigma}_j^2$$

$$\sigma_j^{*2}$$

$$\hat{\sigma}_j^2 \quad : \quad \alpha = 0.01$$

$$\hat{\sigma}_j^2 \quad : \quad \alpha = 0.05$$

$$\hat{\sigma}_j^2 \quad : \quad \alpha = 0.30$$

$$\hat{\sigma}_j^2 \quad : \quad \alpha = 0.75$$

$$\hat{\sigma}_j^2 \quad : \quad c = 1 \quad ,$$

j=L, ML. While, for the M component estimator tables (Tables A5.2.5, A5.2.6, and Table A5.2.9) the order of the estimators is the same except that the last pre-test estimator is for a critical value of c=v/(v+2) rather than c=1. We have omitted these legends from the tables because of space constraints. We note that the following values of α correspond to a nominal critical value of c=1 and of c=v/(v+2):

Degrees of Freedom		c=1	c=v/(v+2)
m	v	α	α
1	30	0.325	0.341
3	30	0.406	0.435
1	16	0.332	0.360

TABLE A5.2.1: Relative Risks of $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$

$v = 30$, $k = 5$, $m = 1$

Estimator	λ_m														
	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111
	6.0931	6.1011	6.1111	6.1233	6.1375	6.1722	6.2153	6.2667	6.3264	6.3942	6.4706	6.5553	6.6481	6.8592	7.1033
	6.0708	6.0817	6.0992	6.1214	6.1458	6.1969	6.2456	6.2889	6.3261	6.3578	6.3844	6.4069	6.4256	6.4542	6.4742
$v=5$, $\lambda_d=0$	6.0161	6.0236	6.0353	6.0481	6.0608	6.0847	6.1047	6.1206	6.1336	6.1439	6.1525	6.1592	6.1647	6.1728	6.1783
	5.9144	5.9169	5.9211	5.9258	5.9306	5.9397	5.9481	5.9556	5.9622	5.9681	5.9733	5.9781	5.9825	5.9903	5.9964
	6.0108	6.0128	6.0153	6.0181	6.0211	6.0264	6.0311	6.0353	6.0389	6.0419	6.0447	6.0472	6.0494	6.0533	6.0564
	5.9139	5.9164	5.9206	5.9253	5.9300	5.9392	5.9475	5.9550	5.9617	5.9675	5.9731	5.9778	5.9819	5.9894	5.9961
	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597
	0.6553	0.6616	0.6698	0.6802	0.6927	0.7239	0.7634	0.8114	0.8675	0.9320	1.0048	1.0861	1.1755	1.3795	1.6167
	0.6534	0.6603	0.6698	0.6808	0.6923	0.7139	0.7314	0.7439	0.7519	0.7561	0.7575	0.7567	0.7545	0.7477	0.7394
$v=10$, $\lambda_d=0$	0.6484	0.6528	0.6583	0.6638	0.6686	0.6763	0.6809	0.6834	0.6842	0.6841	0.6831	0.6819	0.6806	0.6780	0.6753
	0.6380	0.6394	0.6413	0.6431	0.6450	0.6480	0.6503	0.6522	0.6536	0.6547	0.6556	0.6563	0.6567	0.6575	0.6583
	0.6486	0.6495	0.6508	0.6519	0.6530	0.6545	0.6558	0.6566	0.6572	0.6578	0.6581	0.6584	0.6586	0.6589	0.6592
	0.6380	0.6394	0.6413	0.6431	0.6448	0.6478	0.6503	0.6522	0.6536	0.6547	0.6556	0.6561	0.6567	0.6573	0.6581
	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926
	0.0903	0.0956	0.1029	0.1123	0.1239	0.1532	0.1909	0.2368	0.2911	0.3537	0.4246	0.5039	0.5914	0.7915	1.0250
	0.0905	0.0951	0.1001	0.1049	0.1091	0.1151	0.1177	0.1175	0.1154	0.1122	0.1088	0.1057	0.1027	0.0983	0.0957
$v=100$, $\lambda_d=0$	0.0903	0.0929	0.0951	0.0967	0.0978	0.0986	0.0982	0.0973	0.0962	0.0952	0.0945	0.0938	0.0935	0.0930	0.0927
	0.0892	0.0900	0.0906	0.0911	0.0915	0.0919	0.0923	0.0924	0.0925	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926
	0.0908	0.0913	0.0917	0.0920	0.0921	0.0924	0.0925	0.0925	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926
	0.0892	0.0900	0.0906	0.0911	0.0914	0.0919	0.0923	0.0924	0.0925	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926
	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	0.0645	0.0697	0.0770	0.0864	0.0978	0.1270	0.1644	0.2102	0.2643	0.3267	0.3975	0.4766	0.5640	0.7638	0.9969
	0.0648	0.0692	0.0738	0.0780	0.0816	0.0863	0.0877	0.0867	0.0842	0.0812	0.0781	0.0753	0.0729	0.0699	0.0679
$v=\infty$, $\lambda_d=0$	0.0648	0.0673	0.0692	0.0705	0.0712	0.0715	0.0709	0.0700	0.0691	0.0684	0.0678	0.0674	0.0670	0.0668	0.0667
	0.0642	0.0649	0.0654	0.0658	0.0660	0.0664	0.0665	0.0666	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	0.0644	0.0651	0.0656	0.0659	0.0662	0.0665	0.0666	0.0666	0.0666	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	0.0642	0.0649	0.0654	0.0658	0.0660	0.0664	0.0665	0.0666	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303
	6.1111	6.1233	6.1375	6.1539	6.1722	6.2153	6.2667	6.3264	6.3942	6.4706	6.5556	6.6481	6.7494	6.9769	7.2378
	6.0883	6.1008	6.1194	6.1422	6.1678	6.2208	6.2717	6.3169	6.3556	6.3883	6.4161	6.4392	6.4581	6.4869	6.5067
$v=5$, $\lambda_d=1$	6.0328	6.0411	6.0533	6.0667	6.0803	6.1053	6.1261	6.1428	6.1561	6.1667	6.1750	6.1819	6.1875	6.1953	6.2006
	5.9294	5.9322	5.9367	5.9417	5.9467	5.9564	5.9653	5.9731	5.9797	5.9858	5.9914	5.9964	6.0008	6.0086	6.0150
	6.0278	6.0300	6.0328	6.0358	6.0389	6.0447	6.0494	6.0536	6.0572	6.0606	6.0633	6.0661	6.0683	6.0722	6.0756
	5.9289	5.9319	5.9361	5.9411	5.9461	5.9558	5.9644	5.9725	5.9794	5.9853	5.9908	5.9958	6.0003	6.0081	6.0147
	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753
	0.6698	0.6802	0.6927	0.7073	0.7239	0.7634	0.8114	0.8675	0.9320	1.0048	1.0861	1.1755	1.2733	1.4939	1.7478
	0.6678	0.6772	0.6888	0.7012	0.7142	0.7380	0.7570	0.7706	0.7789	0.7831	0.7842	0.7830	0.7802	0.7720	0.7622
$v=10$, $\lambda_d=1$	0.6620	0.6680	0.6744	0.6806	0.6861	0.6945	0.6995	0.7020	0.7027	0.7023	0.7012	0.6998	0.6983	0.6950	0.6920
	0.6503	0.6523	0.6547	0.6570	0.6591	0.6625	0.6652	0.6673	0.6688	0.6700	0.6709	0.6717	0.6722	0.6730	0.6738
	0.6625	0.6639	0.6653	0.6667	0.6678	0.6697	0.6711	0.6720	0.6727	0.6731	0.6736	0.6739	0.6741	0.6745	0.6747
	0.6502	0.6523	0.6545	0.6569	0.6589	0.6625	0.6652	0.6672	0.6688	0.6698	0.6709	0.6716	0.6722	0.6730	0.6736
	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061
	0.1029	0.1123	0.1239	0.1375	0.1532	0.1909	0.2368	0.2911	0.3537	0.4246	0.5039	0.5914	0.6873	0.9041	1.1542
	0.1040	0.1107	0.1183	0.1252	0.1311	0.1391	0.1422	0.1416	0.1385	0.1341	0.1293	0.1247	0.1206	0.1143	0.1106
$v=100$, $\lambda_d=1$	0.1021	0.1067	0.1102	0.1126	0.1141	0.1151	0.1143	0.1130	0.1115	0.1101	0.1089	0.1081	0.1075	0.1065	0.1062
	0.1000	0.1016	0.1028	0.1037	0.1043	0.1052	0.1057	0.1058	0.1059	0.1060	0.1060	0.1060	0.1060	0.1061	0.1061
	0.1029	0.1039	0.1046	0.1052	0.1055	0.1058	0.1059	0.1060	0.1060	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061
	0.1000	0.1015	0.1028	0.1037	0.1043	0.1052	0.1057	0.1058	0.1059	0.1060	0.1060	0.1060	0.1060	0.1061	0.1061
	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
	0.0770	0.0864	0.0978	0.1113	0.1270	0.1644	0.2102	0.2643	0.3267	0.3975	0.4766	0.5640	0.6597	0.8762	1.1259
	0.0770	0.0847	0.0919	0.0983	0.1036	0.1103	0.1121	0.1105	0.1068	0.1022	0.0977	0.0934	0.0899	0.0850	0.0823
$v=\infty$, $\lambda_d=1$	0.0766	0.0809	0.0841	0.0862	0.0874	0.0879	0.0869	0.0854	0.0839	0.0828	0.0818	0.0812	0.0806	0.0804	0.0802
	0.0748	0.0764	0.0775	0.0782	0.0788	0.0795	0.0797	0.0799	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
	0.0752	0.0768	0.0779	0.0786	0.0790	0.0796	0.0798	0.0799	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
	0.0748	0.0764	0.0775	0.0782	0.0788	0.0794	0.0797	0.0799	0.0799	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956
	6.1722	6.1928	6.2153	6.2400	6.2667	6.3264	6.3942	6.4706	6.5556	6.6481	6.7494	6.8592	6.9769	7.2378	7.5319
	6.1492	6.1658	6.1872	6.2125	6.2400	6.2975	6.3528	6.4019	6.4442	6.4794	6.5089	6.5328	6.5522	6.5814	6.6006
$v=5$, $\lambda_d=3$	6.0922	6.1025	6.1161	6.1311	6.1461	6.1736	6.1964	6.2142	6.2283	6.2392	6.2481	6.2550	6.2600	6.2675	6.2728
	5.9847	5.9886	5.9939	5.9997	6.0056	6.0161	6.0261	6.0344	6.0417	6.0483	6.0542	6.0592	6.0642	6.0719	6.0789
	6.0881	6.0906	6.0942	6.0978	6.1011	6.1075	6.1128	6.1172	6.1211	6.1247	6.1275	6.1303	6.1328	6.1367	6.1400
	5.9842	5.9881	5.9936	5.9992	6.0050	6.0158	6.0253	6.0339	6.0411	6.0478	6.0536	6.0589	6.0633	6.0717	6.0786
	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331
	0.7239	0.7427	0.7634	0.7864	0.8114	0.8675	0.9320	1.0048	1.0861	1.1755	1.2733	1.3795	1.4939	1.7478	2.0350
	0.7216	0.7370	0.7534	0.7702	0.7866	0.8158	0.8388	0.8545	0.8637	0.8681	0.8684	0.8661	0.8619	0.8503	0.8375
$v=10$, $\lambda_d=3$	0.7148	0.7242	0.7334	0.7417	0.7488	0.7589	0.7647	0.7672	0.7677	0.7666	0.7648	0.7630	0.7609	0.7566	0.7527
	0.7002	0.7039	0.7075	0.7108	0.7136	0.7181	0.7214	0.7241	0.7258	0.7272	0.7281	0.7291	0.7297	0.7306	0.7313
	0.7161	0.7186	0.7208	0.7227	0.7242	0.7266	0.7283	0.7294	0.7302	0.7308	0.7313	0.7316	0.7319	0.7322	0.7323
	0.7002	0.7039	0.7075	0.7106	0.7134	0.7180	0.7214	0.7239	0.7258	0.7272	0.7281	0.7289	0.7297	0.7305	0.7313
	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	3.1597	0.1597	0.1597	0.1597	0.1597
	0.1532	0.1710	0.1909	0.2127	0.2368	0.2911	0.3537	0.4246	0.5039	0.5914	0.6873	0.7915	0.9041	1.1542	1.4376
	0.1530	0.1676	0.1814	0.1935	0.2037	0.2174	0.2231	0.2224	0.2174	0.2100	0.2019	0.1940	0.1869	0.1759	0.1686
$v=100$, $\lambda_d=3$	0.1514	0.1603	0.1671	0.1717	0.1745	0.1764	0.1751	0.1725	0.1697	0.1672	0.1650	0.1635	0.1623	0.1610	0.1603
	0.1469	0.1506	0.1532	0.1550	0.1564	0.1581	0.1589	0.1594	0.1595	0.1596	0.1597	0.1597	0.1597	0.1597	0.1597
	0.1531	0.1554	0.1569	0.1580	0.1586	0.1592	0.1595	0.1596	0.1597	0.1591	0.1597	0.1597	0.1597	0.1597	0.1597
	0.1469	0.1506	0.1532	0.1550	0.1564	0.1581	0.1589	0.1593	0.1595	0.1596	0.1597	0.1597	0.1597	0.1597	0.1597
	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333
	0.1270	0.1446	0.1644	0.1863	0.2102	0.2643	0.3267	0.3975	0.4766	0.5640	0.6597	0.7638	0.8762	1.1259	1.4090
	0.1267	0.1414	0.1549	0.1666	0.1762	0.1886	0.1929	0.1907	0.1848	0.1769	0.1686	0.1608	0.1541	0.1445	0.1386
$v=\infty$, $\lambda_d=3$	0.1254	0.1344	0.1409	0.1452	0.1477	0.1488	0.1471	0.1443	0.1416	0.1393	0.1372	0.1359	0.1349	0.1339	0.1334
	0.1215	0.1251	0.1277	0.1294	0.1307	0.1321	0.1327	0.1331	0.1332	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333
	0.1225	0.1261	0.1285	0.1301	0.1312	0.1324	0.1329	0.1332	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333
	0.1215	0.1251	0.1276	0.1294	0.1306	0.1321	0.1327	0.1331	0.1332	0.1332	0.1333	0.1333	0.1333	0.1333	0.1333
	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036
	6.6481	6.6978	6.7494	6.8033	6.8592	6.9769	7.1033	7.2378	7.3808	7.5319	7.6914	7.8594	8.0356	8.4131	8.8236
	6.6256	6.6664	6.7078	6.7494	6.7911	6.8728	6.9481	7.0142	7.0697	7.1144	7.1497	7.1772	7.1978	7.2247	7.2381
$v=5$, $\lambda_d=10$	6.5658	6.5917	6.6169	6.6411	6.6636	6.7028	6.7328	6.7556	6.7719	6.7839	6.7925	6.7986	6.8028	6.8078	6.8100
	6.4453	6.4564	6.4672	6.4772	6.4864	6.5028	6.5161	6.5275	6.5372	6.5453	6.5525	6.5589	6.5644	6.5742	6.5817
	6.5711	6.5781	6.5850	6.5911	6.5967	6.6058	6.6131	6.6192	6.6242	6.6281	6.6319	6.6350	6.6378	6.6425	6.6461
	6.4447	6.4556	6.4664	6.4764	6.4856	6.5019	6.5153	6.5267	6.5364	6.5447	6.5519	6.5583	6.5639	6.5731	6.5811
	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153
	1.1755	1.2234	1.2733	1.3253	1.3795	1.4939	1.6167	1.7478	1.8873	2.0350	2.1911	2.3556	2.5283	2.8988	3.3025
	1.1733	1.2158	1.2572	1.2964	1.3330	1.3959	1.4431	1.4745	1.4919	1.4978	1.4952	1.4867	1.4744	1.4444	1.4128
$v=10$, $\lambda_d=10$	1.1650	1.1944	1.2194	1.2400	1.2563	1.2780	1.2881	1.2909	1.2891	1.2848	1.2794	1.2738	1.2681	1.2578	1.2495
	1.1413	1.1544	1.1648	1.1731	1.1800	1.1898	1.1966	1.2011	1.2042	1.2067	1.2084	1.2097	1.2106	1.2120	1.2130
	1.1764	1.1852	1.1916	1.1964	1.2000	1.2050	1.2080	1.2098	1.2113	1.2122	1.2128	1.2133	1.2138	1.2142	1.2145
	1.1411	1.1541	1.1645	1.1730	1.1797	1.1897	1.1964	1.2009	1.2042	1.2067	1.2083	1.2097	1.2106	1.2119	1.2128
	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277
	0.5914	0.6384	0.6873	0.7384	0.7915	0.9041	1.0250	1.1542	1.2918	1.4376	1.5917	1.7543	1.9251	2.2918	2.6917
	0.5911	0.6343	0.6756	0.7136	0.7474	0.8003	0.8322	0.8443	0.8409	0.8261	0.8047	0.7802	0.7553	0.7114	0.6787
$v=100$, $\lambda_d=10$	0.5865	0.6200	0.6448	0.6634	0.6763	0.6885	0.6882	0.6812	0.6717	0.6621	0.6537	0.6467	0.6413	0.6342	0.6307
	0.5773	0.5914	0.6017	0.6091	0.6144	0.6210	0.6242	0.6260	0.6268	0.6272	0.6274	0.6274	0.6276	0.6277	0.6277
	0.6006	0.6104	0.6166	0.6206	0.6231	0.6258	0.6268	0.6273	0.6276	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277
	0.5772	0.5913	0.6015	0.6090	0.6143	0.6209	0.6242	0.6259	0.6268	0.6272	0.6273	0.6274	0.6276	0.6277	0.6277
	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000
	0.5640	0.6108	0.6597	0.7107	0.7638	0.8762	0.9969	1.1259	1.2633	1.4090	1.5630	1.7253	1.8959	2.2622	2.6618
	0.5637	0.6070	0.6482	0.6861	0.7197	0.7717	0.8022	0.8128	0.8074	0.7910	0.7680	0.7423	0.7168	0.6726	0.6416
$v=\infty$, $\lambda_d=10$	0.5614	0.5931	0.6179	0.6363	0.6489	0.6601	0.6589	0.6512	0.6413	0.6314	0.6229	0.6162	0.6110	0.6049	0.6019
	0.5508	0.5651	0.5754	0.5827	0.5879	0.5942	0.5972	0.5987	0.5994	0.5996	0.5998	0.6000	0.6000	0.6000	0.6000
	0.5547	0.5691	0.5790	0.5857	0.5903	0.5956	0.5980	0.5992	0.5996	0.5998	0.5999	0.6000	0.6000	0.6000	0.6000
	0.5507	0.5650	0.5753	0.5826	0.5878	0.5940	0.5971	0.5987	0.5993	0.5996	0.5998	0.5999	0.5999	0.6000	0.6000

TABLE A5.2.2: Relative Risks of $\tilde{\sigma}^2_L$, σ^{*2}_L, and θ^2_L

$v = 30$, $k = 5$, $m = 3$

Estimator	λ_n 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111	6.1111
	6.0606	6.0678	6.0764	6.0872	6.0997	6.1303	6.1683	6.2136	6.2664	6.3264	6.3936	6.4683	6.5503	6.7364	6.9519
	6.0331	6.0425	6.0572	6.0772	6.1017	6.1578	6.2192	6.2803	6.3386	6.3925	6.4417	6.4856	6.5247	6.5906	6.6422
$\nu=5,\ \lambda_d=0$	5.9558	5.9644	5.9778	5.9942	6.0122	6.0497	6.0853	6.1172	6.1450	6.1689	6.1897	6.2072	6.2222	6.2458	6.2631
	5.7531	5.7567	5.7622	5.7689	5.7761	5.7900	5.8033	5.8161	5.8272	5.8375	5.8469	5.8556	5.8631	5.8769	5.8889
	5.8053	5.8089	5.8142	5.8203	5.8269	5.8397	5.8517	5.8625	5.8722	5.8814	5.8892	5.8967	5.9031	5.9150	5.9367
	5.7383	5.7414	5.7467	5.7531	5.7597	5.7736	5.7869	5.7992	5.8106	5.8208	5.8306	5.8392	5.8472	5.8614	5.8733
	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597	0.6597
	0.6470	0.6527	0.6600	0.6691	0.6802	0.7077	0.7427	0.7848	0.8344	0.8914	0.9556	1.0272	1.1063	1.2863	1.4956
	0.6453	0.6517	0.6609	0.6723	0.6855	0.7145	0.7436	0.7702	0.7923	0.8102	0.8233	0.8322	0.8378	0.8408	0.8364
$\nu=10,\ \lambda_d=0$	0.6388	0.6442	0.6514	0.6595	0.6680	0.6838	0.6969	0.7064	0.7128	0.7167	0.7184	0.7189	0.7181	0.7148	0.7103
	0.6191	0.6211	0.6238	0.6267	0.6295	0.6348	0.6392	0.6428	0.6458	0.6481	0.6500	0.6514	0.6527	0.6545	0.6558
	0.6248	0.6269	0.6295	0.6320	0.6347	0.6394	0.6431	0.6461	0.6486	0.6505	0.6519	0.6530	0.6541	0.6556	0.6564
	0.6175	0.6194	0.6220	0.6247	0.6275	0.6327	0.6372	0.6409	0.6439	0.6464	0.6483	0.6500	0.6514	0.6534	0.6547
	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926	0.0926
	0.0861	0.0908	0.0973	0.1056	0.1158	0.1416	0.1748	0.2154	0.2633	0.3186	0.3812	0.4511	0.5284	0.7050	0.9110
	0.0866	0.0914	0.0973	0.1039	0.1107	0.1239	0.1345	0.1418	0.1454	0.1456	0.1433	0.1392	0.1342	0.1233	0.1136
$\nu=100,\ \lambda_d=0$	0.0868	0.0906	0.0943	0.0980	0.1011	0.1057	0.1080	0.1084	0.1074	0.1056	0.1036	0.1015	0.0998	0.0966	0.0951
	0.0856	0.0868	0.0879	0.0888	0.0897	0.0908	0.0915	0.0919	0.0923	0.0925	0.0925	0.0926	0.0926	0.0926	0.0926
	0.0865	0.0877	0.0887	0.0895	0.0903	0.0912	0.0918	0.0921	0.0924	0.0925	0.0925	0.0925	0.0926	0.0926	0.0926
	0.0854	0.0864	0.0876	0.0884	0.0892	0.0905	0.0912	0.0917	0.0921	0.0923	0.0924	0.0925	0.0925	0.0925	0.0926
	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667	0.0667
	0.0606	0.0652	0.0716	0.0799	0.0900	0.1157	0.1488	0.1892	0.2369	0.2920	0.3545	0.4242	0.5014	0.6777	0.8834
	0.0612	0.0658	0.0714	0.0776	0.0839	0.0955	0.1045	0.1100	0.1120	0.1110	0.1078	0.1032	0.0982	0.0881	0.0799
$\nu=\infty,\ \lambda_d=0$	0.0618	0.0653	0.0688	0.0720	0.0747	0.0783	0.0798	0.0795	0.0782	0.0764	0.0745	0.0727	0.0712	0.0689	0.0677
	0.0612	0.0624	0.0633	0.0641	0.0648	0.0656	0.0661	0.0664	0.0666	0.0666	0.0666	0.0667	0.0667	0.0667	0.0667
	0.0619	0.0631	0.0639	0.0647	0.0652	0.0659	0.0663	0.0664	0.0666	0.0666	0.0666	0.0667	0.0667	0.0667	0.0667
	0.0611	0.0621	0.0630	0.0638	0.0645	0.0654	0.0659	0.0662	0.0664	0.0666	0.0666	0.0667	0.0667	0.0667	0.0667
	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303	6.1303
	6.0764	6.0875	6.0997	6.1142	6.1303	6.1683	6.2136	6.2664	6.3264	6.3936	6.4683	6.5503	6.6397	6.8406	7.0708
	6.0317	6.0594	6.0753	6.0956	6.1203	6.1775	6.2408	6.3044	6.3656	6.4222	6.4739	6.5200	6.5611	6.6297	6.6828
$\nu=5,\ \lambda_d=1$	5.9703	5.9792	5.9928	6.0094	6.0283	6.0672	6.1047	6.1383	6.1678	6.1931	6.2144	6.2328	6.2483	6.2725	6.2900
	5.7644	5.7681	5.7739	5.7808	5.7883	5.8033	5.8175	5.8306	5.8425	5.8533	5.8628	5.8717	5.8800	5.8939	5.9061
	5.8175	5.8214	5.8269	5.8336	5.8403	5.8539	5.8664	5.8778	5.8883	5.8975	5.9058	5.9136	5.9203	5.9325	5.9425
	5.7492	5.7525	5.7581	5.7647	5.7719	5.7864	5.8006	5.8136	5.8253	5.8361	5.8461	5.8550	5.8633	5.8781	5.8906
	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753	0.6753
	0.6600	0.6691	0.6802	0.6930	0.7077	0.7427	0.7848	0.8344	0.8914	0.9556	1.0272	1.1063	1.1925	1.3872	1.6112
	0.6578	0.6669	0.6781	0.6914	0.7063	0.7383	0.7703	0.7994	0.8239	0.8434	0.8578	0.8677	0.8736	0.8766	0.8709
$\nu=10,\ \lambda_d=1$	0.6505	0.6575	0.6661	0.6755	0.6850	0.7028	0.7172	0.7278	0.7350	0.7392	0.7411	0.7414	0.7405	0.7364	0.7311
	0.6283	0.6313	0.6347	0.6383	0.6416	0.6478	0.6528	0.6570	0.6602	0.6627	0.6648	0.6664	0.6678	0.6697	0.6711
	0.6348	0.6378	0.6411	0.6442	0.6473	0.6528	0.6570	0.6605	0.6631	0.6652	0.6669	0.6683	0.6692	0.6709	0.6719
	0.6266	0.6292	0.6325	0.6359	0.6394	0.6453	0.6506	0.6547	0.6581	0.6608	0.6631	0.6648	0.6663	0.6686	0.6703
	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061	0.1061
	0.0973	0.1056	0.1158	0.1278	0.1416	0.1748	0.2154	0.2633	0.3186	0.3812	0.4511	0.5284	0.6130	0.8044	1.0250
	0.0975	0.1054	0.1142	0.1236	0.1330	0.1505	0.1645	0.1740	0.1786	0.1791	0.1762	0.1711	0.1643	0.1496	0.1365
$\nu=100,\ \lambda_d=1$	0.0970	0.1031	0.1089	0.1142	0.1187	0.1252	0.1282	0.1287	0.1272	0.1247	0.1218	0.1190	0.1164	0.1123	0.1095
	0.0938	0.0962	0.0983	0.1000	0.1014	0.1034	0.1045	0.1053	0.1057	0.1059	0.1060	0.1060	0.1061	0.1061	0.1060
	0.0954	0.0978	0.0996	0.1012	0.1024	0.1040	0.1049	0.1054	0.1058	0.1058	0.1059	0.1061	0.1060	0.1060	0.1060
	0.0934	0.0958	0.0978	0.0994	0.1008	0.1029	0.1040	0.1049	0.1054	0.1056	0.1059	0.1059	0.1060	0.1060	0.1060
	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800	0.0800
	0.0716	0.0799	0.0900	0.1019	0.1157	0.1488	0.1892	0.2369	0.2920	0.3545	0.4242	0.5014	0.5859	0.7769	0.9973
	0.0719	0.0797	0.0884	0.0974	0.1063	0.1224	0.1348	0.1424	0.1454	0.1443	0.1402	0.1342	0.1270	0.1129	0.1009
$\nu=\infty,\ \lambda_d=1$	0.0718	0.0778	0.0833	0.0883	0.0923	0.0978	0.1000	0.0997	0.0978	0.0951	0.0922	0.0895	0.0872	0.0839	0.0818
	0.0693	0.0718	0.0738	0.0753	0.0765	0.0782	0.0791	0.0796	0.0798	0.0799	0.0799	0.0800	0.0800	0.0800	0.0800
	0.0707	0.0731	0.0749	0.0762	0.0773	0.0786	0.0793	0.0797	0.0798	0.0799	0.0800	0.0800	0.0800	0.0800	0.0800
	0.0690	0.0713	0.0732	0.0748	0.0760	0.0777	0.0787	0.0793	0.0796	0.0798	0.0799	0.0800	0.0800	0.0800	0.0800
	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956	6.1956
	6.1303	6.1483	6.1683	6.1900	6.2136	6.2664	6.3264	6.3936	6.4683	6.5503	6.6397	6.7364	6.8406	7.0708	7.3303
	6.1022	6.1181	6.1372	6.1600	6.1858	6.2464	6.3136	6.3825	6.4492	6.5114	6.5686	6.6192	6.6644	6.7386	6.7953
$\nu=5,\ \lambda_d=3$	6.0225	6.0342	6.0494	6.0675	6.0875	6.1300	6.1714	6.2083	6.2411	6.2686	6.2922	6.3117	6.3283	6.3536	6.3717
	5.8103	5.8153	5.8222	5.8300	5.8386	5.8556	5.8714	5.8858	5.8986	5.9106	5.9211	5.9308	5.9394	5.9544	5.9672
	5.8661	5.8706	5.8775	5.8850	5.8928	5.9081	5.9222	5.9350	5.9461	5.9561	5.9653	5.9733	5.9808	5.9939	6.0047
	5.7944	5.7992	5.8056	5.8133	5.8217	5.8381	5.8536	5.8681	5.8811	5.8928	5.9036	5.9133	5.9222	5.9381	5.9514
	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331	0.7331
	0.7077	0.7242	0.7427	0.7628	0.7848	0.8344	0.8914	0.9556	1.0272	1.1063	1.1925	1.2863	1.3872	1.6112	1.8647
	0.7053	0.7203	0.7372	0.7553	0.7745	0.8144	0.8534	0.8887	0.9189	0.9427	0.9602	0.9722	0.9792	0.9817	0.9739
$\nu=10,\ \lambda_d=3$	0.6969	0.7086	0.7209	0.7336	0.7459	0.7684	0.7866	0.7997	0.8083	0.8131	0.8152	0.8153	0.8136	0.8080	0.8006
	0.6703	0.6756	0.6811	0.6862	0.6911	0.6994	0.7061	0.7112	0.7155	0.7184	0.7209	0.7230	0.7244	0.7269	0.7283
	0.6789	0.6841	0.6892	0.6939	0.6983	0.7056	0.7112	0.7155	0.7189	0.7214	0.7233	0.7250	0.7263	0.7280	0.7294
	0.6680	0.6731	0.6784	0.6834	0.6881	0.6964	0.7033	0.7086	0.7128	0.7161	0.7189	0.7211	0.7228	0.7253	0.7273
	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597	0.1597
	0.1416	0.1573	0.1748	0.1942	0.2154	0.2633	0.3186	0.3812	0.4511	0.5284	0.6130	0.7050	0.8044	1.0250	1.2751
	0.1415	0.1563	0.1717	0.1874	0.2027	0.2312	0.2542	0.2704	0.2795	0.2821	0.2791	0.2720	0.2622	0.2393	0.2170
$\nu=100,\ \lambda_d=3$	0.1403	0.1519	0.1627	0.1724	0.1805	0.1923	0.1983	0.1995	0.1975	0.1937	0.1888	0.1840	0.1793	0.1718	0.1668
	0.1336	0.1392	0.1438	0.1474	0.1504	0.1544	0.1568	0.1582	0.1589	0.1593	0.1595	0.1596	0.1597	0.1598	0.1598
	0.1369	0.1423	0.1465	0.1497	0.1522	0.1556	0.1574	0.1585	0.1591	0.1594	0.1596	0.1596	0.1597	0.1597	0.1597
	0.1329	0.1382	0.1426	0.1463	0.1492	0.1534	0.1559	0.1575	0.1585	0.1589	0.1593	0.1594	0.1595	0.1597	0.1897
	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333	0.1333
	0.1157	0.1313	0.1488	0.1680	0.1892	0.2369	0.2920	0.3545	0.4242	0.5014	0.5859	0.6778	0.7769	0.9973	1.2470
	0.1157	0.1304	0.1457	0.1611	0.1762	0.2036	0.2251	0.2396	0.2468	0.2474	0.2428	0.2343	0.2235	0.1997	0.1781
$\nu=\infty,\ \lambda_d=3$	0.1147	0.1264	0.1371	0.1465	0.1543	0.1650	0.1699	0.1703	0.1676	0.1632	0.1581	0.1532	0.1488	0.1419	0.1377
	0.1089	0.1146	0.1191	0.1227	0.1254	0.1291	0.1312	0.1322	0.1328	0.1331	0.1332	0.1333	0.1333	0.1333	0.1333
	0.1120	0.1175	0.1216	0.1247	0.1271	0.1300	0.1316	0.1324	0.1329	0.1331	0.1332	0.1333	0.1333	0.1333	0.1333
	0.1083	0.1137	0.1181	0.1216	0.1243	0.1281	0.1304	0.1317	0.1325	0.1328	0.1331	0.1332	0.1332	0.1333	0.1333
	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036	6.7036
	6.5503	6.5942	6.6397	6.6872	6.7364	6.8406	6.9519	7.0708	7.1969	7.3303	7.4711	7.6192	7.7747	8.1078	8.4703
	6.5239	6.5639	6.6050	6.6475	6.6908	6.7808	6.8736	6.9672	7.0586	7.1453	7.2244	7.2956	7.3578	7.4578	7.5283
$\nu=5,\ \lambda_d=10$	6.4444	6.4758	6.5069	6.5389	6.5708	6.6339	6.6933	6.7472	6.7933	6.8322	6.8639	6.8894	6.9097	6.9386	6.9564
	6.2169	6.2322	6.2478	6.2633	6.2781	6.3061	6.3308	6.3519	6.3708	6.3867	6.4011	6.4136	6.4250	6.4442	6.4600
	6.2850	6.2994	6.3144	6.3289	6.3431	6.3681	6.3900	6.4086	6.4247	6.4383	6.4506	6.4611	6.4706	6.4867	6.5000
	6.1986	6.2136	6.2286	6.2433	6.2581	6.2850	6.3094	6.3308	6.3497	6.3658	6.3806	6.3936	6.4053	6.4253	6.4419
	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153	1.2153
	1.1063	1.1484	1.1925	1.2384	1.2863	1.3872	1.4956	1.6112	1.7344	1.8647	2.0025	2.1475	2.3000	2.6269	2.9831
	1.1042	1.1448	1.1863	1.2284	1.2708	1.3552	1.4358	1.5095	1.5736	1.6263	1.6669	1.6955	1.7131	1.7216	1.7041
$\nu=10,\ \lambda_d=10$	1.0959	1.1306	1.1644	1.1966	1.2270	1.2803	1.3225	1.3531	1.3733	1.3844	1.3886	1.3873	1.3830	1.3675	1.3491
	1.0600	1.0800	1.0977	1.1131	1.1266	1.1481	1.1641	1.1758	1.1844	1.1908	1.1956	1.1992	1.2022	1.2061	1.2088
	1.0788	1.0986	1.1156	1.1300	1.1420	1.1609	1.1742	1.1838	1.1906	1.1958	1.1997	1.2023	1.2048	1.2080	1.2098
	1.0563	1.0753	1.0925	1.1075	1.1206	1.1420	1.1580	1.1702	1.1792	1.1863	1.1916	1.1958	1.1989	1.2036	1.2067
	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277	0.6277
	0.5284	0.5698	0.6130	0.6581	0.7050	0.8044	0.9110	1.0250	1.1464	1.2751	1.4111	1.5546	1.7052	2.0288	2.3817
	0.5282	0.5688	0.6104	0.6524	0.6944	0.7756	0.8496	0.9126	0.9616	0.9952	1.0134	1.0177	1.0095	0.9665	0.9047
$\nu=100,\ \lambda_d=10$	0.5267	0.5634	0.5984	0.6309	0.6603	0.7088	0.7413	0.7594	0.7652	0.7618	0.7518	0.7383	0.7230	0.6938	0.6701
	0.5140	0.5365	0.5554	0.5708	0.5834	0.6019	0.6127	0.6193	0.6232	0.6254	0.6265	0.6271	0.6274	0.6277	0.6277
	0.5259	0.5488	0.5671	0.5813	0.5924	0.6076	0.6163	0.6213	0.6241	0.6257	0.6266	0.6271	0.6272	0.6276	0.6277
	0.5121	0.5337	0.5519	0.5668	0.5792	0.5973	0.6089	0.6163	0.6209	0.6236	0.6253	0.6262	0.6268	0.6274	0.6277
	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000	0.6000
	0.5014	0.5427	0.5859	0.6309	0.6777	0.7769	0.8834	0.9973	1.1189	1.2470	1.3829	1.5262	1.6768	2.0000	2.3526
	0.5012	0.5419	0.5835	0.6255	0.6673	0.7484	0.8219	0.8838	0.9314	0.9633	0.9796	0.9814	0.9709	0.9235	0.8580
$\nu=\infty,\ \lambda_d=10$	0.5000	0.5369	0.5720	0.6045	0.6339	0.6815	0.7134	0.7304	0.7350	0.7302	0.7192	0.7047	0.6888	0.6589	0.6357
	0.4885	0.5113	0.5303	0.5458	0.5582	0.5760	0.5868	0.5931	0.5965	0.5984	0.5993	0.5998	0.6000	0.6000	0.6000
	0.5001	0.5233	0.5416	0.5559	0.5669	0.5816	0.5900	0.5947	0.5973	0.5985	0.5993	0.5996	0.5998	0.5999	0.6000
	0.4867	0.5086	0.5268	0.5419	0.5541	0.5719	0.5832	0.5901	0.5943	0.5967	0.5981	0.5989	0.5994	0.5998	0.5999

TABLE A5.2.3: Relative Risks of $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$

$v = 30,\ k = 5,\ m = 1$

Estimator	λ_n 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$\nu=5,\ \lambda_d=0$	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464
	4.8164	4.8117	4.8086	4.8072	4.8075	4.8131	4.8250	4.8436	4.8686	4.9000	4.9383	4.9828	5.0339	5.1558	5.3039
	4.7983	4.8008	4.8108	4.8258	4.8442	4.8839	4.9225	4.9569	4.9864	5.0111	5.0314	5.0481	5.0614	5.0803	5.0919
	4.7503	4.7542	4.7622	4.7719	4.7817	4.7986	4.8119	4.8211	4.8278	4.8317	4.8342	4.8353	4.8353	4.8333	4.8300
	4.6108	4.6114	4.6119	4.6117	4.6117	4.6106	4.6092	4.6075	4.6058	4.6042	4.6028	4.6011	4.5997	4.5975	4.5950
	4.5489	4.5489	4.5489	4.5489	4.5489	4.5489	4.5486	4.5486	4.5483	4.5483	4.5483	4.5481	4.5481	4.5481	4.5481
	4.6033	4.6039	4.6039	4.6039	4.6036	4.6025	4.6008	4.5994	4.5978	4.5964	4.5953	4.5939	4.5925	4.5903	4.5883
$\nu=10,\ \lambda_d=0$	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166
	0.5344	0.5313	0.5295	0.5295	0.5313	0.5394	0.5541	0.5753	0.6030	0.6373	0.6781	0.7255	0.7797	0.9067	1.0602
	0.5338	0.5345	0.5388	0.5456	0.5536	0.5703	0.5850	0.5958	0.6028	0.6067	0.6081	0.6075	0.6055	0.5992	0.5916
	0.5311	0.5330	0.5364	0.5405	0.5442	0.5498	0.5528	0.5538	0.5531	0.5517	0.5498	0.5475	0.5453	0.5411	0.5373
	0.5216	0.5219	0.5222	0.5222	0.5222	0.5217	0.5211	0.5205	0.5200	0.5195	0.5192	0.5189	0.5184	0.5181	0.5178
	0.5167	0.5169	0.5167	0.5167	0.5167	0.5167	0.5167	0.5167	0.5167	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166
	0.5209	0.5213	0.5214	0.5214	0.5214	0.5209	0.5203	0.5200	0.5194	0.5191	0.5188	0.5183	0.5181	0.5177	0.5175
$\nu=100,\ \lambda_d=0$	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892
	0.0844	0.0819	0.0810	0.0817	0.0841	0.0938	0.1100	0.1328	0.1620	0.1977	0.2401	0.2889	0.3443	0.4747	0.6311
	0.0852	0.0849	0.0861	0.0882	0.0907	0.0956	0.0991	0.1009	0.1011	0.1003	0.0989	0.0975	0.0958	0.0932	0.0914
	0.0864	0.0870	0.0882	0.0894	0.0904	0.0918	0.0923	0.0921	0.0916	0.0911	0.0907	0.0903	0.0900	0.0895	0.0893
	0.0886	0.0889	0.0891	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0892	0.0892	0.0892	0.0892	0.0892
	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892
	0.0887	0.0890	0.0891	0.0892	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0892	0.0892	0.0892	0.0892	0.0892
$\nu=\infty,\ \lambda_d=0$	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0637	0.0612	0.0604	0.0612	0.0637	0.0735	0.0898	0.1127	0.1420	0.1780	0.2204	0.2694	0.3249	0.4555	0.6122
	0.0645	0.0641	0.0650	0.0667	0.0687	0.0727	0.0754	0.0767	0.0767	0.0761	0.0751	0.0740	0.0728	0.0713	0.0702
	0.0659	0.0664	0.0673	0.0683	0.0691	0.0703	0.0707	0.0707	0.0705	0.0702	0.0700	0.0698	0.0696	0.0694	0.0694
	0.0685	0.0688	0.0691	0.0692	0.0693	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0691	0.0692	0.0693	0.0693	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0686	0.0689	0.0691	0.0692	0.0693	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
$\nu=5,\ \lambda_d=1$	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333
	4.8086	4.8072	4.8075	4.8094	4.8131	4.8250	4.8436	4.8686	4.9000	4.9383	4.9828	5.0339	5.0917	5.2267	5.3878
	4.7903	4.7933	4.8033	4.8183	4.8369	4.8778	4.9181	4.9542	4.9850	5.0106	5.0317	5.0486	5.0619	5.0811	5.0925
	4.7414	4.7450	4.7531	4.7628	4.7728	4.7908	4.8044	4.8142	4.8208	4.8250	4.8272	4.8281	4.8281	4.8258	4.8217
	4.5992	4.5994	4.6000	4.6000	4.5997	4.5986	4.5969	4.5953	4.5936	4.5919	4.5903	4.5889	4.5875	4.5850	4.5822
	4.5361	4.5361	4.5358	4.5358	4.5358	4.5358	4.5356	4.5356	4.5353	4.5353	4.5353	4.5350	4.5350	4.5350	4.5350
	4.5914	4.5919	4.5919	4.5917	4.5914	4.5903	4.5886	4.5872	4.5856	4.5839	4.5825	4.5811	4.5800	4.5775	4.5631
$\nu=10,\ \lambda_d=1$	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077
	0.5295	0.5295	0.5313	0.5344	0.5394	0.5541	0.5753	0.6030	0.6373	0.6781	0.7255	0.7794	0.8397	0.9802	1.1467
	0.5286	0.5308	0.5361	0.5436	0.5523	0.5703	0.5856	0.5972	0.6044	0.6083	0.6092	0.6081	0.6058	0.5983	0.5894
	0.5252	0.5275	0.5314	0.5356	0.5395	0.5455	0.5484	0.5491	0.5481	0.5463	0.5441	0.5414	0.5389	0.5341	0.5298
	0.5134	0.5137	0.5139	0.5139	0.5139	0.5133	0.5125	0.5119	0.5114	0.5108	0.5103	0.5100	0.5097	0.5091	0.5089
	0.5078	0.5078	0.5078	0.5078	0.5078	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077
	0.5128	0.5131	0.5131	0.5131	0.5130	0.5123	0.5117	0.5113	0.5106	0.5102	0.5098	0.5095	0.5092	0.5089	0.5086
$\nu=100,\ \lambda_d=1$	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
	0.0810	0.0817	0.0841	0.0882	0.0938	0.1100	0.1328	0.1620	0.1977	0.2401	0.2889	0.3443	0.4062	0.5497	0.7192
	0.0814	0.0833	0.0861	0.0894	0.0928	0.0985	0.1019	0.1029	0.1021	0.1002	0.0977	0.0951	0.0925	0.0884	0.0858
	0.0822	0.0837	0.0854	0.0867	0.0878	0.0886	0.0884	0.0877	0.0867	0.0857	0.0849	0.0841	0.0837	0.0830	0.0827
	0.0827	0.0829	0.0830	0.0831	0.0830	0.0829	0.0828	0.0827	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
	0.0827	0.0829	0.0829	0.0830	0.0829	0.0828	0.0828	0.0827	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
$\nu=\infty,\ \lambda_d=1$	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
	0.0604	0.0612	0.0637	0.0678	0.0735	0.0898	0.1127	0.1420	0.1780	0.2204	0.2694	0.3249	0.3869	0.5306	0.7006
	0.0609	0.0627	0.0653	0.0682	0.0711	0.0758	0.0782	0.0786	0.0775	0.0756	0.0733	0.0710	0.0691	0.0661	0.0644
	0.0617	0.0632	0.0647	0.0658	0.0666	0.0672	0.0669	0.0662	0.0654	0.0647	0.0641	0.0637	0.0633	0.0631	0.0630
	0.0628	0.0630	0.0631	0.0631	0.0631	0.0630	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
	0.0628	0.0630	0.0630	0.0631	0.0630	0.0630	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
$\nu=5,\ \lambda_d=3$	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269
	4.8131	4.8183	4.8250	4.8333	4.8436	4.8686	4.9000	4.9383	4.9828	5.0339	5.0917	5.1558	5.2267	5.3878	5.5750
	4.7942	4.7994	4.8103	4.8261	4.8450	4.8883	4.9314	4.9706	5.0039	5.0317	5.0539	5.0719	5.0856	5.1047	5.1153
	4.7433	4.7472	4.7556	4.7658	4.7767	4.7958	4.8108	4.8211	4.8281	4.8319	4.8339	4.8347	4.8339	4.8306	4.8258
	4.5953	4.5958	4.5958	4.5958	4.5956	4.5942	4.5925	4.5906	4.5886	4.5869	4.5853	4.5836	4.5822	4.5792	4.5769
	4.5297	4.5294	4.5294	4.5294	4.5292	4.5292	4.5292	4.5289	4.5289	4.5289	4.5286	4.5286	4.5286	4.5286	4.5283
	4.5875	4.5875	4.5878	4.5875	4.5869	4.5856	4.5839	4.5822	4.5806	4.5789	4.5772	4.5758	4.5744	4.5719	4.5700
$\nu=10,\ \lambda_d=3$	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092
	0.5394	0.5459	0.5541	0.5638	0.5753	0.6030	0.6373	0.6781	0.7255	0.7794	0.8397	0.9067	0.9802	1.1467	1.3394
	0.5380	0.5441	0.5523	0.5620	0.5725	0.5933	0.6108	0.6234	0.6309	0.6347	0.6348	0.6328	0.6291	0.6186	0.6069
	0.5333	0.5372	0.5422	0.5472	0.5516	0.5578	0.5606	0.5606	0.5589	0.5561	0.5530	0.5497	0.5464	0.5402	0.5350
	0.5173	0.5175	0.5175	0.5173	0.5170	0.5161	0.5152	0.5144	0.5136	0.5128	0.5123	0.5119	0.5116	0.5111	0.5106
	0.5095	0.5095	0.5095	0.5095	0.5094	0.5094	0.5094	0.5094	0.5094	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092
	0.5164	0.5166	0.5164	0.5163	0.5158	0.5150	0.5142	0.5134	0.5128	0.5122	0.5117	0.5114	0.5111	0.5106	0.5103
$\nu=100,\ \lambda_d=3$	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886
	0.0938	0.1011	0.1100	0.1206	0.1328	0.1620	0.1977	0.2401	0.2889	0.3443	0.4062	0.4747	0.5497	0.7192	0.9148
	0.0939	0.1006	0.1074	0.1139	0.1197	0.1283	0.1322	0.1322	0.1293	0.1246	0.1192	0.1137	0.1088	0.1008	0.0954
	0.0936	0.0977	0.1008	0.1030	0.1041	0.1042	0.1026	0.1001	0.0976	0.0953	0.0934	0.0920	0.0910	0.0898	0.0891
	0.0907	0.0908	0.0906	0.0904	0.0902	0.0897	0.0892	0.0890	0.0888	0.0888	0.0887	0.0886	0.0886	0.0886	0.0886
	0.0887	0.0887	0.0887	0.0887	0.0887	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886
	0.0905	0.0905	0.0903	0.0901	0.0899	0.0894	0.0891	0.0889	0.0888	0.0887	0.0886	0.0886	0.0886	0.0886	0.0886
$\nu=\infty,\ \lambda_d=3$	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0735	0.0808	0.0898	0.1004	0.1127	0.1420	0.1780	0.2204	0.2694	0.3249	0.3869	0.4555	0.5306	0.7004	0.8963
	0.0736	0.0803	0.0870	0.0932	0.0986	0.1061	0.1090	0.1080	0.1044	0.0994	0.0940	0.0887	0.0842	0.0775	0.0733
	0.0735	0.0776	0.0806	0.0824	0.0834	0.0831	0.0811	0.0786	0.0762	0.0742	0.0726	0.0715	0.0707	0.0699	0.0695
	0.0712	0.0712	0.0711	0.0708	0.0706	0.0701	0.0698	0.0696	0.0695	0.0695	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0701	0.0699	0.0698	0.0697	0.0696	0.0695	0.0695	0.0695	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0710	0.0710	0.0708	0.0705	0.0703	0.0700	0.0697	0.0696	0.0695	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
$\nu=5,\ \lambda_d=10$	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097
	5.0339	5.0619	5.0917	5.1231	5.1558	5.2267	5.3039	5.3878	5.4781	5.5750	5.6783	5.7881	5.9047	6.1572	6.4358
	5.0153	5.0372	5.0606	5.0856	5.1117	5.1664	5.2203	5.2700	5.3119	5.3461	5.3728	5.3928	5.4072	5.4233	5.4278
	4.9608	4.9725	4.9858	4.9994	5.0133	5.0375	5.0553	5.0669	5.0733	5.0761	5.0758	5.0742	5.0708	5.0622	5.0528
	4.7919	4.7914	4.7911	4.7906	4.7894	4.7872	4.7844	4.7814	4.7786	4.7761	4.7739	4.7717	4.7697	4.7661	4.7631
	4.7131	4.7131	4.7128	4.7128	4.7125	4.7122	4.7122	4.7122	4.7119	4.7117	4.7117	4.7117	4.7114	4.7114	4.7114
	4.7825	4.7817	4.7811	4.7806	4.7794	4.7769	4.7742	4.7717	4.7692	4.7669	4.7647	4.7628	4.7611	4.7578	4.7553
$\nu=10,\ \lambda_d=10$	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206
	0.7794	0.8088	0.8397	0.8723	0.9067	0.9802	1.0602	1.1467	1.2397	1.3394	1.4455	1.5581	1.6773	1.9353	2.2194
	0.7777	0.8034	0.8291	0.8536	0.8770	0.9180	0.9494	0.9702	0.9809	0.9834	0.9794	0.9708	0.9591	0.9316	0.9028
	0.7708	0.7872	0.8011	0.8122	0.8206	0.8302	0.8316	0.8278	0.8211	0.8128	0.8042	0.7959	0.7881	0.7745	0.7638
	0.7400	0.7400	0.7392	0.7380	0.7367	0.7341	0.7316	0.7295	0.7280	0.7266	0.7256	0.7247	0.7241	0.7231	0.7227
	0.7216	0.7214	0.7213	0.7211	0.7211	0.7209	0.7208	0.7208	0.7208	0.7208	0.7208	0.7208	0.7208	0.7206	0.7206
	0.7380	0.7375	0.7367	0.7355	0.7342	0.7317	0.7297	0.7280	0.7266	0.7256	0.7247	0.7241	0.7234	0.7225	0.7220
$\nu=100,\ \lambda_d=10$	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158
	0.3443	0.3744	0.4062	0.4396	0.4747	0.5497	0.6311	0.7192	0.8137	0.9148	1.0224	1.1366	1.2573	1.5182	1.8053
	0.3441	0.3719	0.3985	0.4232	0.4449	0.4786	0.4975	0.5026	0.4965	0.4822	0.4633	0.4425	0.4216	0.3850	0.3581
	0.3423	0.3613	0.3755	0.3850	0.3907	0.3920	0.3850	0.3742	0.3625	0.3515	0.3424	0.3351	0.3294	0.3224	0.3187
	0.3284	0.3280	0.3261	0.3243	0.3229	0.3206	0.3186	0.3177	0.3167	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158
	0.3164	0.3162	0.3161	0.3160	0.3159	0.3159	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158
	0.3267	0.3264	0.3253	0.3238	0.3225	0.3200	0.3183	0.3173	0.3166	0.3162	0.3160	0.3160	0.3159	0.3158	0.3158
$\nu=\infty,\ \lambda_d=10$	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980
	0.3249	0.3551	0.3869	0.4204	0.4555	0.5306	0.6122	0.7004	0.7951	0.8963	1.0041	1.1184	1.2392	1.5004	1.7878
	0.3247	0.3527	0.3794	0.4040	0.4258	0.4587	0.4766	0.4803	0.4726	0.4570	0.4370	0.4152	0.3939	0.3575	0.3320
	0.3231	0.3424	0.3566	0.3661	0.3714	0.3721	0.3644	0.3529	0.3409	0.3300	0.3209	0.3140	0.3088	0.3027	0.2998
	0.3097	0.3098	0.3087	0.3072	0.3057	0.3028	0.3008	0.2996	0.2989	0.2984	0.2982	0.2982	0.2980	0.2980	0.2980
	0.3027	0.3017	0.3008	0.3001	0.2995	0.2988	0.2983	0.2982	0.2981	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980
	0.3085	0.3082	0.3071	0.3057	0.3042	0.3018	0.3001	0.2992	0.2986	0.2983	0.2981	0.2980	0.2980	0.2980	0.2980

TABLE A5.2.4: Relative Risks of $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$

$v = 30$, $k = 5$, $m = 3$

Estimator	λ_n 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464	4.5464
	5.3969	5.3978	5.4000	5.4042	5.4100	5.4261	5.4492	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.8669	6.0369
	5.3697	5.3750	5.3869	5.4047	5.4272	5.4814	5.5411	5.6006	5.6561	5.7069	5.7519	5.7914	5.8256	5.8794	5.9181
$v=5,\ \lambda_d=0$	5.2825	5.2903	5.3036	5.3206	5.3392	5.3758	5.4086	5.4356	5.4569	5.4728	5.4844	5.4925	5.4975	5.5006	5.4975
	4.9239	4.9272	4.9300	4.9322	4.9331	4.9322	4.9286	4.9239	4.9178	4.9114	4.9047	4.8981	4.8911	4.8786	4.8669
	4.6489	4.6489	4.6481	4.6469	4.6456	4.6422	4.6389	4.6358	4.6328	4.6300	4.6272	4.6247	4.6225	4.6186	4.6153
	4.8194	4.8211	4.8219	4.8217	4.8208	4.8172	4.8122	4.8064	4.8006	4.7944	4.7886	4.7831	4.7778	4.7678	4.7589
	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166	0.5166
	0.5803	0.5813	0.5836	0.5877	0.5934	0.6097	0.6327	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	1.0506	1.2203
	0.5794	0.5828	0.5895	0.5992	0.6109	0.6386	0.6672	0.6931	0.7150	0.7320	0.7441	0.7516	0.7555	0.7544	0.7456
$v=10,\ \lambda_d=0$	0.5745	0.5792	0.5864	0.5947	0.6034	0.6194	0.6314	0.6391	0.6427	0.6431	0.6411	0.6377	0.6331	0.6222	0.6109
	0.5486	0.5508	0.5525	0.5536	0.5539	0.5533	0.5511	0.5484	0.5456	0.5427	0.5402	0.5377	0.5355	0.5319	0.5289
	0.5258	0.5259	0.5258	0.5253	0.5248	0.5238	0.5227	0.5217	0.5209	0.5203	0.5197	0.5192	0.5189	0.5184	0.5178
	0.5402	0.5414	0.5420	0.5422	0.5419	0.5403	0.5381	0.5358	0.5336	0.5317	0.5298	0.5283	0.5269	0.5247	0.5230
	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892	0.0892
	0.0800	0.0808	0.0832	0.0873	0.0930	0.1093	0.1322	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.5502	0.7199
	0.0811	0.0834	0.0873	0.0921	0.0979	0.1100	0.1209	0.1291	0.1340	0.1358	0.1349	0.1322	0.1282	0.1188	0.1099
$v=100,\ \lambda_d=0$	0.0834	0.0864	0.0900	0.0938	0.0973	0.1028	0.1059	0.1068	0.1060	0.1043	0.1021	0.0999	0.0978	0.0942	0.0923
	0.0881	0.0897	0.0907	0.0914	0.0917	0.0918	0.0915	0.0910	0.0906	0.0902	0.0900	0.0897	0.0897	0.0892	0.0892
	0.0893	0.0895	0.0897	0.0897	0.0897	0.0895	0.0894	0.0893	0.0893	0.0893	0.0892	0.0892	0.0892	0.0892	0.0892
	0.0887	0.0898	0.0903	0.0907	0.0908	0.0907	0.0904	0.0901	0.0899	0.0895	0.0894	0.0894	0.0893	0.0892	0.0892
	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0571	0.0580	0.0604	0.0645	0.0702	0.0865	0.1094	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.5274	0.6971
	0.0584	0.0605	0.0641	0.0687	0.0738	0.0844	0.0937	0.1003	0.1040	0.1048	0.1034	0.1004	0.0967	0.0866	0.0815
$v=\infty,\ \lambda_d=0$	0.0609	0.0638	0.0671	0.0704	0.0734	0.0781	0.0806	0.0812	0.0805	0.0791	0.0773	0.0757	0.0742	0.0719	0.0705
	0.0668	0.0683	0.0693	0.0700	0.0704	0.0707	0.0705	0.0702	0.0700	0.0698	0.0696	0.0696	0.0695	0.0694	0.0694
	0.0691	0.0693	0.0694	0.0695	0.0695	0.0695	0.0695	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0679	0.0688	0.0694	0.0698	0.0699	0.0700	0.0699	0.0697	0.0696	0.0696	0.0694	0.0694	0.0694	0.0694	0.0694
	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333	4.5333
	5.4000	5.4042	5.4100	5.4172	5.4261	5.4492	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.7919	5.9486	6.1314
	5.3722	5.3783	5.3903	5.4075	5.4294	5.4836	5.5444	5.6058	5.6642	5.7175	5.7650	5.8064	5.8422	5.8986	5.9381
$v=5,\ \lambda_d=1$	5.2833	5.2903	5.3033	5.3197	5.3383	5.3761	5.4106	5.4392	5.4614	5.4786	5.4908	5.4989	5.5042	5.5072	5.5031
	4.9181	4.9208	4.9236	4.9258	4.9267	4.9261	4.9225	4.9175	4.9111	4.9044	4.8972	4.8903	4.8833	4.8700	4.8578
	4.6378	4.6378	4.6369	4.6358	4.6342	4.6308	4.6275	4.6239	4.6208	4.6178	4.6153	4.6128	4.6106	4.6061	4.6025
	4.8117	4.8131	4.8139	4.8136	4.8128	4.8089	4.8039	4.7978	4.7914	4.7853	4.7792	4.7733	4.7678	4.7575	4.7481
	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077	0.5077
	0.5836	0.5877	0.5934	0.6008	0.6097	0.6327	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	0.9755	1.1322	1.3150
	0.5820	0.5873	0.5955	0.6063	0.6189	0.6480	0.6783	0.7061	0.7295	0.7478	0.7606	0.7686	0.7725	0.7706	0.7602
$v=10,\ \lambda_d=1$	0.5759	0.5814	0.5891	0.5977	0.6069	0.6236	0.6363	0.6442	0.6478	0.6480	0.6455	0.6414	0.6361	0.6236	0.6108
	0.5452	0.5469	0.5484	0.5494	0.5495	0.5484	0.5459	0.5428	0.5394	0.5363	0.5331	0.5303	0.5281	0.5239	0.5209
	0.5183	0.5183	0.5180	0.5175	0.5167	0.5155	0.5142	0.5133	0.5123	0.5116	0.5109	0.5106	0.5100	0.5095	0.5089
	0.5353	0.5361	0.5366	0.5364	0.5359	0.5339	0.5316	0.5287	0.5263	0.5241	0.5220	0.5203	0.5188	0.5164	0.5147
	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826	0.0826
	0.0832	0.0873	0.0930	0.1004	0.1093	0.1322	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.4750	0.6318	0.8147
	0.0839	0.0887	0.0948	0.1017	0.1091	0.1239	0.1367	0.1459	0.1509	0.1520	0.1499	0.1455	0.1394	0.1257	0.1130
$v=100,\ \lambda_d=1$	0.0851	0.0897	0.0943	0.0989	0.1029	0.1086	0.1112	0.1111	0.1090	0.1058	0.1021	0.0987	0.0954	0.0904	0.0868
	0.0859	0.0873	0.0880	0.0883	0.0883	0.0876	0.0866	0.0856	0.0848	0.0840	0.0836	0.0832	0.0830	0.0827	0.0826
	0.0839	0.0838	0.0837	0.0836	0.0834	0.0831	0.0829	0.0828	0.0827	0.0826	0.0826	0.0827	0.0826	0.0826	0.0826
	0.0854	0.0860	0.0862	0.0862	0.0860	0.0853	0.0845	0.0839	0.0835	0.0831	0.0830	0.0828	0.0827	0.0826	0.0825
	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
	0.0604	0.0645	0.0702	0.0776	0.0865	0.1094	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.4522	0.6090	0.7918
	0.0611	0.0659	0.0718	0.0784	0.0854	0.0989	0.1100	0.1175	0.1210	0.1208	0.1178	0.1128	0.1067	0.0939	0.0829
$v=\infty,\ \lambda_d=1$	0.0626	0.0671	0.0716	0.0757	0.0793	0.0841	0.0859	0.0853	0.0831	0.0801	0.0768	0.0738	0.0712	0.0673	0.0650
	0.0647	0.0660	0.0668	0.0670	0.0670	0.0664	0.0655	0.0648	0.0642	0.0637	0.0633	0.0632	0.0630	0.0630	0.0629
	0.0638	0.0638	0.0637	0.0635	0.0634	0.0632	0.0631	0.0630	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629	0.0629
	0.0646	0.0652	0.0654	0.0654	0.0653	0.0647	0.0641	0.0637	0.0634	0.0632	0.0631	0.0630	0.0629	0.0629	0.0629
	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269	4.5269
	5.4261	5.4369	5.4492	5.4631	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.7919	5.8669	5.9514	6.1314	6.3406
	5.3978	5.4075	5.4208	5.4386	5.4606	5.5144	5.5769	5.6419	5.7053	5.7633	5.8158	5.8614	5.9008	5.9622	6.0042
$v=5,\ \lambda_d=3$	5.3067	5.3139	5.3261	5.3422	5.3608	5.4003	5.4375	5.4692	5.4939	5.5128	5.5261	5.5350	5.5403	5.5422	5.5369
	4.9283	4.9300	4.9322	4.9342	4.9356	4.9350	4.9311	4.9250	4.9181	4.9106	4.9028	4.8950	4.8875	4.8731	4.8597
	4.6361	4.6358	4.6350	4.6336	4.6319	4.6281	4.6242	4.6289	4.6169	4.6136	4.6108	4.6081	4.6056	4.6014	4.5978
	4.8175	4.8181	4.8186	4.8183	4.8172	4.8133	4.8072	4.8008	4.7939	4.7869	4.7803	4.7739	4.7678	4.7567	4.7467
	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092	0.5092
	0.6097	0.6203	0.6327	0.6464	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	0.9755	1.0506	1.1322	1.3150	1.5241
	0.6077	0.6178	0.6300	0.6439	0.6594	0.6931	0.7277	0.7594	0.7866	0.8077	0.8225	0.8313	0.8356	0.8319	0.8175
$v=10,\ \lambda_d=3$	0.5997	0.6078	0.6173	0.6275	0.6380	0.6567	0.6708	0.6794	0.6828	0.6822	0.6784	0.6727	0.6655	0.6491	0.6325
	0.5597	0.5611	0.5622	0.5627	0.5623	0.5602	0.5564	0.5522	0.5478	0.5434	0.5397	0.5363	0.5331	0.5283	0.5244
	0.5239	0.5234	0.5227	0.5219	0.5208	0.5189	0.5173	0.5158	0.5148	0.5139	0.5131	0.5127	0.5120	0.5113	0.5109
	0.5466	0.5469	0.5467	0.5459	0.5450	0.5419	0.5384	0.5348	0.5316	0.5287	0.5263	0.5241	0.5222	0.5191	0.5172
	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886	0.0886
	0.1093	0.1200	0.1322	0.1461	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.4750	0.5502	0.6318	0.8147	1.0236
	0.1095	0.1198	0.1312	0.1431	0.1549	0.1775	0.1961	0.2092	0.2160	0.2170	0.2129	0.2051	0.1948	0.1710	0.1481
$v=100,\ \lambda_d=3$	0.1092	0.1178	0.1257	0.1328	0.1386	0.1463	0.1486	0.1465	0.1415	0.1349	0.1278	0.1209	0.1144	0.1043	0.0978
	0.1029	0.1042	0.1045	0.1040	0.1031	0.1006	0.0978	0.0953	0.0933	0.0918	0.0908	0.0901	0.0895	0.0890	0.0888
	0.0935	0.0928	0.0920	0.0914	0.0909	0.0900	0.0894	0.0891	0.0889	0.0888	0.0888	0.0887	0.0886	0.0887	0.0886
	0.0998	0.0999	0.0993	0.0985	0.0975	0.0952	0.0933	0.0918	0.0907	0.0900	0.0894	0.0891	0.0889	0.0888	0.0887
	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0865	0.0971	0.1094	0.1233	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.4522	0.5274	0.6090	0.7918	1.0008
	0.0868	0.0972	0.1084	0.1201	0.1317	0.1533	0.1704	0.1817	0.1868	0.1860	0.1805	0.1715	0.1605	0.1364	0.1146
$v=\infty,\ \lambda_d=3$	0.0868	0.0953	0.1032	0.1100	0.1154	0.1221	0.1234	0.1205	0.1151	0.1083	0.1012	0.0946	0.0889	0.0802	0.0749
	0.0819	0.0833	0.0835	0.0830	0.0820	0.0794	0.0768	0.0745	0.0728	0.0715	0.0707	0.0703	0.0699	0.0696	0.0694
	0.0738	0.0731	0.0724	0.0717	0.0712	0.0705	0.0700	0.0697	0.0696	0.0694	0.0694	0.0694	0.0694	0.0694	0.0694
	0.0793	0.0794	0.0788	0.0779	0.0769	0.0748	0.0730	0.0717	0.0708	0.0702	0.0699	0.0697	0.0695	0.0694	0.0694
	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097	4.7097
	5.7233	5.7569	5.7919	5.8286	5.8669	5.9486	6.0369	6.1314	6.2328	6.3406	6.4547	6.5756	6.7031	6.9772	7.2778
	5.6964	5.7261	5.7572	5.7897	5.8236	5.8953	5.9717	6.0508	6.1297	6.2053	6.2747	6.3364	6.3892	6.4697	6.5200
$v=5,\ \lambda_d=10$	5.6039	5.6247	5.6464	5.6692	5.6928	5.7408	5.7872	5.8281	5.8606	5.8853	5.9019	5.9117	5.9156	5.9114	5.8964
	5.1897	5.1894	5.1903	5.1911	5.1911	5.1886	5.1819	5.1722	5.1614	5.1497	5.1381	5.1267	5.1156	5.0953	5.0769
	4.8394	4.8425	4.8397	4.8369	4.8342	4.8275	4.8217	4.8161	4.8111	4.8064	4.8025	4.7989	4.7956	4.7900	4.7861
	5.0614	5.0589	5.0569	5.0550	5.0525	5.0450	5.0358	5.0256	5.0150	5.0047	4.9953	4.9861	4.9778	4.9625	4.9494
	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206	0.7206
	0.9069	0.9403	0.9755	1.0122	1.0506	1.1322	1.2203	1.3150	1.4163	1.5241	1.6383	1.7591	1.8864	2.1608	2.4612
	0.9050	0.9370	0.9698	1.0033	1.0372	1.1050	1.1702	1.2295	1.2806	1.3216	1.3514	1.3700	1.3781	1.3692	1.3358
$v=10,\ \lambda_d=10$	0.8959	0.9222	0.9473	0.9713	0.9934	1.0309	1.0575	1.0730	1.0781	1.0745	1.0642	1.0494	1.0317	0.9928	0.9544
	0.8336	0.8377	0.8394	0.8389	0.8367	0.8289	0.8183	0.8070	0.7961	0.7861	0.7773	0.7697	0.7633	0.7530	0.7455
	0.7589	0.7555	0.7520	0.7488	0.7458	0.7405	0.7363	0.7330	0.7306	0.7288	0.7272	0.7259	0.7252	0.7238	0.7230
	0.8084	0.8081	0.8061	0.8030	0.7989	0.7897	0.7800	0.7713	0.7634	0.7567	0.7513	0.7467	0.7428	0.7370	0.7330
	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158	0.3158
	0.4065	0.4399	0.4750	0.5118	0.5502	0.6318	0.7199	0.8147	0.9159	1.0236	1.1380	1.2587	1.3861	1.6603	1.9607
	0.4064	0.4392	0.4729	0.5071	0.5410	0.6068	0.6659	0.7147	0.7504	0.7718	0.7788	0.7729	0.7556	0.6972	0.6233
$v=100,\ \lambda_d=10$	0.4050	0.4339	0.4611	0.4856	0.5070	0.5387	0.5549	0.5566	0.5466	0.5284	0.5051	0.4797	0.4541	0.4088	0.3744
	0.3835	0.3900	0.3922	0.3912	0.3878	0.3768	0.3641	0.3522	0.3423	0.3345	0.3288	0.3246	0.3216	0.3183	0.3168
	0.3428	0.3389	0.3351	0.3315	0.3285	0.3237	0.3206	0.3186	0.3175	0.3167	0.3163	0.3162	0.3159	0.3158	0.3158
	0.3713	0.3722	0.3703	0.3665	0.3616	0.3511	0.3415	0.3337	0.3280	0.3238	0.3210	0.3191	0.3179	0.3167	0.3162
	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980	0.2980
	0.3837	0.4171	0.4522	0.4890	0.5274	0.6090	0.6971	0.7918	0.8931	1.0008	1.1151	1.2359	1.3633	1.6376	1.9380
	0.3836	0.4165	0.4503	0.4845	0.5185	0.5841	0.6427	0.6905	0.7248	0.7445	0.7496	0.7412	0.7216	0.6589	0.5819
$v=\infty,\ \lambda_d=10$	0.3824	0.4117	0.4390	0.4636	0.4849	0.5160	0.5312	0.5318	0.5205	0.5010	0.4767	0.4504	0.4246	0.3794	0.3464
	0.3631	0.3699	0.3722	0.3710	0.3675	0.3563	0.3434	0.3316	0.3218	0.3143	0.3088	0.3051	0.3025	0.2997	0.2987
	0.3243	0.3204	0.3165	0.3140	0.3099	0.3053	0.3022	0.3004	0.2994	0.2987	0.2984	0.2981	0.2980	0.2980	0.2980
	0.3517	0.3528	0.3508	0.3469	0.3420	0.3314	0.3217	0.3141	0.3085	0.3047	0.3021	0.3004	0.2994	0.2985	0.2981

TABLE A5.2.5: Relative Risks of $\tilde{\sigma}^2_M$, σ^{*2}_M, and ∂^2_M

$v = 30$, $k = 5$, $m = 1$

Estimator							λ_n								
	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
$v=5$, $\lambda_d=0$	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819
	5.3872	5.3881	5.3908	5.3956	5.4019	5.4203	5.4461	5.4792	5.5194	5.5672	5.6222	5.6847	5.7544	5.9161	6.1072
	5.3681	5.3736	5.3864	5.4036	5.4239	5.4672	5.5097	5.5475	5.5808	5.6089	5.6331	5.6533	5.6700	5.6958	5.7139
	5.3206	5.3256	5.3342	5.3447	5.3556	5.3761	5.3933	5.4072	5.4183	5.4275	5.4347	5.4406	5.4450	5.4519	5.4564
	5.2289	5.2303	5.2328	5.2361	5.2397	5.2467	5.2531	5.2589	5.2642	5.2686	5.2731	5.2767	5.2800	5.2861	5.2911
	5.3017	5.3025	5.3044	5.3064	5.3086	5.3131	5.3167	5.3197	5.3228	5.3253	5.3275	5.3294	5.3314	5.3344	5.3369
	5.2278	5.2289	5.2317	5.2350	5.2383	5.2453	5.2517	5.2578	5.2628	5.2675	5.2717	5.2756	5.2789	5.2850	5.2900
$v=10$, $\lambda_d=0$	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859
	0.5839	0.5848	0.5877	0.5922	0.5986	0.6170	0.6428	0.6758	0.7162	0.7639	0.8191	0.8816	0.9513	1.1130	1.3039
	0.5828	0.5859	0.5920	0.6003	0.6094	0.6275	0.6430	0.6545	0.6620	0.6666	0.6684	0.6683	0.6669	0.6616	0.6548
	0.5792	0.5816	0.5853	0.5897	0.5938	0.6003	0.6044	0.6067	0.6075	0.6075	0.6069	0.6056	0.6047	0.6022	0.6000
	0.5711	0.5716	0.5727	0.5738	0.5750	0.5772	0.5789	0.5813	0.5822	0.5828	0.5833	0.5836	0.5842	0.5848	0.5848
	0.5781	0.5786	0.5792	0.5800	0.5808	0.5819	0.5828	0.5834	0.5839	0.5844	0.5847	0.5848	0.5850	0.5853	0.5855
	0.5711	0.5716	0.5725	0.5738	0.5748	0.5770	0.5788	0.5802	0.5813	0.5820	0.5827	0.5833	0.5836	0.5842	0.5847
$v=100$, $\lambda_d=0$	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
	0.0835	0.0844	0.0872	0.0917	0.0982	0.1165	0.1422	0.1753	0.2157	0.2634	0.3185	0.3810	0.4507	0.6125	0.8034
	0.0839	0.0856	0.0882	0.0912	0.0941	0.0990	0.1017	0.1025	0.1016	0.0998	0.0975	0.0954	0.0932	0.0900	0.0879
	0.0843	0.0855	0.0866	0.0877	0.0884	0.0892	0.0892	0.0887	0.0881	0.0874	0.0869	0.0864	0.0861	0.0857	0.0855
	0.0844	0.0845	0.0847	0.0848	0.0850	0.0851	0.0853	0.0853	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
	0.0849	0.0850	0.0851	0.0852	0.0853	0.0853	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
	0.0844	0.0845	0.0847	0.0848	0.0849	0.0851	0.0853	0.0853	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
$v=\infty$, $\lambda_d=0$	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0606	0.0615	0.0643	0.0689	0.0753	0.0937	0.1194	0.1524	0.1928	0.2406	0.2957	0.3518	0.4279	0.5895	0.7300
	0.0611	0.0626	0.0649	0.0675	0.0699	0.0736	0.0754	0.0755	0.0742	0.0725	0.0705	0.0687	0.0671	0.0649	0.0635
	0.0617	0.0627	0.0636	0.0644	0.0649	0.0653	0.0651	0.0647	0.0641	0.0637	0.0633	0.0630	0.0627	0.0626	0.0625
	0.0622	0.0622	0.0622	0.0623	0.0623	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0622	0.0622	0.0623	0.0623	0.0624	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0622	0.0622	0.0622	0.0623	0.0623	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
$v=5$, $\lambda_d=1$	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858
	5.3908	5.3956	5.4019	5.4103	5.4203	5.4461	5.4792	5.5194	5.5672	5.6222	5.6847	5.7544	5.8317	6.0081	6.2136
	5.3711	5.3781	5.3911	5.4089	5.4297	5.4747	5.5189	5.5589	5.5933	5.6225	5.6475	5.6681	5.6853	5.7117	5.7297
	5.3231	5.3281	5.3375	5.3483	5.3597	5.3811	5.3992	5.4136	5.4253	5.4344	5.4417	5.4475	5.4522	5.4589	5.4631
	5.2297	5.2311	5.2339	5.2375	5.2414	5.2486	5.2553	5.2614	5.2667	5.2714	5.2758	5.2797	5.2833	5.2894	5.2944
	5.3039	5.3050	5.3069	5.3092	5.3114	5.3158	5.3197	5.3231	5.3261	5.3286	5.3308	5.3331	5.3350	5.3381	5.3408
	5.2283	5.2300	5.2328	5.2364	5.2400	5.2472	5.2542	5.2603	5.2656	5.2703	5.2747	5.2786	5.2822	5.2883	5.2936
$v=10$, $\lambda_d=1$	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898
	0.5877	0.5922	0.5986	0.6069	0.6170	0.6428	0.6758	0.7162	0.7639	0.8191	0.8816	0.9513	1.0284	1.2047	1.4105
	0.5861	0.5914	0.5991	0.6084	0.6186	0.6386	0.6552	0.6677	0.6756	0.6802	0.6817	0.6813	0.6794	0.6730	0.6650
	0.5819	0.5853	0.5900	0.5948	0.5994	0.6066	0.6111	0.6133	0.6141	0.6138	0.6130	0.6117	0.6103	0.6075	0.6048
	0.5727	0.5736	0.5750	0.5764	0.5778	0.5803	0.5823	0.5839	0.5850	0.5859	0.5866	0.5872	0.5875	0.5881	0.5886
	0.5808	0.5814	0.5823	0.5833	0.5841	0.5855	0.5864	0.5872	0.5878	0.5881	0.5884	0.5888	0.5889	0.5892	0.5894
	0.5725	0.5734	0.5748	0.5763	0.5777	0.5802	0.5822	0.5838	0.5848	0.5858	0.5864	0.5870	0.5875	0.5881	0.5886
$v=100$, $\lambda_d=1$	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893
	0.0872	0.0917	0.0982	0.1064	0.1165	0.1422	0.1753	0.2157	0.2634	0.3185	0.3810	0.4507	0.5279	0.7042	0.9099
	0.0874	0.0916	0.0964	0.1011	0.1054	0.1117	0.1150	0.1152	0.1135	0.1105	0.1071	0.1038	0.1007	0.0959	0.0930
	0.0873	0.0899	0.0920	0.0937	0.0949	0.0957	0.0954	0.0944	0.0935	0.0924	0.0915	0.0909	0.0905	0.0897	0.0894
	0.0863	0.0870	0.0876	0.0880	0.0884	0.0888	0.0891	0.0891	0.0892	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893
	0.0877	0.0882	0.0885	0.0888	0.0889	0.0891	0.0892	0.0892	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893
	0.0863	0.0870	0.0876	0.0880	0.0883	0.0888	0.0890	0.0891	0.0892	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893
$v=\infty$, $\lambda_d=1$	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664
	0.0643	0.0689	0.0753	0.0836	0.0937	0.1194	0.1524	0.1928	0.2406	0.2957	0.3581	0.4279	0.5051	0.6814	0.8871
	0.0645	0.0687	0.0732	0.0775	0.0812	0.0865	0.0885	0.0880	0.0857	0.0827	0.0796	0.0765	0.0740	0.0703	0.0682
	0.0646	0.0671	0.0691	0.0705	0.0713	0.0718	0.0712	0.0703	0.0692	0.0684	0.0677	0.0673	0.0669	0.0667	0.0666
	0.0641	0.0647	0.0652	0.0656	0.0658	0.0661	0.0663	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664
	0.0642	0.0649	0.0654	0.0657	0.0659	0.0662	0.0663	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664
	0.0640	0.0647	0.0652	0.0656	0.0658	0.0661	0.0663	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664
$v=5$, $\lambda_d=3$	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172
	5.4203	5.4322	5.4461	5.4617	5.4792	5.5194	5.5672	5.6222	5.6847	5.7544	5.8317	5.9161	6.0081	6.2136	6.4486
	5.4003	5.4106	5.4256	5.4447	5.4672	5.5156	5.5633	5.6067	5.6442	5.6758	5.7022	5.7239	5.7500	5.7683	5.7858
	5.3506	5.3569	5.3672	5.3794	5.3919	5.4153	5.4350	5.4506	5.4631	5.4725	5.4800	5.4858	5.4903	5.4967	5.5006
	5.2536	5.2556	5.2592	5.2633	5.2675	5.2756	5.2831	5.2897	5.2956	5.3006	5.3050	5.3092	5.3131	5.3192	5.3247
	5.3311	5.3328	5.3350	5.3378	5.3403	5.3453	5.3494	5.3531	5.3561	5.3589	5.3614	5.3636	5.3656	5.3689	5.3717
	5.2522	5.2544	5.2581	5.2619	5.2664	5.2744	5.2819	5.2883	5.2942	5.2994	5.3039	5.3081	5.3117	5.3183	5.3236
$v=10$, $\lambda_d=3$	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211
	0.6170	0.6289	0.6428	0.6583	0.6758	0.7162	0.7639	0.8191	0.8816	0.9513	1.0284	1.1130	1.2047	1.4105	1.6455
	0.6152	0.6255	0.6372	0.6497	0.6627	0.6867	0.7066	0.7208	0.7297	0.7344	0.7355	0.7342	0.7311	0.7219	0.7114
	0.6098	0.6163	0.6230	0.6294	0.6352	0.6436	0.6488	0.6511	0.6516	0.6508	0.6494	0.6477	0.6459	0.6420	0.6388
	0.5980	0.6002	0.6025	0.6047	0.6067	0.6100	0.6123	0.6142	0.6156	0.6166	0.6173	0.6180	0.6186	0.6194	0.6198
	0.6089	0.6103	0.6119	0.6131	0.6142	0.6161	0.6173	0.6181	0.6188	0.6192	0.6197	0.6198	0.6202	0.6203	0.6206
	0.5978	0.6000	0.6023	0.6045	0.6064	0.6097	0.6122	0.6141	0.6155	0.6166	0.6173	0.6180	0.6184	0.6192	0.6197
$v=100$, $\lambda_d=3$	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206
	0.1165	0.1285	0.1422	0.1579	0.1753	0.2157	0.2634	0.3185	0.3810	0.4507	0.5279	0.6125	0.7042	0.9099	1.1450
	0.1164	0.1266	0.1364	0.1454	0.1531	0.1641	0.1691	0.1691	0.1658	0.1602	0.1542	0.1480	0.1424	0.1337	0.1279
	0.1156	0.1218	0.1266	0.1299	0.1321	0.1336	0.1327	0.1307	0.1286	0.1266	0.1248	0.1236	0.1227	0.1215	0.1210
	0.1125	0.1147	0.1164	0.1176	0.1185	0.1195	0.1201	0.1204	0.1205	0.1205	0.1206	0.1206	0.1206	0.1206	0.1206
	0.1162	0.1177	0.1187	0.1193	0.1197	0.1202	0.1204	0.1205	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206
	0.1125	0.1146	0.1163	0.1176	0.1184	0.1194	0.1200	0.1203	0.1205	0.1205	0.1206	0.1206	0.1206	0.1206	0.1206
$v=\infty$, $\lambda_d=3$	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977
	0.0937	0.1056	0.1194	0.1350	0.1524	0.1928	0.2406	0.2957	0.3581	0.4279	0.5051	0.5895	0.6814	0.8871	1.1221
	0.0936	0.1038	0.1135	0.1293	0.1351	0.1392	0.1429	0.1418	0.1376	0.1317	0.1254	0.1194	0.1142	0.1067	0.1019
	0.0929	0.0992	0.1038	0.1069	0.1088	0.1097	0.1084	0.1063	0.1041	0.1022	0.1007	0.0997	0.0989	0.0981	0.0977
	0.0903	0.0926	0.0941	0.0952	0.0960	0.0969	0.0973	0.0975	0.0976	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977
	0.0908	0.0930	0.0946	0.0956	0.0963	0.0970	0.0974	0.0976	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977
	0.0902	0.0925	0.0941	0.0952	0.0960	0.0969	0.0973	0.0975	0.0976	0.0976	0.0976	0.0976	0.0976	0.0977	0.0977
$v=5$, $\lambda_d=10$	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725
	5.7544	5.7922	5.8317	5.8731	5.9161	6.0081	6.1072	6.2136	6.3275	6.4486	6.5772	6.7131	6.8564	7.1650	7.5028
	5.7347	5.7653	5.7969	5.8294	5.8625	5.9292	5.9928	6.0500	6.0989	6.1389	6.1711	6.1961	6.2156	6.2408	6.2542
	5.6825	5.7011	5.7206	5.7394	5.7578	5.7906	5.8167	5.8364	5.8508	5.8611	5.8686	5.8736	5.8775	5.8811	5.8828
	5.5728	5.5803	5.5878	5.5953	5.6022	5.6144	5.6250	5.6336	5.6411	5.6475	5.6531	5.6581	5.6625	5.6700	5.6758
	5.6672	5.6719	5.6769	5.6814	5.6858	5.6928	5.6986	5.7033	5.7072	5.7108	5.7136	5.7161	5.7183	5.7222	5.7253
	5.5714	5.5786	5.5864	5.5936	5.6006	5.6128	5.6233	5.6322	5.6397	5.6461	5.6517	5.6567	5.6611	5.6686	5.6750
$v=10$, $\lambda_d=10$	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766
	0.9513	0.9889	1.0284	1.0697	1.1130	1.2047	1.3039	1.4105	1.5242	1.6455	1.7741	1.9100	2.0533	2.3617	2.6997
	0.9494	0.9828	1.0155	1.0469	1.0764	1.1280	1.1675	1.1944	1.2100	1.2159	1.2147	1.2084	1.1986	1.1741	1.1477
	0.9425	0.9655	0.9852	1.0016	1.0147	1.0323	1.0409	1.0433	1.0417	1.0381	1.0333	1.0284	1.0236	1.0145	1.0072
	0.9219	0.9312	0.9389	0.9452	0.9502	0.9575	0.9625	0.9659	0.9683	0.9700	0.9714	0.9723	0.9731	0.9741	0.9748
	0.9470	0.9534	0.9583	0.9619	0.9645	0.9684	0.9708	0.9722	0.9733	0.9741	0.9745	0.9750	0.9753	0.9756	0.9759
	0.9214	0.9309	0.9386	0.9447	0.9497	0.9572	0.9622	0.9656	0.9681	0.9698	0.9713	0.9722	0.9730	0.9741	0.9747
$v=100$, $\lambda_d=10$	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761
	0.4507	0.4884	0.5279	0.5692	0.6125	0.7042	0.8034	0.9099	1.0238	1.1450	1.2735	1.4095	1.5527	1.8613	2.1992
	0.4505	0.4853	0.5186	0.5495	0.5771	0.6205	0.6468	0.6572	0.6549	0.6424	0.6247	0.6044	0.5837	0.5469	0.5193
	0.4485	0.4738	0.4938	0.5086	0.5190	0.5285	0.5279	0.5219	0.5137	0.5055	0.4982	0.4923	0.4877	0.4817	0.4786
	0.4389	0.4495	0.4572	0.4627	0.4666	0.4714	0.4737	0.4749	0.4754	0.4757	0.4759	0.4759	0.4759	0.4761	0.4761
	0.4556	0.4629	0.4676	0.4706	0.4725	0.4746	0.4754	0.4757	0.4759	0.4759	0.4761	0.4761	0.4761	0.4761	0.4761
	0.4387	0.4493	0.4569	0.4624	0.4664	0.4712	0.4736	0.4748	0.4754	0.4757	0.4758	0.4759	0.4759	0.4761	0.4761
$v=\infty$, $\lambda_d=10$	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531
	0.4279	0.4656	0.5051	0.5464	0.5895	0.6814	0.7805	0.8871	1.0009	1.1221	1.2507	1.3866	1.5298	1.8384	2.1763
	0.4277	0.4626	0.4960	0.5268	0.5542	0.5969	0.6221	0.6310	0.6267	0.6132	0.5941	0.5727	0.5514	0.5144	0.4883
	0.4259	0.4514	0.4714	0.4861	0.4962	0.5050	0.5036	0.4969	0.4883	0.4799	0.4726	0.4669	0.4625	0.4572	0.4548
	0.4169	0.4277	0.4353	0.4408	0.4446	0.4491	0.4512	0.4523	0.4528	0.4529	0.4530	0.4531	0.4531	0.4531	0.4531
	0.4192	0.4300	0.4374	0.4425	0.4459	0.4499	0.4516	0.4525	0.4529	0.4530	0.4531	0.4531	0.4531	0.4531	0.4531
	0.4167	0.4274	0.4351	0.4405	0.4443	0.4489	0.4511	0.4522	0.4527	0.4529	0.4530	0.4531	0.4531	0.4531	0.4531

TABLE A5.2.6: Relative Risks of $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$

$v = 30,\ k = 5,\ m = 3$

Estimator	λ_n 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819	5.3819
	5.3969	5.3978	5.4000	5.4042	5.4100	5.4261	5.4492	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.8669	6.0369
	5.3728	5.3769	5.3867	5.4017	5.4208	5.4675	5.5203	5.5739	5.6256	5.6733	5.7172	5.7567	5.7919	5.8514	5.8978
$v=5,\ \lambda_d=0$	5.3090	5.3100	5.3200	5.3331	5.3483	5.3806	5.4114	5.4394	5.4639	5.4847	5.5031	5.5183	5.5314	5.5517	5.5661
	5.1181	5.1197	5.1236	5.1283	5.1336	5.1442	5.1542	5.1639	5.1725	5.1803	5.1875	5.1939	5.1997	5.2103	5.2192
	5.1417	5.1436	5.1467	5.1511	5.1558	5.1656	5.1747	5.1831	5.1908	5.1978	5.2042	5.2100	5.2150	5.2244	5.2325
	5.0981	5.0994	5.1028	5.1072	5.1119	5.1222	5.1322	5.1417	5.1503	5.1581	5.1656	5.1722	5.1783	5.1892	5.1986
	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859	0.5859
	0.5803	0.5813	0.5836	0.5877	0.5934	0.6097	0.6327	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	1.0506	1.2203
	0.5792	0.5817	0.5872	0.5952	0.6050	0.6286	0.6534	0.6767	0.6967	0.7131	0.7255	0.7342	0.7400	0.7439	0.7411
$v=10,\ \lambda_d=0$	0.5747	0.5773	0.5823	0.5886	0.5955	0.6088	0.6202	0.6287	0.6345	0.6381	0.6398	0.6403	0.6397	0.6367	0.6327
	0.5592	0.5600	0.5616	0.5633	0.5653	0.5691	0.5722	0.5747	0.5767	0.5783	0.5797	0.5806	0.5814	0.5827	0.5834
	0.5617	0.5623	0.5638	0.5655	0.5672	0.5705	0.5733	0.5753	0.5772	0.5788	0.5798	0.5806	0.5814	0.5828	0.5833
	0.5575	0.5580	0.5594	0.5609	0.5628	0.5664	0.5695	0.5722	0.5744	0.5761	0.5777	0.5788	0.5798	0.5813	0.5823
	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
	0.0800	0.0808	0.0832	0.0873	0.0930	0.1093	0.1322	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.5502	0.7199
	0.0807	0.0824	0.0854	0.0894	0.0940	0.1038	0.1128	0.1194	0.1233	0.1246	0.1238	0.1213	0.1180	0.1100	0.1025
$v=100,\ \lambda_d=0$	0.0818	0.0835	0.0856	0.0880	0.0903	0.0939	0.0961	0.0969	0.0965	0.0955	0.0940	0.0925	0.0912	0.0887	0.0875
	0.0828	0.0831	0.0834	0.0838	0.0841	0.0847	0.0850	0.0852	0.0854	0.0854	0.0854	0.0854	0.0855	0.0854	0.0854
	0.0832	0.0833	0.0836	0.0839	0.0842	0.0847	0.0850	0.0852	0.0853	0.0854	0.0854	0.0854	0.0854	0.0854	0.0854
	0.0828	0.0829	0.0832	0.0835	0.0838	0.0843	0.0848	0.0850	0.0852	0.0853	0.0853	0.0854	0.0854	0.0854	0.0854
	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0571	0.0580	0.0604	0.0645	0.0702	0.0865	0.1094	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.5274	0.6971
	0.0580	0.0596	0.0624	0.0660	0.0702	0.0787	0.0861	0.0912	0.0939	0.0942	0.0915	0.0897	0.0863	0.0791	0.0729
$v=\infty,\ \lambda_d=0$	0.0594	0.0608	0.0628	0.0648	0.0667	0.0696	0.0711	0.0713	0.0707	0.0696	0.0683	0.0671	0.0660	0.0643	0.0633
	0.0610	0.0612	0.0614	0.0617	0.0618	0.0622	0.0624	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0611	0.0613	0.0615	0.0617	0.0619	0.0622	0.0623	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	0.0610	0.0611	0.0613	0.0614	0.0616	0.0619	0.0622	0.0623	0.0624	0.0625	0.0625	0.0625	0.0625	0.0625	0.0625
	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858	5.3858
	5.4000	5.4042	5.4100	5.4172	5.4261	5.4492	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.7919	5.9486	6.1314
	5.3756	5.3811	5.3911	5.4061	5.4250	5.4725	5.5264	5.5819	5.6358	5.6864	5.7325	5.7739	5.8108	5.8728	5.9206
$v=5,\ \lambda_d=1$	5.3067	5.3117	5.3217	5.3347	5.3503	5.3836	5.4161	5.4456	5.4714	5.4936	5.5125	5.5283	5.5419	5.5628	5.5775
	5.1164	5.1181	5.1219	5.1269	5.1325	5.1436	5.1544	5.1644	5.1733	5.1817	5.1889	5.1958	5.2019	5.2125	5.2214
	5.1406	5.1422	5.1458	5.1503	5.1553	5.1653	5.1750	5.1839	5.1922	5.1992	5.2058	5.2119	5.2172	5.2269	5.2350
	5.0961	5.0975	5.1008	5.1056	5.1106	5.1214	5.1319	5.1417	5.1508	5.1589	5.1667	5.1736	5.1800	5.1911	5.2008
	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898	0.5898
	0.5836	0.5877	0.5934	0.6008	0.6097	0.6327	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	0.9755	1.1322	1.3150
	0.5820	0.5867	0.5939	0.6033	0.6144	0.6402	0.6670	0.6925	0.7144	0.7322	0.7458	0.7552	0.7613	0.7653	0.7616
$v=10,\ \lambda_d=1$	0.5766	0.5806	0.5866	0.5938	0.6013	0.6161	0.6286	0.6380	0.6445	0.6483	0.6502	0.6505	0.6497	0.6461	0.6413
	0.5588	0.5602	0.5622	0.5645	0.5667	0.5711	0.5747	0.5777	0.5798	0.5817	0.5831	0.5842	0.5852	0.5864	0.5873
	0.5616	0.5628	0.5647	0.5669	0.5689	0.5727	0.5759	0.5784	0.5803	0.5820	0.5833	0.5844	0.5850	0.5864	0.5870
	0.5567	0.5578	0.5597	0.5619	0.5641	0.5681	0.5719	0.5748	0.5773	0.5792	0.5809	0.5822	0.5833	0.5848	0.5861
	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893	0.0893
	0.0832	0.0873	0.0930	0.1004	0.1093	0.1322	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.4750	0.6318	0.8147
	0.0836	0.0880	0.0934	0.0996	0.1064	0.1196	0.1311	0.1393	0.1440	0.1455	0.1440	0.1407	0.1358	0.1246	0.1143
$v=100,\ \lambda_d=1$	0.0840	0.0876	0.0913	0.0950	0.0983	0.1034	0.1060	0.1068	0.1059	0.1041	0.1020	0.0999	0.0978	0.0945	0.0923
	0.0831	0.0842	0.0853	0.0861	0.0869	0.0880	0.0886	0.0890	0.0892	0.0893	0.0894	0.0893	0.0893	0.0893	0.0893
	0.0836	0.0847	0.0857	0.0864	0.0872	0.0881	0.0886	0.0889	0.0891	0.0892	0.0893	0.0893	0.0893	0.0893	0.0893
	0.0828	0.0838	0.0848	0.0856	0.0863	0.0875	0.0882	0.0886	0.0889	0.0891	0.0891	0.0892	0.0892	0.0893	0.0893
	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664	0.0664
	0.0604	0.0645	0.0702	0.0776	0.0865	0.1094	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.4522	0.6090	0.7918
	0.0609	0.0652	0.0705	0.0765	0.0828	0.0949	0.1049	0.1116	0.1149	0.1150	0.1125	0.1084	0.1032	0.0926	0.0832
$v=\infty,\ \lambda_d=1$	0.0615	0.0650	0.0686	0.0720	0.0749	0.0791	0.0811	0.0812	0.0800	0.0781	0.0759	0.0739	0.0722	0.0696	0.0678
	0.0613	0.0624	0.0634	0.0642	0.0648	0.0656	0.0661	0.0663	0.0664	0.0664	0.0664	0.0665	0.0664	0.0664	0.0664
	0.0617	0.0628	0.0637	0.0644	0.0649	0.0656	0.0660	0.0662	0.0662	0.0663	0.0664	0.0664	0.0664	0.0664	0.0664
	0.0611	0.0621	0.0630	0.0637	0.0643	0.0652	0.0658	0.0660	0.0662	0.0663	0.0663	0.0664	0.0664	0.0664	0.0664
	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172	5.4172
	5.4261	5.4369	5.4492	5.4631	5.4783	5.5144	5.5569	5.6058	5.6614	5.7233	5.7919	5.8669	5.9486	6.1314	6.3406
	5.4014	5.4108	5.4233	5.4400	5.4597	5.5083	5.5650	5.6244	5.6831	5.7383	5.7892	5.8347	5.8756	5.9425	5.9939
$v=5,\ \lambda_d=3$	5.3308	5.3375	5.3483	5.3622	5.3783	5.4142	5.4500	5.4822	5.5108	5.5353	5.5561	5.5731	5.5878	5.6092	5.6244
	5.1342	5.1367	5.1411	5.1467	5.1531	5.1656	5.1778	5.1886	5.1983	5.2075	5.2153	5.2228	5.2292	5.2406	5.2500
	5.1597	5.1617	5.1658	5.1711	5.1767	5.1881	5.1992	5.2089	5.2175	5.2253	5.2325	5.2389	5.2450	5.2553	5.2639
	5.1131	5.1153	5.1192	5.1244	5.1303	5.1425	5.1542	5.1650	5.1750	5.1839	5.1922	5.1994	5.2064	5.2183	5.2286
	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211	0.6211
	0.6097	0.6203	0.6327	0.6464	0.6620	0.6980	0.7403	0.7894	0.8448	0.9069	0.9755	1.0506	1.1322	1.3150	1.5241
	0.6078	0.6177	0.6294	0.6427	0.6572	0.6888	0.7213	0.7514	0.7778	0.7992	0.8155	0.8269	0.8341	0.8380	0.8323
$v=10,\ \lambda_d=3$	0.6011	0.6089	0.6177	0.6272	0.6370	0.6555	0.6708	0.6822	0.6898	0.6944	0.6963	0.6964	0.6948	0.6898	0.6834
	0.5789	0.5820	0.5855	0.5891	0.5923	0.5983	0.6030	0.6067	0.6095	0.6116	0.6133	0.6147	0.6156	0.6173	0.6181
	0.5825	0.5855	0.5888	0.5919	0.5950	0.6002	0.6044	0.6075	0.6102	0.6120	0.6134	0.6148	0.6158	0.6170	0.6181
	0.5761	0.5791	0.5823	0.5856	0.5888	0.5945	0.5994	0.6033	0.6063	0.6088	0.6108	0.6123	0.6136	0.6156	0.6169
	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206	0.1206
	0.1093	0.1200	0.1322	0.1461	0.1616	0.1975	0.2400	0.2889	0.3444	0.4065	0.4750	0.5502	0.6318	0.8147	1.0236
	0.1093	0.1195	0.1306	0.1421	0.1538	0.1759	0.1944	0.2080	0.2162	0.2191	0.2173	0.2122	0.2047	0.1865	0.1685
$v=100,\ \lambda_d=3$	0.1088	0.1170	0.1248	0.1320	0.1382	0.1473	0.1521	0.1534	0.1518	0.1488	0.1448	0.1410	0.1370	0.1308	0.1266
	0.1043	0.1080	0.1110	0.1133	0.1152	0.1178	0.1191	0.1200	0.1204	0.1205	0.1206	0.1206	0.1206	0.1207	0.1206
	0.1058	0.1091	0.1118	0.1139	0.1156	0.1177	0.1190	0.1196	0.1202	0.1204	0.1205	0.1205	0.1205	0.1206	0.1206
	0.1036	0.1069	0.1097	0.1120	0.1139	0.1165	0.1182	0.1191	0.1197	0.1201	0.1203	0.1204	0.1205	0.1206	0.1206
	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977	0.0977
	0.0865	0.0971	0.1094	0.1233	0.1388	0.1747	0.2171	0.2661	0.3216	0.3837	0.4522	0.5274	0.6090	0.7918	1.0008
	0.0866	0.0968	0.1078	0.1192	0.1306	0.1518	0.1692	0.1812	0.1877	0.1889	0.1858	0.1795	0.1712	0.1522	0.1347
$v=\infty,\ \lambda_d=3$	0.0863	0.0946	0.1023	0.1093	0.1152	0.1235	0.1274	0.1278	0.1257	0.1222	0.1181	0.1142	0.1105	0.1049	0.1014
	0.0827	0.0864	0.0892	0.0915	0.0933	0.0955	0.0967	0.0973	0.0975	0.0976	0.0977	0.0977	0.0977	0.0977	0.0977
	0.0840	0.0874	0.0900	0.0920	0.0935	0.0955	0.0965	0.0970	0.0974	0.0975	0.0976	0.0976	0.0976	0.0977	0.0977
	0.0820	0.0854	0.0882	0.0903	0.0921	0.0944	0.0959	0.0966	0.0971	0.0973	0.0975	0.0976	0.0976	0.0976	0.0976
	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725	5.7725
	5.7233	5.7569	5.7919	5.8286	5.8669	5.9486	6.0369	6.1314	6.2328	6.3406	6.4547	6.5756	6.7031	6.9772	7.2778
	5.7000	5.7303	5.7617	5.7944	5.8283	5.8997	5.9753	6.0531	6.1308	6.2056	6.2750	6.3378	6.3936	6.4839	6.5483
$v=5,\ \lambda_d=10$	5.6289	5.6519	5.6756	5.7000	5.7250	5.7758	5.8256	5.8714	5.9111	5.9453	5.9728	5.9953	6.0131	6.0378	6.0528
	5.4161	5.4261	5.4369	5.4481	5.4589	5.4797	5.4983	5.5144	5.5286	5.5406	5.5514	5.5608	5.5692	5.5833	5.5950
	5.4478	5.4569	5.4672	5.4772	5.4875	5.5061	5.5228	5.5369	5.5497	5.5603	5.5700	5.5783	5.5858	5.5986	5.6092
	5.3919	5.4014	5.4114	5.4219	5.4322	5.4522	5.4703	5.4864	5.5006	5.5131	5.5242	5.5342	5.5428	5.5583	5.5708
	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766	0.9766
	0.9069	0.9403	0.9755	1.0122	1.0506	1.1322	1.2203	1.3150	1.4163	1.5241	1.6383	1.7591	1.8864	2.1608	2.4612
	0.9052	0.9373	0.9703	1.0041	1.0383	1.1067	1.1731	1.2345	1.2886	1.3338	1.3689	1.3942	1.4102	1.4192	1.4056
$v=10,\ \lambda_d=10$	0.8981	0.9255	0.9522	0.9778	1.0023	1.0456	1.0803	1.1058	1.1227	1.1319	1.1353	1.1341	1.1298	1.1159	1.0992
	0.8659	0.8806	0.8937	0.9052	0.9152	0.9309	0.9425	0.9508	0.9569	0.9612	0.9644	0.9669	0.9688	0.9713	0.9728
	0.8750	0.8892	0.9016	0.9120	0.9209	0.9348	0.9448	0.9522	0.9575	0.9614	0.9644	0.9664	0.9684	0.9708	0.9722
	0.8609	0.8748	0.8872	0.8981	0.9077	0.9231	0.9348	0.9436	0.9503	0.9553	0.9592	0.9623	0.9647	0.9681	0.9703
	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761	0.4761
	0.4065	0.4399	0.4750	0.5118	0.5502	0.6318	0.7199	0.8147	0.9159	1.0236	1.1380	1.2587	1.3861	1.6603	1.9607
	0.4064	0.4393	0.4730	0.5073	0.5416	0.6087	0.6701	0.7226	0.7637	0.7922	0.8076	0.8113	0.8045	0.7680	0.7153
$v=100,\ \lambda_d=10$	0.4051	0.4348	0.4632	0.4898	0.5138	0.5531	0.5798	0.5942	0.5984	0.5949	0.5857	0.5736	0.5602	0.5344	0.5135
	0.3942	0.4116	0.4261	0.4377	0.4470	0.4600	0.4677	0.4721	0.4743	0.4755	0.4761	0.4762	0.4763	0.4762	0.4761
	0.4004	0.4174	0.4310	0.4416	0.4498	0.4611	0.4675	0.4713	0.4733	0.4746	0.4752	0.4756	0.4757	0.4759	0.4759
	0.3918	0.4081	0.4216	0.4327	0.4418	0.4548	0.4631	0.4682	0.4714	0.4733	0.4744	0.4751	0.4755	0.4758	0.4759
	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531	0.4531
	0.3837	0.4171	0.4522	0.4890	0.5274	0.6090	0.6971	0.7918	0.8931	1.0008	1.1151	1.2359	1.3633	1.6376	1.9380
	0.3836	0.4166	0.4504	0.4847	0.5191	0.5860	0.6470	0.6987	0.7387	0.7655	0.7793	0.7809	0.7719	0.7316	0.6756
$v=\infty,\ \lambda_d=10$	0.3826	0.4125	0.4411	0.4677	0.4917	0.5306	0.5565	0.5700	0.5731	0.5683	0.5583	0.5453	0.5313	0.5049	0.4846
	0.3728	0.3905	0.4050	0.4167	0.4259	0.4387	0.4461	0.4501	0.4520	0.4530	0.4533	0.4534	0.4534	0.4533	0.4533
	0.3788	0.3961	0.4097	0.4204	0.4285	0.4395	0.4457	0.4492	0.4511	0.4520	0.4526	0.4528	0.4529	0.4531	0.4531
	0.3705	0.3871	0.4007	0.4118	0.4208	0.4337	0.4416	0.4465	0.4493	0.4511	0.4519	0.4524	0.4528	0.4531	0.4531

TABLE A5.2.7: Relative Risks of $\tilde{\sigma}_L^2$, σ_L^{*2}, and $\hat{\sigma}_L^2$

$v = 16,\ k = 4,\ m = 1$

						λ_d									
Estimator	0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	6.5972	6.6272	6.6650	6.7106	6.7639	6.8942	7.0556	7.2483	7.4722	7.7275	8.0139	8.3317	8.6806	9.4722	10.3889
	6.5358	6.5625	6.5958	6.6364	6.6836	6.7989	6.9419	7.1128	7.3111	7.5372	7.7908	8.0722	8.3814	9.0828	9.8947
	6.5003	6.5261	6.5594	6.5994	6.6467	6.7619	6.9053	7.0761	7.2750	7.5017	7.7558	8.0378	8.3475	9.0497	9.8628
$\nu=5,\ \lambda_n=0$	6.4028	6.4275	6.4594	6.4986	6.5447	6.6589	6.8011	6.9717	7.1700	7.3964	7.6506	7.9328	8.2428	8.9458	9.7600
	6.2017	6.2231	6.2519	6.2878	6.3308	6.4383	6.5744	6.7389	6.9317	7.1525	7.4017	7.6792	7.9847	8.6797	9.4869
	6.3919	6.4175	6.4506	6.4908	6.5386	6.6567	6.8044	6.9819	7.1886	7.4250	7.6908	7.9858	8.3100	9.0458	9.8983
	6.2000	6.2214	6.2500	6.2858	6.3289	6.4364	6.5722	6.7367	6.9292	7.1503	7.3994	7.6767	7.9819	8.6769	9.4836
	0.7813	0.8047	0.8359	0.8750	0.9219	1.0391	1.1875	1.3672	1.5781	1.8203	2.0938	2.3984	2.7344	3.5000	4.3906
	0.7659	0.7867	0.8144	0.8489	0.8905	0.9944	1.1258	1.2850	1.4719	1.6864	1.9286	2.1984	2.4961	3.1742	3.9631
	0.7636	0.7839	0.8113	0.8458	0.8872	0.9908	1.1223	1.2817	1.4686	1.6833	1.9289	2.1958	2.4936	3.1722	3.9613
$\nu=10,\ \lambda_n=0$	0.7547	0.7741	0.8006	0.8344	0.8752	0.9780	1.1088	1.2678	1.4547	1.6694	1.9119	2.1823	2.4803	3.1597	3.9497
	0.7331	0.7500	0.7742	0.8055	0.8441	0.9423	1.0689	1.2239	1.4072	1.6186	1.8581	2.1256	2.4213	3.0966	3.8836
	0.7561	0.7761	0.8036	0.8384	0.8809	0.9878	1.1244	1.2903	1.4858	1.7106	1.9647	2.2481	2.5686	3.2730	4.1013
	0.7328	0.7498	0.7739	0.8053	0.8438	0.9419	1.0684	1.2234	1.4066	1.6178	1.8573	2.1248	2.4205	3.0955	3.8823
	0.1545	0.1744	0.2021	0.2375	0.2808	0.3909	0.5321	0.7046	0.9084	1.1434	1.4096	1.7072	2.0359	2.7872	3.6635
	0.1467	0.1643	0.1889	0.2202	0.2586	0.3561	0.4812	0.6340	0.8145	1.0227	1.2585	1.5221	1.8133	2.4788	3.2550
	0.1474	0.1646	0.1889	0.2201	0.2583	0.3556	0.4807	0.6335	0.8140	1.0223	1.2582	1.5218	1.8131	2.4787	3.2549
$\nu=100,\ \lambda_n=0$	0.1472	0.1637	0.1872	0.2179	0.2556	0.3524	0.4770	0.6296	0.8101	1.0184	1.2545	1.5183	1.8099	2.4759	3.2527
	0.1442	0.1588	0.1804	0.2092	0.2451	0.3383	0.4598	0.6095	0.7875	0.9936	1.2278	1.4900	1.7802	2.4444	3.2201
	0.1491	0.1661	0.1905	0.2224	0.2618	0.3626	0.4929	0.6526	0.8418	1.0603	1.3080	1.5850	1.8910	2.5902	3.4054
	0.1442	0.1587	0.1803	0.2091	0.2450	0.3381	0.4596	0.6092	0.7872	0.9932	1.2273	1.4895	1.7796	2.4437	3.2194
	0.1250	0.1445	0.1719	0.2070	0.2500	0.3594	0.5000	0.6719	0.8750	1.1094	1.3750	1.6719	2.0000	2.7500	3.6250
	0.1177	0.1350	0.1592	0.1903	0.2284	0.3253	0.4498	0.6021	0.7820	0.9896	1.2249	1.4879	1.7786	2.4429	3.2180
	0.1184	0.1354	0.1594	0.1903	0.2282	0.3250	0.4495	0.6017	0.7817	0.9893	1.2247	1.4877	1.7784	2.4428	3.2179
$\nu=\infty,\ \lambda_n=0$	0.1186	0.1348	0.1581	0.1885	0.2260	0.3221	0.4462	0.5983	0.7782	0.9860	1.2215	1.4847	1.7756	2.4406	3.2162
	0.1165	0.1308	0.1522	0.1808	0.2165	0.3092	0.4303	0.5795	0.7570	0.9627	1.1963	1.4581	1.7478	2.4110	3.1858
	0.1205	0.1372	0.1614	0.1930	0.2321	0.3323	0.4621	0.6213	0.8099	1.0279	1.2750	1.5513	1.8568	2.5549	3.3688
	0.1164	0.1307	0.1521	0.1807	0.2164	0.3091	0.4300	0.5793	0.7567	0.9623	1.1959	1.4576	1.7472	2.4103	3.1849
	6.5972	6.6272	6.6650	6.7106	6.7639	6.8942	7.0556	7.2483	7.4722	7.7275	8.0139	8.3317	8.6806	9.4722	10.3889
	6.5958	6.6364	6.6836	6.7378	6.7989	6.9419	7.1128	7.3111	7.5372	7.7908	8.0722	8.3814	8.7181	9.4747	10.3422
	6.5736	6.6050	6.6447	6.6928	6.7486	6.8842	7.0503	7.2461	7.4711	7.7244	8.0067	8.3167	8.6547	9.4142	10.2844
$\nu=5,\ \lambda_n=1$	6.4522	6.4797	6.5156	6.5589	6.6106	6.7367	6.8942	7.0819	7.3003	7.5483	7.8258	8.1328	8.4686	9.2261	10.0969
	6.2175	6.2403	6.2708	6.3086	6.3539	6.4669	6.6094	6.7814	6.9828	7.2136	7.4739	7.7633	8.0817	8.8058	9.6453
	6.4028	6.4294	6.4633	6.5050	6.5544	6.6761	6.8283	7.0108	7.2239	7.4672	7.7411	8.0453	8.3797	9.1394	10.0203
	6.2156	6.2383	6.2689	6.3067	6.3519	6.4647	6.6072	6.7792	6.9806	7.2111	7.4711	7.7603	8.0786	8.8022	9.6414
	0.7813	0.8047	0.8359	0.8750	0.9219	1.0391	1.1875	1.3672	1.5781	1.8203	2.0938	2.3984	2.7344	3.5000	4.3906
	0.8144	0.8489	0.8905	0.9389	0.9944	1.1258	1.2850	1.4719	1.6864	1.9286	2.1984	2.4961	2.8213	3.5548	4.3992
	0.8109	0.8403	0.8777	0.9227	0.9753	1.1033	1.2608	1.4472	1.6620	1.9053	2.1764	2.4755	2.8020	3.5386	4.3855
$\nu=10,\ \lambda_n=1$	0.7831	0.8078	0.8405	0.8809	0.9291	1.0488	1.1991	1.3794	1.5897	1.8294	2.0981	2.3958	2.7220	3.4598	4.3100
	0.7414	0.7608	0.7875	0.8216	0.8631	0.9686	1.1034	1.2680	1.4619	1.6850	1.9372	2.2188	2.5292	3.2372	4.0605
	0.7619	0.7833	0.8125	0.8492	0.8936	1.0053	1.1475	1.3203	1.5234	1.7570	2.0211	2.3153	2.6400	3.3800	4.2409
	0.7413	0.7605	0.7872	0.8212	0.8628	0.9681	1.1030	1.2673	1.4611	1.6841	1.9361	2.2175	2.5278	3.2355	4.0581
	0.1545	0.1744	0.2021	0.2375	0.2808	0.3909	0.5321	0.7046	0.9084	1.1434	1.4096	1.7072	2.0359	2.7872	3.6635
	0.1889	0.2202	0.2586	0.3038	0.3561	0.4812	0.6340	0.8145	1.0227	1.2585	1.5221	1.8133	2.1322	2.8531	3.6846
	0.1804	0.2087	0.2446	0.2880	0.3389	0.4627	0.6154	0.7966	1.0060	1.2433	1.5085	1.8013	2.1217	2.8454	3.6791
$\nu=100,\ \lambda_n=1$	0.1643	0.1873	0.2182	0.2570	0.3034	0.4194	0.5656	0.7419	0.9476	1.1827	1.4467	1.7394	2.0605	2.7875	3.6264
	0.1487	0.1659	0.1905	0.2226	0.2622	0.3634	0.4943	0.6545	0.8441	1.0630	1.3110	1.5882	1.8944	2.5935	3.4080
	0.1520	0.1708	0.1971	0.2312	0.2728	0.3790	0.5158	0.6832	0.8809	1.1091	1.3678	1.6567	1.9759	2.7053	3.5557
	0.1487	0.1659	0.1904	0.2225	0.2620	0.3632	0.4939	0.6540	0.8435	1.0623	1.3101	1.5872	1.8932	2.5919	3.4060
	0.1250	0.1445	0.1719	0.2070	0.2500	0.3594	0.5000	0.6719	0.8750	1.1094	1.3750	1.6719	2.0000	2.7500	3.6250
	0.1592	0.1903	0.2284	0.2734	0.3253	0.4498	0.6021	0.7820	0.9896	1.2249	1.4879	1.7786	2.0969	2.8166	3.6471
	0.1501	0.1783	0.2141	0.2574	0.3081	0.4316	0.5839	0.7646	0.9735	1.2103	1.4749	1.7672	2.0871	2.8095	3.6420
$\nu=\infty,\ \lambda_n=1$	0.1347	0.1576	0.1883	0.2269	0.2732	0.3888	0.5347	0.7106	0.9160	1.1506	1.4142	1.7064	2.0271	2.7532	3.5910
	0.1206	0.1376	0.1621	0.1940	0.2333	0.3342	0.4647	0.6245	0.8138	1.0323	1.2799	1.5567	1.8625	2.5610	3.3746
	0.1232	0.1416	0.1678	0.2015	0.2430	0.3487	0.4850	0.6519	0.8492	1.0769	1.3350	1.6235	1.9423	2.6707	3.5202
	0.1206	0.1376	0.1620	0.1939	0.2332	0.3340	0.4643	0.6241	0.8132	1.0316	1.2791	1.5557	1.8614	2.5593	3.3726
	6.5972	6.6272	6.6650	6.7106	6.7639	6.8942	7.0556	7.2483	7.4722	7.7275	8.0139	8.3317	8.6806	9.4722	10.3889
	7.1128	7.2083	7.3111	7.4206	7.5372	7.7908	8.0722	8.3814	8.7181	9.0828	9.4747	9.8947	10.3422	11.3203	12.4092
	7.0219	7.0728	7.1328	7.2025	7.2817	7.4694	7.6961	7.9619	8.2661	8.6081	8.9864	9.4008	9.8494	10.8442	11.9617
$\nu=5,\ \lambda_n=5$	6.6528	6.6897	6.7353	6.7892	6.8511	7.0011	7.1856	7.4044	7.6583	7.9469	8.2711	8.6306	9.0253	9.9214	10.9569
	6.2878	6.3144	6.3486	6.3906	6.4400	6.5628	6.7161	6.9000	7.1153	7.3608	7.6375	7.9450	8.2828	9.0508	9.9419
	6.4447	6.4733	6.5094	6.5536	6.6053	6.7322	6.8900	7.0794	7.2994	7.5508	7.8333	8.1472	8.4919	9.2747	10.1814
	6.2861	6.3125	6.3467	6.3886	6.4383	6.5606	6.7139	6.8978	7.1128	7.3583	7.6347	7.9417	8.2797	9.0475	9.9381
	0.7813	0.8047	0.8359	0.8750	0.9219	1.0391	1.1875	1.3672	1.5781	1.8203	2.0938	2.3984	2.7344	3.5000	4.3906
	1.2850	1.3750	1.4719	1.5756	1.6864	1.9286	2.1984	2.4961	2.8213	3.1742	3.5548	3.9631	4.3992	5.3542	6.4198
	1.0191	1.0702	1.1312	1.2028	1.2845	1.4792	1.7141	1.9889	2.3022	2.6530	3.0389	3.4598	3.9131	4.9136	6.0317
$\nu=10,\ \lambda_n=5$	0.8495	0.8812	0.9214	0.9703	1.0280	1.1697	1.3469	1.5598	1.8087	2.0939	2.4152	2.7728	3.1664	4.0611	5.0969
	0.7653	0.7878	0.8181	0.8559	0.9017	1.0164	1.1620	1.3386	1.5461	1.7845	2.0539	2.3539	2.6850	3.4395	4.3172
	0.7748	0.7978	0.8288	0.8673	0.9139	1.0302	1.1777	1.3564	1.5663	1.8073	2.0795	2.3830	2.7175	3.4802	4.3673
	0.7653	0.7877	0.8180	0.8558	0.9014	1.0159	1.1616	1.3380	1.5456	1.7839	2.0531	2.3544	2.6841	3.4383	4.3155
	0.1545	0.1744	0.2021	0.2375	0.2808	0.3909	0.5321	0.7046	0.9084	1.1434	1.4096	1.7072	2.0359	2.7872	3.6635
	0.6340	0.7207	0.8145	0.9151	1.0227	1.2585	1.5221	1.8133	2.1322	2.4788	2.8531	3.2550	3.6846	4.6269	5.6799
	0.2600	0.3096	0.3705	0.4423	0.5251	0.7229	0.9628	1.2429	1.5616	1.9170	2.3076	2.7317	3.1877	4.1914	5.3095
$\nu=100,\ \lambda_n=5$	0.1745	0.2010	0.2364	0.2807	0.3342	0.4683	0.6392	0.8475	1.0918	1.3738	1.6928	2.0487	2.4415	3.3352	4.3709
	0.1541	0.1737	0.2013	0.2366	0.2796	0.3889	0.5295	0.7011	0.9037	1.1374	1.4021	1.6977	2.0244	2.7704	3.6399
	0.1544	0.1743	0.2019	0.2373	0.2805	0.3905	0.5315	0.7039	0.9074	1.1422	1.4083	1.7054	2.0339	2.7845	3.6598
	0.1540	0.1738	0.2012	0.2364	0.2795	0.3889	0.5294	0.7009	0.9035	1.1370	1.4017	1.6973	2.0237	2.7696	3.6387
	0.1250	0.1445	0.1719	0.2070	0.2500	0.3594	0.5000	0.6719	0.8750	1.1094	1.3750	1.6719	2.0000	2.7500	3.6250
	0.6021	0.6886	0.7820	0.8824	0.9896	1.2249	1.4879	1.7786	2.0969	2.4429	2.8166	3.2180	3.6471	4.5882	5.6401
	0.2187	0.2679	0.3287	0.4006	0.4833	0.6816	0.9219	1.2025	1.5216	1.8776	2.2684	2.6928	3.1492	4.1526	5.2709
$\nu=\infty,\ \lambda_n=5$	0.1412	0.1670	0.2018	0.2457	0.2985	0.4318	0.6019	0.8092	1.0536	1.3352	1.6540	2.0098	2.4022	3.2959	4.3318
	0.1247	0.1442	0.1714	0.2064	0.2491	0.3580	0.4980	0.6690	0.8712	1.1045	1.3688	1.6640	1.9902	2.7353	3.6042
	0.1250	0.1445	0.1718	0.2069	0.2498	0.3591	0.4996	0.6714	0.8744	1.1086	1.3740	1.6706	1.9985	2.7479	3.6221
	0.1247	0.1441	0.1714	0.2063	0.2491	0.3580	0.4979	0.6690	0.8710	1.1042	1.3683	1.6635	1.9896	2.7347	3.6030

TABLE A5.2.8: Relative Risks of $\tilde{\sigma}^2_{ML}$, σ^{*2}_{ML}, and $\hat{\sigma}^2_{ML}$

$v = 16,\ k = 4,\ m = 1$

Estimator	0.0	0.5	1.0	1.5	2.0	3.0	λ_d 4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	4.3333	4.3192	4.3100	4.3058	4.3067	4.3233	4.3600	4.4167	4.4933	4.5900	4.7067	4.8433	5.0000	5.3733	5.8267
	4.7847	4.7789	4.7781	4.7822	4.7914	4.8247	4.8781	4.9514	5.0447	5.1581	5.2914	5.4447	5.6181	6.0247	6.5114
	4.7603	4.7536	4.7525	4.7561	4.7650	4.7981	4.8514	4.9247	5.0183	5.1317	5.2653	5.4189	5.5928	6.0000	6.4872
$\nu=5,\ \lambda_n=0$	4.6861	4.6783	4.6756	4.6783	4.6861	4.7172	4.7692	4.8417	4.9344	5.0475	5.1806	5.3342	5.5078	5.9153	6.4033
	4.4494	4.4375	4.4306	4.4289	4.4322	4.4544	4.4975	4.5611	4.6453	4.7500	4.8753	5.0208	5.1872	5.5808	6.0564
	4.3381	4.3239	4.3150	4.3108	4.3117	4.3286	4.3656	4.4225	4.4997	4.5967	4.7139	4.8511	5.0081	5.3828	5.8375
	4.4328	4.4203	4.4131	4.4111	4.4142	4.4358	4.4778	4.5406	4.6236	4.7275	4.8517	4.9964	5.1617	5.5533	6.0264
	0.5625	0.5525	0.5475	0.5475	0.5525	0.5775	0.6225	0.6875	0.7725	0.8775	1.0025	1.1475	1.3125	1.7025	2.1725
	0.5886	0.5848	0.5861	0.5923	0.6036	0.6411	0.6986	0.7761	0.8736	0.9911	1.1286	1.2861	1.4636	1.8786	2.3736
	0.5886	0.5842	0.5852	0.5911	0.6020	0.6392	0.6966	0.7741	0.8716	0.9891	1.1267	1.2844	1.4619	1.8770	2.3722
$\nu=10,\ \lambda_n=0$	0.5859	0.5805	0.5805	0.5855	0.5958	0.6317	0.6881	0.7650	0.8620	0.9794	1.1169	1.2745	1.4522	1.8677	2.3633
	0.5714	0.5630	0.5597	0.5616	0.5686	0.5981	0.6484	0.7192	0.8105	0.9222	1.0545	1.2073	1.3805	1.7881	2.2770
	0.5628	0.5530	0.5480	0.5481	0.5531	0.5784	0.6236	0.6889	0.7742	0.8797	1.0052	1.1506	1.3161	1.7072	2.1786
	0.5702	0.5614	0.5580	0.5597	0.5664	0.5953	0.6450	0.7150	0.8056	0.9167	1.0481	1.2002	1.3725	1.7784	2.2659
	0.1406	0.1329	0.1302	0.1324	0.1397	0.1693	0.2190	0.2885	0.3781	0.4877	0.6173	0.7669	0.9365	1.3357	1.8149
	0.1294	0.1268	0.1292	0.1366	0.1490	0.1889	0.2486	0.3284	0.4283	0.5480	0.6878	0.8476	1.0274	1.4470	1.9466
	0.1312	0.1282	0.1302	0.1373	0.1495	0.1890	0.2486	0.3283	0.4281	0.5479	0.6876	0.8475	1.0273	1.4469	1.9466
$\nu=100,\ \lambda_n=0$	0.1343	0.1304	0.1316	0.1381	0.1496	0.1883	0.2472	0.3265	0.4260	0.5456	0.6854	0.8453	1.0252	1.4450	1.9449
	0.1394	0.1330	0.1317	0.1357	0.1447	0.1783	0.2325	0.3072	0.4024	0.5181	0.6543	0.8109	0.9879	1.4031	1.8995
	0.1406	0.1329	0.1303	0.1327	0.1399	0.1698	0.2196	0.2895	0.3794	0.4894	0.6193	0.7694	0.9395	1.3399	1.8204
	0.1396	0.1331	0.1316	0.1353	0.1441	0.1772	0.2308	0.3050	0.3596	0.5147	0.6502	0.8061	0.9825	1.3964	1.8916
	0.1200	0.1125	0.1100	0.1125	0.1200	0.1500	0.2000	0.2700	0.3600	0.4700	0.6000	0.7500	0.9200	1.3200	1.8000
	0.1075	0.1050	0.1075	0.1150	0.1275	0.1675	0.2275	0.3075	0.4075	0.5275	0.6675	0.8275	1.0075	1.4275	1.9275
	0.1093	0.1064	0.1085	0.1157	0.1280	0.1677	0.2275	0.3075	0.4074	0.5274	0.6674	0.8274	1.0074	1.4274	1.9275
$\nu=\infty,\ \lambda_n=0$	0.1126	0.1088	0.1102	0.1167	0.1285	0.1673	0.2264	0.3059	0.4057	0.5255	0.6655	0.8255	1.0056	1.4259	1.9262
	0.1184	0.1122	0.1111	0.1152	0.1245	0.1584	0.2129	0.2880	0.3836	0.4996	0.6362	0.7931	0.9705	1.3863	1.8833
	0.1200	0.1125	0.1101	0.1126	0.1202	0.1504	0.2007	0.2709	0.3612	0.4716	0.6020	0.7525	0.9230	1.3242	1.8055
	0.1187	0.1123	0.1110	0.1149	0.1239	0.1574	0.2114	0.2859	0.3808	0.4963	0.6322	0.7885	0.9652	1.3798	1.8757
	4.3333	4.3192	4.3100	4.3058	4.3067	4.3233	4.3600	4.4167	4.4933	4.5900	4.7067	4.8433	5.0000	5.3733	5.8267
	4.7781	4.7822	4.7914	4.8056	4.8247	4.8781	4.9514	5.0447	5.1581	5.2914	5.4447	5.6181	5.8114	6.2581	6.7847
	4.7867	4.7817	4.7831	4.7903	4.8039	4.8483	4.9156	5.0047	5.1156	5.2475	5.4003	5.5733	5.7669	6.2147	6.7431
$\nu=5,\ \lambda_n=1$	4.7136	4.7056	4.7036	4.7069	4.7164	4.7522	4.8108	4.8919	4.9956	5.1211	5.2686	5.4375	5.6275	6.0711	6.5972
	4.4522	4.4403	4.4331	4.4311	4.4344	4.4564	4.4989	4.5622	4.6464	4.7511	4.8769	5.0233	5.1908	5.5883	6.0694
	4.3381	4.3239	4.3147	4.3106	4.3117	4.3283	4.3653	4.4222	4.4992	4.5961	4.7131	4.8500	5.0069	5.3811	5.8353
	4.4347	4.4222	4.4147	4.4125	4.4156	4.4364	4.4783	4.5406	4.6233	4.7269	4.8511	4.9961	5.1617	5.5553	6.0322
	0.5625	0.5525	0.5475	0.5475	0.5525	0.5775	0.6225	0.6875	0.7725	0.8775	1.0025	1.1475	1.3125	1.7025	2.1725
	0.5861	0.5923	0.6036	0.6198	0.6411	0.6986	0.7761	0.8736	0.9911	1.1286	1.2861	1.4636	1.6611	2.1161	2.6511
	0.6025	0.6030	0.6092	0.6214	0.6392	0.6917	0.7659	0.8616	0.9781	1.1153	1.2730	1.4508	1.6489	2.1053	2.6417
$\nu=10,\ \lambda_n=1$	0.6000	0.5966	0.5991	0.6072	0.6209	0.6656	0.7328	0.8222	0.9336	1.0666	1.2209	1.3966	1.5931	2.0484	2.5859
	0.5731	0.5645	0.5611	0.5628	0.5697	0.5991	0.6491	0.7200	0.8117	0.9242	1.0575	1.2119	1.3872	1.8005	2.2975
	0.5628	0.5530	0.5480	0.5480	0.5531	0.5781	0.6233	0.6884	0.7736	0.8789	1.0041	1.1494	1.3145	1.7052	2.1759
	0.5714	0.5627	0.5589	0.5603	0.5669	0.5955	0.6445	0.7144	0.8050	0.9161	1.0481	1.2008	1.3744	1.7838	2.2764
	0.1406	0.1329	0.1302	0.1324	0.1397	0.1693	0.2190	0.2885	0.3781	0.4877	0.6173	0.7669	0.9365	1.3357	1.8149
	0.1292	0.1366	0.1490	0.1664	0.1889	0.2486	0.3284	0.4283	0.5480	0.6878	0.8476	1.0274	1.2272	1.6868	2.2264
	0.1382	0.1417	0.1509	0.1657	0.1859	0.2427	0.3208	0.4198	0.5395	0.6795	0.8399	1.0203	1.2208	1.6819	2.2227
$\nu=100,\ \lambda_n=1$	0.1412	0.1403	0.1451	0.1558	0.1721	0.2217	0.2935	0.3873	0.5029	0.6400	0.7983	0.9777	1.1778	1.6399	2.1836
	0.1407	0.1341	0.1327	0.1363	0.1451	0.1786	0.2327	0.3078	0.4037	0.5204	0.6582	0.8168	0.9965	1.4185	1.9244
	0.1406	0.1329	0.1302	0.1325	0.1398	0.1695	0.2193	0.2889	0.3787	0.4884	0.6182	0.7680	0.9378	1.3376	1.8174
	0.1407	0.1338	0.1321	0.1356	0.1441	0.1768	0.2301	0.3040	0.3988	0.5143	0.6506	0.8076	0.9854	1.4036	1.9052
	0.1200	0.1125	0.1100	0.1125	0.1200	0.1500	0.2000	0.2700	0.3600	0.4700	0.6000	0.7500	0.9200	1.3200	1.8000
	0.1075	0.1150	0.1275	0.1450	0.1675	0.2275	0.3075	0.4075	0.5275	0.6675	0.8275	1.0075	1.2075	1.6675	2.2075
	0.1157	0.1194	0.1288	0.1439	0.1643	0.2215	0.3000	0.3993	0.5192	0.6595	0.8201	1.0008	1.2015	1.6630	2.2042
$\nu=\infty,\ \lambda_n=1$	0.1188	0.1182	0.1233	0.1341	0.1506	0.2006	0.2729	0.3671	0.4831	0.6205	0.7792	0.9589	1.1593	1.6221	2.1663
	0.1197	0.1132	0.1119	0.1158	0.1249	0.1586	0.2132	0.2885	0.3848	0.5020	0.6401	0.7991	0.9791	1.4020	1.9087
	0.1200	0.1125	0.1100	0.1126	0.1201	0.1502	0.2003	0.2704	0.3605	0.4707	0.6009	0.7511	0.9213	1.3218	1.8025
	0.1197	0.1131	0.1115	0.1151	0.1239	0.1569	0.2105	0.2849	0.3801	0.4959	0.6325	0.7900	0.9682	1.3872	1.8896
	4.3333	4.3192	4.3100	4.3058	4.3067	4.3233	4.3600	4.4167	4.4933	4.5900	4.7067	4.8433	5.0000	5.3733	5.8267
	4.9514	4.9956	5.0447	5.0989	5.1581	5.2914	5.4447	5.6181	5.8114	6.0247	6.2581	6.5114	6.7847	7.3914	8.0781
	5.0961	5.1011	5.1125	5.1306	5.1550	5.2250	5.3228	5.4489	5.6031	5.7856	5.9953	6.2319	6.4950	7.0947	7.7875
$\nu=5,\ \lambda_n=5$	4.8486	4.8461	4.8492	4.8581	4.8728	4.9194	4.9900	5.0847	5.2039	5.3478	5.5169	5.7114	5.9314	6.4494	7.0700
	4.4467	4.4342	4.4267	4.4244	4.4269	4.4475	4.4883	4.5494	4.6311	4.7331	4.8553	4.9981	5.1611	5.5489	6.0192
	4.3372	4.3231	4.3139	4.3097	4.3106	4.3275	4.3642	4.4208	4.4975	4.5944	4.7111	4.8481	5.0047	5.3783	5.8317
	4.4281	4.4150	4.4072	4.4044	4.4069	4.4264	4.4667	4.5267	4.6072	4.7081	4.8289	4.9703	5.1319	5.5167	5.9828
	0.5625	0.5525	0.5475	0.5475	0.5525	0.5775	0.6225	0.6875	0.7725	0.8775	1.0025	1.1475	1.3125	1.7025	2.1725
	0.7761	0.8223	0.8736	0.9298	0.9911	1.1286	1.2861	1.4636	1.6611	1.8786	2.1161	2.3736	2.6511	3.2661	3.9611
	0.7441	0.7559	0.7748	0.8013	0.8347	0.9242	1.0431	1.1914	1.3681	1.5728	1.8039	2.0613	2.3431	2.9777	3.7005
$\nu=10,\ \lambda_n=5$	0.6455	0.6445	0.6495	0.6605	0.6775	0.7297	0.8070	0.9095	1.0378	1.1920	1.3725	1.5794	1.8127	2.3584	3.0091
	0.5705	0.5613	0.5570	0.5577	0.5636	0.5905	0.6375	0.7050	0.7928	0.9009	1.0295	1.1784	1.3478	1.7484	2.2316
	0.5627	0.5527	0.5477	0.5477	0.5527	0.5777	0.6227	0.6878	0.7728	0.8778	1.0028	1.1478	1.3128	1.7030	2.1731
	0.5689	0.5594	0.5550	0.5555	0.5611	0.5875	0.6341	0.7008	0.7880	0.8953	1.0228	1.1709	1.3392	1.7369	2.2162
	0.1406	0.1329	0.1302	0.1324	0.1397	0.1693	0.2190	0.2885	0.3781	0.4877	0.6173	0.7669	0.9365	1.3357	1.8149
	0.3284	0.3758	0.4282	0.4856	0.5480	0.6878	0.8476	1.0274	1.2272	1.4470	1.6868	1.9466	2.2264	2.8460	3.5456
	0.1924	0.2064	0.2284	0.2584	0.2964	0.3961	0.5268	0.6873	0.8767	1.0936	1.3368	1.6053	1.8978	2.5513	3.2905
$\nu=100,\ \lambda_n=5$	0.1527	0.1509	0.1552	0.1657	0.1826	0.2354	0.3142	0.4194	0.5514	0.7103	0.8963	1.1093	1.3496	1.9107	2.5776
	0.1409	0.1333	0.1308	0.1334	0.1410	0.1713	0.2218	0.2926	0.3837	0.4952	0.6269	0.7792	0.9518	1.3589	1.8483
	0.1406	0.1329	0.1302	0.1324	0.1397	0.1693	0.2190	0.2885	0.3782	0.4877	0.6173	0.7669	0.9365	1.3357	1.8150
	0.1407	0.1332	0.1306	0.1331	0.1406	0.1708	0.2209	0.2913	0.3820	0.4928	0.6240	0.7754	0.9471	1.3517	1.8379
	0.1200	0.1125	0.1100	0.1125	0.1200	0.1500	0.2000	0.2700	0.3600	0.4700	0.6000	0.7500	0.9200	1.3200	1.8000
	0.3075	0.3550	0.4075	0.4650	0.5275	0.6675	0.8275	1.0075	1.2075	1.4275	1.6675	1.9275	2.2075	2.8275	3.5275
	0.1625	0.1764	0.1988	0.2291	0.2674	0.3680	0.4997	0.6614	0.8518	1.0698	1.3140	1.5835	1.8770	2.5319	3.2727
$\nu=\infty,\ \lambda_n=5$	0.1284	0.1264	0.1305	0.1409	0.1576	0.2102	0.2891	0.3945	0.5267	0.6859	0.8724	1.0861	1.3268	1.8891	2.5577
	0.1201	0.1128	0.1104	0.1132	0.1208	0.1514	0.2023	0.2734	0.3647	0.4765	0.6087	0.7611	0.9341	1.3415	1.8316
	0.1200	0.1125	0.1100	0.1125	0.1200	0.1500	0.2000	0.2700	0.3600	0.4700	0.6000	0.7500	0.9200	1.3200	1.8000
	0.1200	0.1126	0.1103	0.1129	0.1206	0.1510	0.2016	0.2723	0.3632	0.4744	0.6059	0.7576	0.9296	1.3349	1.8216

TABLE A5.2.9: Relative Risks of $\tilde{\sigma}^2_M$, σ^{*2}_M, and $\hat{\sigma}^2_M$

$v = 16,\ k = 4,\ m = 1$

Estimator	λ_d 0.0	0.5	1.0	1.5	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	12.0	14.0
	5.2469	5.2500	5.2592	5.2747	5.2964	5.3581	5.4444	5.5556	5.6914	5.8519	6.0369	6.2469	6.4814	7.0247	7.6667
	5.2631	5.2658	5.2742	5.2881	5.3075	5.3628	5.4406	5.5403	5.6619	5.8061	5.9722	6.1606	6.3711	6.8586	7.4350
	5.2364	5.2383	5.2464	5.2600	5.2792	5.3342	5.4119	5.5117	5.6339	5.7783	5.9447	6.1333	6.3444	6.8325	7.4094
$\nu=5,\ \lambda_n=0$	5.1617	5.1625	5.1694	5.1819	5.2000	5.2539	5.3306	5.4294	5.5512	5.6953	5.8614	6.0503	6.2611	6.7500	7.3275
	4.9958	4.9936	4.9978	5.0072	5.0228	5.0708	5.1417	5.2353	5.3517	5.4906	5.6522	5.8364	6.0431	6.5242	7.0950
	5.1092	5.1094	5.1158	5.1281	5.1461	5.2003	5.2778	5.3792	5.5039	5.6519	5.8236	6.0183	6.2367	6.7428	7.3417
	4.9914	4.9894	4.9933	5.0028	5.0181	5.0658	5.1367	5.2300	5.3461	5.4850	5.6464	5.8303	6.0367	6.5172	7.0875
	0.6366	0.6397	0.6489	0.6644	0.6859	0.7477	0.8341	0.9452	1.0811	1.2416	1.4267	1.6366	1.8711	2.4144	3.0563
	0.6305	0.6333	0.6416	0.6555	0.6748	0.7302	0.8078	0.9075	1.0294	1.1734	1.3397	1.5592	1.7384	2.2261	2.8022
	0.6297	0.6320	0.6400	0.6536	0.6728	0.7280	0.8055	0.9052	1.0272	1.1713	1.3377	1.5261	1.7367	2.2245	2.8008
$\nu=10,\ \lambda_n=0$	0.6252	0.6266	0.6338	0.6466	0.6652	0.7194	0.7961	0.8953	1.0170	1.1611	1.3273	1.5159	1.7267	2.2150	2.7919
	0.6117	0.6109	0.6161	0.6269	0.6433	0.6934	0.7664	0.8619	0.9802	1.1209	1.2844	1.4703	1.6788	2.1628	2.7366
	0.6228	0.6238	0.6308	0.6436	0.6623	0.7175	0.7961	0.8983	1.0239	1.1730	1.3453	1.5409	1.7597	2.2670	2.8667
	0.6113	0.6106	0.6155	0.6263	0.6427	0.6927	0.7653	0.8608	0.9789	1.1195	1.2827	1.4684	1.6766	2.1603	2.7338
	0.1349	0.1381	0.1473	0.1627	0.1844	0.2460	0.3325	0.4436	0.5794	0.7399	0.9251	1.1349	1.3695	1.9127	2.5548
	0.1290	0.1318	0.1400	0.1539	0.1734	0.2288	0.3063	0.4060	0.5279	0.6719	0.8382	1.0266	1.2371	1.7246	2.3007
	0.1303	0.1327	0.1407	0.1543	0.1736	0.2287	0.3061	0.4058	0.5277	0.6717	0.8380	1.0263	1.2369	1.7245	2.3007
$\nu=100,\ \lambda_n=0$	0.1319	0.1336	0.1410	0.1540	0.1727	0.2272	0.3040	0.4035	0.5252	0.6692	0.8355	1.0239	1.2346	1.7225	2.2990
	0.1330	0.1329	0.1386	0.1499	0.1670	0.2183	0.2924	0.3890	0.5083	0.6501	0.8145	1.0014	1.2106	1.6965	2.2717
	0.1338	0.1352	0.1423	0.1556	0.1746	0.2303	0.3097	0.4123	0.5384	0.6879	0.8607	1.0566	1.2758	1.7836	2.3839
	0.1330	0.1329	0.1385	0.1498	0.1668	0.2180	0.2919	0.3884	0.5075	0.6492	0.8134	1.0001	1.2092	1.6947	2.2696
	0.1111	0.1142	0.1235	0.1389	0.1605	0.2222	0.3086	0.4198	0.5556	0.7161	0.9012	1.1111	1.3457	1.8889	2.5309
	0.1053	0.1080	0.1163	0.1302	0.1496	0.2050	0.2826	0.3823	0.5042	0.6482	0.8144	1.0028	1.2133	1.7008	2.2770
	0.1066	0.1090	0.1170	0.1306	0.1499	0.2050	0.2825	0.3821	0.5040	0.6480	0.8143	1.0027	1.2132	1.7007	2.2770
$\nu=\infty,\ \lambda_n=0$	0.1085	0.1102	0.1175	0.1306	0.1494	0.2038	0.2807	0.3801	0.5018	0.6459	0.8121	1.0006	1.2113	1.6992	2.2757
	0.1101	0.1101	0.1158	0.1273	0.1443	0.1958	0.2699	0.3667	0.4861	0.6280	0.7925	0.9794	1.1887	1.6747	2.2500
	0.1105	0.1118	0.1191	0.1323	0.1514	0.2072	0.2866	0.3893	0.5155	0.6650	0.8377	1.0337	1.2530	1.7609	2.3611
	0.1101	0.1101	0.1157	0.1271	0.1442	0.1955	0.2695	0.3661	0.4853	0.6271	0.7194	0.9781	1.1873	1.6730	2.2480
	5.2469	5.2500	5.2592	5.2747	5.2964	5.3581	5.4444	5.5556	5.6914	5.8519	6.0369	6.2469	6.4814	7.0247	7.6667
	5.2742	5.2881	5.3075	5.3325	5.3628	5.4406	5.5403	5.6619	5.8061	5.9722	6.1606	6.3711	6.6039	7.1358	7.7561
	5.2725	5.2778	5.2897	5.3083	5.3336	5.4031	5.4975	5.6161	5.7583	5.9233	6.1111	6.3222	6.5556	7.0892	7.7114
$\nu=5,\ \lambda_n=1$	5.1903	5.1925	5.2011	5.2161	5.2375	5.2989	5.3856	5.4969	5.6328	5.7925	5.9761	6.1836	6.4142	6.9444	7.5661
	5.0003	5.0011	5.0058	5.0167	5.0331	5.0842	5.1589	5.2572	5.3792	5.5244	5.6933	5.8856	6.1011	6.6019	7.1953
	5.1142	5.1150	5.1217	5.1347	5.1536	5.2097	5.2900	5.3942	5.5228	5.6756	5.8522	6.0531	6.2778	6.7994	7.4172
	4.9978	4.9964	5.0011	5.0117	5.0283	5.0792	5.1536	5.2517	5.3733	5.5183	5.6867	5.8786	6.0936	6.5933	7.1858
	0.6366	0.6397	0.6489	0.6644	0.6859	0.7477	0.8341	0.9452	1.0811	1.2416	1.4267	1.6366	1.8711	2.4144	3.0563
	0.6416	0.6555	0.6748	0.6997	0.7302	0.8078	0.9075	1.0294	1.1734	1.3397	1.5280	1.7384	1.9713	2.3031	3.1236
	0.6508	0.6592	0.6742	0.6955	0.7230	0.7963	0.8933	1.0139	1.1575	1.3238	1.5127	1.7239	1.9573	2.4911	3.1133
$\nu=10,\ \lambda_n=1$	0.6400	0.6447	0.6559	0.6733	0.6969	0.7630	0.8538	0.9688	1.1078	1.2706	1.4569	1.6664	1.8989	2.4325	3.0564
	0.6142	0.6148	0.6214	0.6339	0.6523	0.7070	0.7852	0.8870	1.0123	1.1611	1.3331	1.5288	1.7473	2.2545	2.8541
	0.6248	0.6267	0.6345	0.6486	0.6686	0.7267	0.8091	0.9155	1.0461	1.2009	1.3797	1.5827	1.8095	2.3355	2.9573
	0.6138	0.6144	0.6208	0.6333	0.6516	0.7059	0.7841	0.8855	1.0105	1.1589	1.3306	1.5256	1.7441	2.2502	2.8486
	0.1349	0.1381	0.1473	0.1627	0.1844	0.2460	0.3325	0.4436	0.5794	0.7399	0.9251	1.1349	1.3695	1.9127	2.5548
	0.1400	0.1539	0.1734	0.1983	0.2288	0.3063	0.4060	0.5279	0.6719	0.8382	1.0266	1.2371	1.4697	2.0016	2.6221
	0.1430	0.1533	0.1699	0.1926	0.2214	0.2966	0.3953	0.5169	0.6612	0.8281	1.0173	1.2288	1.4624	1.9960	2.6181
$\nu=100,\ \lambda_n=1$	0.1392	0.1453	0.1577	0.1765	0.2015	0.2700	0.3630	0.4800	0.6209	0.7853	0.9730	1.1839	1.4175	1.9529	2.5783
	0.1334	0.1349	0.1425	0.1560	0.1753	0.2319	0.3121	0.4159	0.5430	0.6936	0.8677	1.0649	1.2855	1.7961	2.3992
	0.1343	0.1366	0.1450	0.1596	0.1801	0.2395	0.3230	0.4307	0.5625	0.7183	0.8984	1.1026	1.2786	1.8590	2.4833
	0.1334	0.1348	0.1423	0.1558	0.1750	0.2314	0.3113	0.4148	0.5416	0.6920	0.8656	1.0625	1.2826	1.7923	2.3943
	0.1111	0.1142	0.1235	0.1389	0.1605	0.2222	0.3086	0.4198	0.5556	0.7161	0.9012	1.1111	1.3457	1.8889	2.5309
	0.1163	0.1336	0.1496	0.1745	0.2050	0.2826	0.3823	0.5042	0.6482	0.8144	1.0028	1.2133	1.4460	1.9778	2.5983
	0.1185	0.1315	0.1458	0.1686	0.1974	0.2729	0.3717	0.4934	0.6378	0.8047	0.9940	1.2054	1.4391	1.9727	2.5947
$\nu=\infty,\ \lambda_n=1$	0.1150	0.1213	0.1339	0.1528	0.1779	0.2466	0.3397	0.4569	0.5980	0.7626	0.9504	1.1613	1.3951	1.9308	2.5562
	0.1104	0.1120	0.1196	0.1332	0.1527	0.2094	0.2897	0.3936	0.5210	0.6718	0.8459	1.0434	1.2641	1.7751	2.3785
	0.1108	0.1132	0.1218	0.1363	0.1569	0.2163	0.2999	0.4077	0.5396	0.6957	0.8758	1.0800	1.3083	1.8369	2.4614
	0.1103	0.1119	0.1195	0.1330	0.1524	0.2089	0.2890	0.3926	0.5197	0.6701	0.8440	1.0410	1.2613	1.7713	2.3736
	5.2469	5.2500	5.2592	5.2747	5.2964	5.3581	5.4444	5.5556	5.6914	5.8519	6.0369	6.2469	6.4814	7.0247	7.6667
	5.5403	5.5983	5.6619	5.7314	5.8061	5.9722	6.1606	6.3711	6.6039	6.8586	7.1358	7.4350	7.7561	8.4653	9.2631
	5.6081	5.6258	5.6511	5.6839	5.7242	5.8281	5.9631	6.1294	6.3269	6.5550	6.8128	7.1000	7.4153	8.1258	8.9372
$\nu=5,\ \lambda_n=5$	5.3433	5.3519	5.3678	5.3900	5.4189	5.4969	5.6022	5.7350	5.8953	6.0836	6.3000	6.5447	6.8175	7.4486	8.1917
	5.0453	5.0464	5.0533	5.0667	5.0858	5.1428	5.2242	5.3300	5.4603	5.6150	5.7942	5.9978	6.2256	6.7544	7.3808
	5.1408	5.1428	5.1511	5.1653	5.1858	5.2453	5.3294	5.4381	5.5711	5.7292	5.9117	6.1186	6.3503	6.8875	7.5225
	5.0408	5.0417	5.0486	5.0617	5.0808	5.1375	5.2186	5.3242	5.4542	5.6086	5.7875	5.9906	6.2183	6.7464	7.3717
	0.6366	0.6397	0.6489	0.6644	0.6859	0.7477	0.8341	0.9452	1.0811	1.2416	1.4267	1.6366	1.8711	2.4144	3.0563
	0.9075	0.9656	1.0294	1.0986	1.1734	1.3397	1.5592	1.7384	1.9713	2.2261	2.5031	2.8022	3.1236	3.8327	4.6305
	0.8036	0.8269	0.8581	0.8980	0.9456	1.0659	1.2186	1.4034	1.6192	1.8655	2.1405	2.4438	2.7736	3.5100	4.3425
$\nu=10,\lambda_n=5$	0.6902	0.6997	0.7159	0.7392	0.7694	0.8506	0.9602	1.0980	1.2647	1.4602	1.6845	1.9380	2.2202	2.8711	3.6353
	0.6272	0.6297	0.6384	0.6531	0.6741	0.7344	0.8191	0.9283	1.0623	1.2206	1.4034	1.6106	1.8425	2.3795	3.0144
	0.6323	0.6352	0.6441	0.6594	0.6806	0.7417	0.8275	0.9380	1.0731	1.2328	1.4172	1.6261	1.8597	2.4009	3.0406
	0.6267	0.6292	0.6378	0.6525	0.6734	0.7336	0.8183	0.9273	1.0611	1.2192	1.4019	1.6089	1.8405	2.3767	3.0105
	0.1349	0.1381	0.1473	0.1627	0.1844	0.2460	0.3325	0.4436	0.5794	0.7399	0.9251	1.1349	1.3695	1.9127	2.5548
	0.4060	0.4642	0.5279	0.5971	0.6719	0.8382	1.0266	1.2371	1.4697	1.7246	2.0016	2.3007	2.6221	3.3312	4.1290
	0.1992	0.2241	0.2580	0.3006	0.3520	0.4808	0.6435	0.8385	1.0650	1.3211	1.6058	1.9177	2.2558	3.0065	3.8503
$\nu=100,\ \lambda_n=5$	0.1476	0.1557	0.1708	0.1929	0.2225	0.3031	0.4128	0.5521	0.7208	0.9194	1.1475	1.4052	1.6925	2.3540	3.1296
	0.1348	0.1378	0.1469	0.1624	0.1838	0.2452	0.3312	0.4420	0.5772	0.7372	0.9215	1.1305	1.3639	1.9046	2.5433
	0.1349	0.1380	0.1472	0.1626	0.1842	0.2458	0.3322	0.4431	0.5788	0.7392	0.9242	1.1338	1.3683	1.9109	2.5523
	0.1347	0.1378	0.1469	0.1622	0.1837	0.2451	0.3310	0.4416	0.5767	0.7365	0.9207	1.1294	1.3628	1.9029	2.5407
	0.1111	0.1142	0.1235	0.1389	0.1605	0.2222	0.3086	0.4198	0.5556	0.7161	0.9012	1.1111	1.3457	1.8889	2.5309
	0.3823	0.4404	0.5042	0.5734	0.6482	0.8144	1.0028	1.2133	1.4460	1.7008	1.9778	2.2770	2.5983	3.3075	4.1053
	0.1665	0.1914	0.2254	0.2683	0.3200	0.4496	0.6131	0.8092	1.0364	1.2935	1.5790	1.8918	2.2306	2.9821	3.8271
$\nu=\infty,\ \lambda_n=5$	0.1208	0.1286	0.1435	0.1656	0.1948	0.2751	0.3848	0.5241	0.6930	0.8917	1.1202	1.3783	1.6658	2.3283	3.1052
	0.1110	0.1141	0.1233	0.1387	0.1601	0.2216	0.3077	0.4184	0.5538	0.7138	0.8984	1.1074	1.3411	1.8820	2.5211
	0.1111	0.1142	0.1234	0.1388	0.1604	0.2163	0.3084	0.4195	0.5552	0.7157	0.8985	1.1103	1.3447	1.8875	2.5614
	0.1110	0.1140	0.1232	0.1385	0.1600	0.2215	0.3076	0.4183	0.5535	0.7133	0.8976	1.1064	1.3400	1.8804	2.5186

CHAPTER SIX

SMALL SAMPLE PROPERTIES OF THE MIS-SPECIFIED PRE-TEST HOMOGENEITY ESTIMATOR OF THE ERROR VARIANCE

6.1 Introduction

In the last two chapters we investigated some sampling properties of estimators after a pre-test for exact linear restrictions on the coefficient vector. Another pre-test which is frequently undertaken in applied research, and which has received attention in the literature, is for homogeneity of a regression error variance. We consider here, as has much of the literature, the problem of the estimation of the error variance of the first sample[1] when it is suspected that the sample regressions have a common coefficient vector but possibly different error variances.

We follow the approach of the last two chapters in considering the sampling properties of the estimators of the error variance when the regression model is possibly mis-specified in two ways. First, we may have wrongly excluded regressors from each of the samples. We assume that these need not be the same regressors. Secondly, the regression disturbances are spherically symmetrically distributed but are wrongly assumed to be normal. This latter assumption does not affect the validity of the usual test statistic for homoscedasticity, J, as $J \sim F_{(v_2, v_1)}$ under the null hypothesis of homoscedasticity (and assuming there are no omitted regressors) for all members of the spherically symmetric family. In fact, the non-null distribution of J is also invariant. This result is shown by Chmielewski

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[1] The case for the estimation of the error variance in the second sample is identical allowing for the change in the sample of interest.

(1981b) and is implied by the results of King (1979). There has been no investigation in the literature of the null or of the non-null distribution of J when regressors are omitted from the model specification.[2] So, in the next section we derive the distribution of J when the model is mis-specified in this way and the errors are SSD_N. In this section we also define the model framework and the estimators we will consider.

In Section 6.3 we consider the bias and the risk functions of the never-pooled, always-pooled and pre-test estimators of the error variance of the first sample. To aid our analysis we consider exact evaluations of the risk functions for the special case of Mt regression disturbances in Sections 6.4 and 6.5. Section 6.4 assumes that the design matrices are correctly specified while Section 6.5 extends the analysis to the mis-specified regressor problem. Some concluding remarks are given in Section 6.6 followed by two Appendices. Appendix 6.1 gives some special cases of the theorems and the corollaries of Section 6.3, while a small sample of the numerical evaluations of the relative risk functions are given in Apendix 6.2 . Further detail regarding their content is contained in the discussion below and in the introduction to each Appendix.

So, we extend the existing literature by deriving the exact risk of the pre-test estimator when the regression model is mis-specified in possibly two ways, and by assuming that the components of the pre-test estimator are from a general family of never-pooled and always-pooled estimators. This last extension is elaborated on in the next section.

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[2] We recall, though, that Ohtani (1987a) derives the distribution of a related test statistic, B, which we discussed in Chapter Two.

6.2 The Model Framework, Estimators and Some Preliminary Results

We consider a simple heteroscedastic linear regression model in which the error variance is constant within each sample but it may be different between the samples,

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}\beta + \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \tag{6.2.1}$$

or

$$y = X\beta + Z\gamma + e. \tag{6.2.2}$$

For i=1,2, y_i is a $(T_i \times 1)$ vector of observations on the dependent variable, X_i is a known $(T_i \times k)$ non-stochastic design matrix of rank k $(<T_i)$, Z_i is a fixed $(T_i \times p_i)$ matrix of full rank, γ_i is a $(p_i \times 1)$ coefficient vector, and β is a $(k \times 1)$ vector of unknown parameters, which we assume is common to both samples, as has the literature we discussed in Chapter Two.[3] Let $T=T_1+T_2$.

We assume that the $(T_i \times 1)$ vector of regression disturbances, $e_i \sim SSD_N(0, I_{T_i})$ and $E(e_i e_i') = \sigma^2_{e_i} I_{T_i}$. Let $\phi = \sigma^2_{e_1}/\sigma^2_{e_2}$, then

$$E(ee') = \begin{bmatrix} \sigma^2_{e_1} I_{T_1} & 0 \\ 0 & \sigma^2_{e_2} I_{T_2} \end{bmatrix} = \sigma^2_{e_2}\begin{bmatrix} \phi I_{T_1} & 0 \\ 0 & I_{T_2} \end{bmatrix} = \sigma^2_{e_2}\Sigma. \tag{6.2.3}$$

So, e has an elliptically symmetric distribution $ESD_N(0,\Sigma)$. We assume that the non-normality of e arises from a random variance such that $f(e)=\int_0^\infty f_N(e)f(\tau)d\tau$ where $f_N(e)$ is f(e) when $e \sim N(0,\tau^2\Sigma)$ and $f(\tau)$ is supported on $[0,\infty)$. Hence, $\sigma^2_{e_2}=E(\tau^2)$ and $\sigma^2_{e_1}=\phi E(\tau^2)$. We could assume that

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[3] We recall that the risk properties of the estimators which arises after a pre-test for equality of the location vectors when the scale parameters are possibly unequal has received little attention in the literature as the traditional test statistics are inexact.

each sample is generated by a different variance mixing distribution. If we assume that the mixing distributions are independent then we can easily extend our analysis. However, it is unclear how we would proceed if they are dependent.

Rather than model (6.2.2) being estimated, we suppose that the proposed model is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \beta + \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} , \tag{6.2.4}$$

or

$$y = X\beta + u \quad , \quad u \sim N\left(0, \sigma^2_{e_2} \Sigma\right) . \tag{6.2.5}$$

We proceed assuming that (6.2.5) is properly specified when in fact $u \sim ESD_N(Z\gamma, \Sigma)$. Note that (6.2.5) reflects the fact that $\sigma^2_{u_i} = \sigma^2_{e_i}$, i=1,2. The researcher is uncertain of the homogeneity of the error variances and so conducts a pre-test of

$$H_0 : \phi = 1 \quad \text{vs} \quad H_1 : \phi < 1. \tag{6.2.6}$$

where $\phi = \left(\sigma^2_{e_1} / \sigma^2_{e_2}\right)$ is a measure of the hypothesis error and we assume, for simplicity, a one-sided alternative hypothesis.[4]

We noted in Chapter Two, within the framework of the linear regression model, that the research on this particular pre-test problem follows the literature associated with the pooling of two normal samples. The never-pooled estimator of the error variance of the first sample, $\sigma^2_{e_1}$, is $s^2_N = s^2_1$ while the always-pooled estimator is $s^2_A = \left(v_1 s^2_1 + v_2 s^2_2\right) / (v_1 + v_2)$ where

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[4] It is straightforward to extend the analysis to the two-sided case. We assume the one-sided alternative to maintain consistency with much of the existing literature.

$s_i^2=(y_i-X_ib_i)'(y_i-X_ib_i)/v_i$, which is the usual unbiased least squares (L) estimator of $\sigma^2_{e_i}$, i=1,2. We term these estimators s^2_{iL}, s^2_{NL}, and s^2_{AL}.

Within the linear regression model framework two other estimators of $\sigma^2_{e_i}$ are commonly used (assuming normality): the maximum likelihood (ML) estimator and the minimum mean squared error (M) estimator. Let these estimators be denoted by s^2_{iML} and s^2_{iM} respectively. They differ from s^2_{iL} by the divisor used, this being T_i for the ML estimators and (v_i+2) for the M estimators. These estimators are members of a family of estimators

$$S_i^2 = (u_i'M_iu_i)/(T_i+\mu) \ , \tag{6.2.7}$$

where, in this chapter, we define u_i as the regression disturbance of the ith sample and $M_i=I_{T_i}-X_iS_i^{-1}X_i'$. We can generate s^2_{iL}, s^2_{iML}, and s^2_{iM} by setting μ to -k, 0, and (-k+2), respectively. Let $S_N^2=S_1^2$ be the family of never-pooled estimators of $\sigma^2_{e_1}$.

In the spirit of s^2_{AL} we can conceive of two feasible alternative always-pooled estimators s^2_{AML} and s^2_{AM}, which have as their components the sample ML and M estimators, s^2_{iML} and s^2_{iM}, respectively. That is, $s^2_{AML}=\left(T_1s^2_{1ML}+T_2s^2_{2ML}\right)/T$ and $s^2_{AM}=\left((v_1+2)s^2_{1M}+(v_2+2)s^2_{2M}\right)/(v_1+v_2+4)$. s^2_{AL}, s^2_{AML}, and s^2_{AM} are always-pooled estimators of the form

$$S_A^2 = \left[(T_1+\mu)\left(u_1'M_1u_1/(T_1+\mu)\right)+(T_2+\mu)\left(u_2'M_2u_2/(T_2+\mu)\right)\right]/(T+2\mu). \tag{6.2.8}$$

We obtain s^2_{AL}, s^2_{AML}, and s^2_{AM} by setting μ to -k, 0, and (-k+2).

Assuming for the moment that the errors are normal, we should note that even though s^2_{AML} is comprised of the sample ML estimators of $\sigma^2_{e_1}$ and $\sigma^2_{e_2}$ it is not itself a ML estimator. Similarly, s^2_{AM} does not possess the M property. Of course, when the errors are non-normal then even the sample estimators are not the ML or M estimators, though the researcher proceeds assuming that they possess these properties. One may then reasonably ask why we do not consider the L, ML, and M pooled estimators assuming that the error variances are equal and the regression disturbances are normally distributed. That is, consider $\sigma_L^2=u'Mu/v_T$, $\sigma_{ML}^2=u'Mu/T$, and $\sigma_M^2= u'Mu/(v_T+2)$ which are the L, ML, M estimators of $\sigma_{e_1}^2$ for the pooled model (6.2.5) assuming that $\Sigma=\sigma_{e_1}^2 I_T$, $v_T=T-k$, $M=I_T-XS^{-1}X'$. These estimators should be more efficient than s_{AL}^2, s_{AML}^2, and s_{AM}^2 as they incorporate the information that β is common to both samples. However, even under a normality assumption, the non-null distribution of $u'Mu$ is not clear, though it is obviously proportional to a non-central Chi-square random variate under the null hypothesis. Consequently, and in common with the related literature, we do not proceed with these estimators.

There is also another obvious family of pooled estimators, $\sigma_A^2=(u_1'M_1u_1+u_2'M_2u_2)/q$ where q is some scalar. So, for instance, under H_0, σ_A^2 is unbiased when $q=v_1+v_2$ and then $\sigma_A^2=s_{AL}^2$. Alternatively, the mean squared error of σ_A^2 is a minimum when $q=v_1+v_2+2$. However, it is unclear as to which member of this family would correspond to a ML estimator.

Accordingly, in this chapter we follow the spirit of the existing literature and we use the family of estimators given by S_A^2 as the always-pooled estimator of $\sigma_{e_1}^2$. So, the researcher pre-tests for the equality of the error variances using the test statistic $J=s_2^2/s_1^2$, which results in the pre-test estimator S_p^2, of $\sigma_{e_1}^2$, where

$$S_p^2 = \begin{cases} S_A^2 & \text{if } J \leq c \\ S_N^2 & \text{if } J > c \end{cases}, \qquad (6.2.9)$$

and c is a critical value of the pre-test associated with some chosen significance level, α. Under these assumptions the researcher will choose c by solving $\int_0^{\infty} F_{(v_2,v_1)}=(1-\alpha)$ as $J\sim F_{(v_2,v_2)}$ under H_0 if (6.2.5) is the true data generating process. However, if the model is mis-specified in the way investigated here then both the null and the non-null distributions of J depend on v_1, v_2, the degree of mis-specification of the design matrix and the variance mixing distribution, $f(\tau)$. We see this in the following theorem and its associated corollaries.

Theorem 6.2.1

Under the above assumptions, the density function of $J=s_2^2/s_1^2$ is

$$f(J) = \frac{1}{\phi}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta_1^s\theta_2^r v_1^{\frac{v_1}{2}+s} v_2^{\frac{v_2}{2}+r} J^{\frac{v_2}{2}+r-1}}{r!s!B\left(\frac{v_2}{2}+r;\frac{v_1}{2}+s\right)\left(v_1+v_2 J\right)^{\frac{1}{2}(v_1+v_2)+r+s}}$$

$$\cdot\int_0^{\infty} e^{(-\theta_1+\theta_2)/\tau^2}\left(\tau^2\right)^{-(r+s)} f(\tau)d\tau \tag{6.2.10}$$

where $\phi=\sigma_{e_1}^2/\sigma_{e_2}^2$; $\theta_1=\gamma_1'Z_1'M_1Z_1\gamma_1/2\phi$; $\theta_2=\gamma_2'Z_2'M_2Z_2\gamma_2/2$.

Proof.

$$f(J) = \int_0^{\infty} f_N(J)f(\tau)d\tau \tag{6.2.11}$$

where $f_N(J)$ is the joint density function of J when $e\sim N(0,\tau^2\Sigma)$. Now, if $e\sim N(0,\tau^2\Sigma)$ then

$$e^* \equiv \begin{bmatrix}(Z_1\gamma_1+e_1)/\sqrt{\phi}\\(Z_2\gamma_2+e_2)\end{bmatrix} \sim N\left(\begin{bmatrix}Z_1\gamma_1/\sqrt{\phi}\\Z_2\gamma_2\end{bmatrix}, \tau^2 I_{T_1+T_2}\right), \tag{6.2.12}$$

so,

$$J = \frac{s_2^2}{s_1^2} = \frac{v_1}{v_2}\cdot\frac{(Z_2\gamma_2+e_2)'M_2(Z_2\gamma_2+e_2)}{(Z_1\gamma_1+e_1)'M_1(Z_1\gamma_1+e_1)} = \frac{v_1\ e^{*\prime}M_2^*e^*}{v_2\phi e^{*\prime}M_1^*e^*} \tag{6.2.13}$$

where M_1^* and M_2^* are $(T\times T)$ idempotent matrices partitioned as $M_1^* = \begin{bmatrix}M_1 & 0\\0 & 0\end{bmatrix}$ and $M_2^* = \begin{bmatrix}0 & 0\\0 & M_2\end{bmatrix}$. Further $r(M_i^*)=r(M_i)=v_i$, $i=1,2$.

Under the normality assumption, it is straightforward to show that the quadratic forms $(e^{*\prime}M_2^*e^*/\tau^2)$ and $(e^{*\prime}M_1^*e^*/\tau^2)$ are independent, and that $(e^{*\prime}M_2^*e^*/\tau^2)\sim\chi^{2\prime}_{v_2;\lambda_{2\tau}}$ and $(e^{*\prime}M_1^*e^*/\tau^2)\sim\chi^{2\prime}_{v_1;\lambda_{1\tau}}$, where

$$\lambda_{1\tau} = \gamma_1'Z_1'M_1Z_1\gamma_1/(2\phi\tau^2) = \theta_1/\tau^2 \tag{6.2.14}$$

and

$$\lambda_{2\tau} = \gamma_2'Z_2'M_2Z_2\gamma_2/(2\tau^2) = \theta_2/\tau^2 . \tag{6.2.15}$$

So, $f_N(J) = \dfrac{v_1 f\left(\chi^{2\prime}_{v_2;\theta_2/\tau^2}\right)}{v_2\phi f\left(\chi^{2\prime}_{v_1;\theta_1/\tau^2}\right)} = \dfrac{1}{\phi}f\left(F''_{(v_2,v_1;\theta_2/\tau^2,\theta_1/\tau^2)}\right)$. Given the density function of a doubly non-central F random variate, and using (6.2.11), (6.2.10) follows directly. #

Corollary 6.2.1

Under the null hypothesis, $\phi=1$,

$$f(J) = \frac{1}{\phi}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta_{10}^{s}\theta_2^{r}v_1^{\frac{v_1}{2}+s}\,v_2^{\frac{v_2}{2}+r}\,J^{\frac{v_2}{2}+r-1}}{r!s!B\left(\frac{v_2}{2}+r;\frac{v_1}{2}+s\right)\left(v_1+v_2J\right)^{\frac{1}{2}(v_1+v_2)+r+s}}$$

$$\cdot\int_0^{\infty}e^{(-\theta_{10}+\theta_2)/\tau^2}\left(\tau^2\right)^{-(r+s)}f(\tau)d\tau \tag{6.2.16}$$

where $\theta_{10} = \gamma_1'Z_1'M_1Z_1\gamma_1/2$.

Proof.

$\theta_1=\theta_{10}$ when $\phi=1$, and so (6.2.16) follows from Theorem 6.2.1 . #

So, if the model is mis-specified by the omission of relevant regressors then both the null and the non-null distribution of J are no longer related to a central F random variate and are dependent on $f(\tau)$. If the design matrices are properly specified $\theta_1=\theta_2=0$ and, from (6.2.10),

$$f(J) = \frac{1}{\phi}\frac{v_1^{\frac{v_1}{2}}v_2^{\frac{v_2}{2}}J^{\frac{v_2}{2}-1}}{B\left(\frac{v_2}{2},\frac{v_1}{2}\right)\left(v_1+v_2J\right)^{\frac{v_1+v_2}{2}}}\int_0^{\infty}f(\tau)d\tau = \frac{1}{\phi}f\left(F_{(v_2,v_1)}\right) . \qquad (6.2.17)$$

which is the result we noted in our discussion in Chapter Two.

As we will illustrate the results assuming that $e\sim Mt\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$, Corollaries 6.2.2 and 6.2.3 derive the non-null distribution of J under this Mt error assumption and that of normality, respectively.

<u>Corollary 6.2.2</u>

If $e\sim Mt\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$ then $\sigma^2_{e_2}=\nu\sigma_2^2/(\nu-2)$, $\sigma^2_{e_1}=\nu\phi\sigma_2^2/(\nu-2)$, and

$$f(J) = \frac{1}{\phi}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2\lambda_1/\nu)^s(2\lambda_2/\nu)^r v_1^{\frac{v_1}{2}+s}v_2^{\frac{v_2}{2}+r}}{r!s!\left[1+2(\lambda_1+\lambda_2)/\nu\right]^{\frac{\nu}{2}+r+s}B\left(\frac{v_2}{2}+r;\frac{v_1}{2}+s\right)}$$

$$\cdot\frac{J^{\frac{v_2}{2}+r-1}\,\Gamma\left(\frac{\nu}{2}+r+s\right)}{\Gamma\left(\frac{\nu}{2}\right)\left(v_1+v_2J\right)^{\frac{1}{2}(v_1+v_2)+r+s}} , \qquad (6.2.18)$$

where $\lambda_i=\theta_i/\sigma_2^2$, i=1,2.

<u>Proof</u>.

$e\sim Mt\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$ when $\tau\sim IG$ with scale parameter σ_2^2, and degrees of freedom parameter ν. So, using Theorem 6.2.1 and $f(\tau)$, which is given by equation (4.2.13),

$$f_{Mt}(J) = \frac{1}{\phi}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{\theta_1^s\theta_2^r v_1^{\frac{v_1}{2}+s}v^{\frac{v_2}{2}+r}J^{\frac{v_2}{2}+r-1}}{r!s!B\left(\frac{v_2}{2}+r;\frac{v_1}{2}+s\right)\left(v_1+v_2J\right)^{\frac{1}{2}(v_1+v_2)+r+s}}$$

$$\cdot\frac{2}{\Gamma\left(\frac{\nu}{2}\right)}\left(\frac{\nu\sigma_2^2}{2}\right)^{\frac{\nu}{2}}\int_0^\infty e^{-[2(\theta_1+\theta_2)+\nu\sigma_2^2]/2\tau^2}\left(\tau^2\right)^{-(\frac{\nu}{2}+\frac{1}{2}+r+s)}f(\tau)d\tau. \qquad (6.2.19)$$

The proof is completed by following an identical procedure to that outlined for the proof to Corollary 4.2.2 . #

Corollary 6.2.3

If $e\sim N(0,\sigma_2^2\Sigma)$ then, $\sigma_{e_2}^2=\sigma_2^2$ and $\sigma_{e_1}^2=\sigma_1^2$ say, and

$$f_N(J) = \frac{1}{\phi}f\left(F''_{(v_2,v_1;\lambda_2,\lambda_1)}\right) , \qquad (6.2.20)$$

where $\lambda_1=\theta_1/\sigma_2^2=\gamma_1'Z_1'M_1Z_1\gamma_1/2\sigma_1^2$, as $\phi=\sigma_{e_1}^2/\sigma_{e_2}^2=\sigma_1^2/\sigma_2^2$, and $\lambda_2=\theta_2/\sigma_2^2=\gamma_2'Z_2'M_2Z_2\gamma_2/2\sigma_2^2$.

Proof.

(6.2.20) is obtained from Corollary 6.2.2 as $e\sim N(0,\sigma_2^2\Sigma)$ when $\nu=\infty$. #

In this section we have established the distribution of the test statistic J, which we use to test for the homogeneity of the error variances. In the next section we derive the exact bias and the exact risk functions of the never-pooled, the always-pooled and the pre-test estimators of $\sigma_{e_1}^2$.

6.3 The Bias and Risk Functions

Now, the pre-test estimator

$$S_P^2 = S_A^2I_{[0,c]}(J) + S_N^2I_{(c,\infty)}(J) , \qquad (6.3.1)$$

reflects the strategy of either pooling the samples if we conclude that there is no heteroscedasticity on the basis of the pre-test, or ignoring the information from the second sample if our test rejects homoscedasticity.

Theorem 6.3.1

If we use the mis-specified model (6.2.5) rather than the true model (6.2.2) when $e\sim ESD_N(0,\Sigma)$ and the pre-test is of H_0 in (6.2.6), then

$$\mathrm{bias}(S_N^2) = \phi\left(2\theta_1-(k+\mu)E(\tau^2)\right)/(T_1+\mu) , \tag{6.3.2}$$

$$\mathrm{bias}(S_A^2) = \left(v_2E(\tau^2)(1-\phi)-2\phi E(\tau^2)(k+\mu)+2(\phi\theta_1+\theta_2)\right)/(T+2\mu), \tag{6.3.3}$$

$$\begin{aligned}\mathrm{bias}(S_P^2) = &\left(\phi(T+2\mu)\left[2\theta_1-(k+\mu)E(\tau^2)\right]+\int_0^{\infty}\left[\tau^2\left(v_2(T_1+\mu)Q_{20}^{d\tau}-\right.\right.\right.\\ &\left.\left. v_1\phi(T_2+\mu)Q_{02}^{d\tau}\right)\right]f(\tau)d\tau+2\int_0^{\infty}\left[\theta_2(T_1+\mu)Q_{40}^{d\tau}-\right.\\ &\left.\left.\theta_1\phi(T_2+\mu)Q_{04}^{d\tau}\right]f(\tau)d\tau\right)/\left((T_1+\mu)(T+2\mu)\right),\end{aligned} \tag{6.3.4}$$

where $\theta_1=\gamma_1'Z_1'M_1Z_1\gamma_1/2\phi$, $\theta_2=\gamma_2'Z_2'M_2Z_2\gamma_2/2$, $M_i=I_{T_i}-X_iS_i^{-1}X_i'$, $i=1,2$, and

$$\begin{aligned}Q_{ij}^{d\tau} &= \Pr.\left[F''_{(v_2+i,v_1+j;\lambda_{1\tau},\lambda_{2\tau})}\le\left(v_2(v_1+j)c\phi\right)/\left(v_1(v_2+i)\right)\right]\\ &= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{-(\lambda_{1\tau}+\lambda_{2\tau})}(\lambda_{2\tau}^r/r!)(\lambda_{1\tau}^s/s!)I_w\left(\tfrac{1}{2}(v_2+i)+r;\tfrac{1}{2}(v_1+j)+s\right) ,\end{aligned} \tag{6.3.5}$$

$i,j=0,1,2\ldots$. $I_w(.;.)$ is Pearson's incomplete beta function with $w=c\phi v_2/(v_1+c\phi v_2)$, $\lambda_{i\tau}=\theta_i/\tau^2$, $i=1,2$.

Proof.

$$\mathrm{bias}(S_N^2)=E(S_N^2)-\sigma_{e_1}^2=E(S_N^2)-\phi\sigma_{e_2}^2=E(S_N^2)-\phi E(\tau^2) , \tag{6.3.6}$$

and $E(S_N^2)=\int_0^{\infty}E_N(S_N^2)f(\tau)d\tau$ where $E_N(S_N^2)=E(S_N^2)$ when $e\sim N(0,\tau^2\Sigma)$. So, $e^*\sim N(0,\tau^2I_{T_1+T_2})$ and $e^{*\prime}M_1^*e^*/\tau^2\sim\chi^{2\prime}_{v_1;\lambda_{1\tau}}$ which gives $E_N(e^{*\prime}M_1^*e^*/\tau^2)=v_1+2\lambda_{1\tau}$, and $E(S_N^2)=\phi\left(v_1E(\tau^2)+2\theta_1\right)/(T_1+\mu)$. Using this and (6.3.6) $\mathrm{bias}(S_N^2)$ follows directly.

Similarly, $S_A^2=(\phi e^{*\prime}M_1^*e^*+e^{*\prime}M_2^*e^*)/(T+2\mu)$, so that

$$E_N(S_A^2) = \tau^2\left(\phi(v_1+2\lambda_{1\tau})+v_2+2\lambda_{2\tau}\right)/(T+2\mu) \qquad (6.3.7)$$

as $(e^{*\prime}M_1^*e^*/\tau^2)\sim\chi^2_{v_1;\lambda_{1\tau}}$, $(e^{*\prime}M_2^*e^*/\tau^2)\sim\chi^2_{v_2;\lambda_{2\tau}}$, and the quadratic forms are independent under the assumption that $e\sim N(0,\tau^2\Sigma)$. Integrating (6.3.7) with respect to τ we have $E(S_A^2)=\left[\phi\left(v_1E(\tau^2)+2\theta_1\right)+v_2E(\tau^2)+2\theta_2\right]/(T+2\mu)$ and so bias(S_A^2) follows directly as bias$(S_A^2)=E(S_A^2)-\phi E(\tau^2)$.

Adopting the same notation we write (6.3.1) as

$$S_P^2 = \left(\phi(T+2\mu)(e^{*\prime}M_1^*e^*)+\left[(T_1+\mu)e^{*\prime}M_2^*e^*-\phi(T_2+\mu)e^{*\prime}M_1^*e^*\right]\right.$$

$$\left.I_{[0,c\phi]}\left((v_1e^{*\prime}M_2^*e^*)/(v_2e^{*\prime}M_1^*e^*)\right)\right)/\left((T_1+\mu)(T+2\mu)\right). \qquad (6.3.8)$$

Now,

$$E(S_P^2) = \int_0^\infty E_N(S_P^2)f(\tau)d\tau \qquad (6.3.9)$$

where $E_N(S_P^2)=E(S_P^2)$ when $e\sim N(0,\tau^2\Sigma)$, so that $e^*\sim N(0,\tau^2I_{T_1+T_2})$. Using Lemma 1 of Clarke *et al.* (1987a)

$$E_N\left[(e^{*\prime}M_2^*e^*/\tau^2)I_{[0,c\phi]}\left((v_1e^{*\prime}M_2^*e^*/\tau^2)/(v_2e^{*\prime}M_1^*e^*/\tau^2)\right)\right]= v_2Q_{20}^{d\tau}+2\lambda_{2\tau}Q_{40}^{d\tau} ,$$

$$E_N\left[(e^{*\prime}M_1^*e^*/\tau^2)I_{[0,c\phi]}\left((v_1e^{*\prime}M_2^*e^*/\tau^2)/(v_2e^{*\prime}M_1^*e^*/\tau^2)\right)\right]= v_1Q_{02}^{d\tau}+ 2\lambda_{1\tau}Q_{04}^{d\tau} ,$$

where $Q_{ij}^{d\tau}$ is given by (6.3.5), $i,j=0,1,2,\ldots$, and $\lambda_{n\tau}$ is as defined previously, $n=1,2$. Hence,

$$E_N(S_P^2)=\left(\phi(T+2\mu)(v_1\tau^2+2\theta_1)+v_2(T_1+\mu)\tau^2Q_{20}^{d\tau}-\right.$$

$$\left.v_1\phi(T_2+\mu)\tau^2Q_{02}^{d\tau}+2\theta_2(T_1+\mu)Q_{40}^{d\tau}-2\theta_1\phi(T_2+\mu)Q_{04}^{d\tau}\right)/\left((T_1+\mu)(T+2\mu)\right) ,$$

and so, using (6.3.9)

$$E(S_P^2) = \left(\phi(T+2\mu)\left[v_1E(\tau^2)+2\theta_1\right]+\int_0^\infty\tau^2\left[v_2(T_1+\mu)Q_{20}^{d\tau}-v_1\phi(T_2+\mu)Q_{02}^{d\tau}\right]f(\tau)d\tau\right.$$

$$\left.+2\int_0^\infty\left[\theta_2(T_1+\mu)Q_{40}^{d\tau}-\theta_1\phi(T_2+\mu)Q_{04}^{d\tau}\right]f(\tau)d\tau\right)/\left((T_1+\mu)(T+2\mu)\right).$$

The proof is completed by noting that $\text{bias}(S_P^2)=E(S_P^2)-\phi E(\tau^2)$. #

We now present three corollaries from Theorem 6.3.1 . Corollary 6.3.1, gives the bias functions of S_N^2, S_A^2 and S_P^2 when we have not excluded regressors $(Z_1\gamma_1=Z_2\gamma_2=0)$, while Corollaries 6.3.2 and 6.3.3 respectively, consider Mt and N regression disturbances.

<u>Corollary 6.3.1</u>

If there are no omitted regressors $(Z_1\gamma_1=Z_2\gamma_2=0)$, and $e\sim ESD_N(0,\Sigma)$, then

$$\text{bias}_0(S_N^2) = -\phi(k+\mu)E(\tau^2)/(T_1+\mu) \;, \tag{6.3.10}$$

$$\text{bias}_0(S_A^2) = E(\tau^2)\Big(v_2(1-\phi)-2\phi(k+\mu)\Big)/(T+2\mu) \;, \tag{6.3.11}$$

$$\text{bias}_0(S_P^2) = E(\tau^2)\Big(-\phi(T+2\mu)(k+\mu)+v_2(T_1+\mu)Q_{20}$$

$$-v_1\phi(T_2+\mu)Q_{02}\Big)\Big/\Big((T_1+\mu)(T+2\mu)\Big) \;, \tag{6.3.12}$$

where

$$Q_{ij} = \Pr.\left[F_{(v_2+i,v_1+j)}\le\Big(v_2(v_1+j)c\phi\Big)\Big/\Big(v_1(v_2+i)\Big)\right]$$

$$= I_w\left(\tfrac{1}{2}(v_2+i);\tfrac{1}{2}(v_1+j)\right) \;, \quad i,j=0,1,\dots \;, \tag{6.3.13}$$

which does not depend on τ.

<u>Proof</u>.

If $Z_1\gamma_1=Z_2\gamma_2=0$ then $\theta_1=\theta_2=0$, and $Q_{ij}^{d\tau}=Q_{ij}$. Then (6.3.10), (6.3.11) and (6.3.12) follow from Theorem 6.3.1 . #

<u>Corollary 6.3.2</u>

If we use the mis-specified model (6.2.5) rather than the true model (6.2.2) when $e\sim Mt\Big(0,\nu\sigma_2^2/(\nu-2)\Sigma\Big)$, and the pre-test is of H_0 in (6.2.6), then for $\nu>2$

$$\text{bias}_{Mt}(S_N^2) = \phi\sigma_2^2\Big(2\lambda_1(\nu-2)-\nu(k+\mu)\Big)\Big/\Big((\nu-2)(T_1+\mu)\Big) \;, \tag{6.3.14}$$

$$\text{bias}_{\text{Mt}}(S_A^2) = \sigma_2^2\left(v_2\nu(1-\phi)-2\phi\nu(k+\mu)+2(\nu-2)(\phi\lambda_1+\lambda_2)\right)/\left((\nu-2)(T+2\mu)\right) \tag{6.3.15}$$

$$\text{bias}_{\text{Mt}}(S_P^2) = \sigma_2^2\left(\phi(T+2\mu)\left[2\lambda_1(\nu-2)-\nu(k+\mu)\right]+v_2\nu(T_1+\mu)Q_{201}^d-v_1\phi\nu(T_2+\mu)Q_{021}^d\right.$$

$$\left.+2\lambda_2(T_1+\mu)(\nu-2)Q_{402}^d-2\lambda_1\phi(T_2+\mu)(\nu-2)Q_{042}^d\right)/\left((\nu-2)(T_1+\mu)(T+2\mu)\right) \tag{6.3.16}$$

where

$$Q_{ijn}^d = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{(2\lambda_2/\nu)^r(2\lambda_1/\nu)^s\Gamma\left(\frac{\nu}{2}+r+s+n-2\right)}{r!s!\left[1+2(\lambda_1+\lambda_2)/\nu\right]^{\frac{\nu}{2}+r+s+n-2}\Gamma\left(\frac{\nu}{2}+n-2\right)}$$

$$.I_w\left[\tfrac{1}{2}(v_2+i)+r;\tfrac{1}{2}(v_1+j)+s\right]. \tag{6.3.17}$$

If there are no omitted regressors, $Z_1\gamma_1=Z_2\gamma_2=0$, then

$$\text{bias}_{\text{OMt}}(S_N^2) = -\phi\sigma_2^2\nu(k+\mu)/\left((\nu-2)(T_1+\mu)\right), \tag{6.3.18}$$

$$\text{bias}_{\text{OMt}}(S_A^2) = \nu\sigma_2^2\left(v_2(1-\phi)-2\phi(k+\mu)\right)/\left((\nu-2)(T+2\mu)\right), \tag{6.3.19}$$

$$\text{bias}_{\text{OMt}}(S_P^2) = \nu\sigma_2^2\left(-\phi(T+2\mu)(k+\mu)+v_2(T_1+\mu)Q_{20}-v_1\phi(T_2+\mu)Q_{02}\right)$$

$$/\left((\nu-2)(T_1+\mu)(T+2\mu)\right) . \tag{6.3.20}$$

<u>Proof</u>.

$e\sim\text{Mt}\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$ when $\tau\sim$IG with scale parameter σ_2^2, and degrees of freedom parameter ν. So, $\sigma_{e_2}^2=\nu\sigma_2^2/(\nu-2)$, $\sigma_{e_1}^2=\phi\sigma_{e_2}^2=\phi\nu\sigma_2^2/(\nu-2)$, and

$$E(\tau^2) = \nu\sigma_2^2/(\nu-2) . \tag{6.3.21}$$

We then obtain (6.3.14) and (6.3.15) from (6.3.2) and (6.3.3), respectively. From (4.3.19) and (4.3.37) we have

$$\int_0^{\infty} Q_{ij}^{d\tau}f(\tau)d\tau = Q_{ij2}^d \tag{6.3.22}$$

and

$$\int_0^\infty \tau^2 Q^{d\tau}_{ij} f(\tau)d\tau = \nu\sigma_2^2 Q^d_{ij1}/(\nu-2) \ , \tag{6.3.23}$$

$i,j=0,1,2,\ldots$. So, we use (6.3.21), (6.3.22) and (6.3.23) in conjunction with (6.3.4) to establish $\text{bias}_{Mt}(S_P^2)$.

To complete the proof we only need to note that if $Z_1\gamma_1=Z_2\gamma_2=0$ then $\lambda_1=\lambda_2=0$ and $Q^d_{ijn}=Q_{ij}$. Then, (6.3.18), (6.3.19) and (6.3.20) follow from (6.3.14), (6.3.15) and (6.3.16) respectively. We could alternatively, obtain these expressions by using Corollary 6.3.1 and (6.3.21). #

Corollary 6.3.3

If we use the mis-specified model (6.2.5) rather than the true model (6.2.2) when $e\sim N(0,\sigma_2^2\Sigma)$, and the pre-test is of H_0 in (6.2.6), then $\sigma^2_{e_2}=\sigma^2_2$, $\sigma^2_{e_1}=\sigma^2_1$ (say), and [5]

$$\text{bias}_N(\sigma_1^2,S_N^2) = \phi\sigma_2^2\Big(2\lambda_1-(k+\mu)\Big)/(T_1+\mu) \ , \tag{6.3.24}$$

$$\text{bias}_N(\sigma_1^2,S_A^2) = \sigma_2^2\Big(v_2(1-\phi)-2\phi(k+\mu)+2(\phi\lambda_1+\lambda_2)\Big)/(T+2\mu) \ , \tag{6.3.25}$$

$$\text{bias}_N(\sigma_1^2,S_P^2) = \sigma_2^2\Big(\phi(T+2\mu)\Big[2\lambda_1-(k+\mu)\Big]+v_2(T_1+\mu)Q^d_{20}$$

$$-v_1\phi(T_2+\mu)Q^d_{02}+2\lambda_2(T_1+\mu)Q^d_{40}-2\lambda_1\phi(T_2+\mu)Q^d_{04}\Big)/\Big((T_1+\mu)(T+2\mu)\Big) \tag{6.3.26}$$

where

$$Q^d_{ij} = \Pr.\Big[F''_{(v_2+i,v_1+j;\lambda_2,\lambda_1)}\le\Big(v_2(v_1+j)c\phi\Big)/\Big(v_1(v_2+i)\Big)\Big] \ , \tag{6.3.27}$$

$i,j=0,1,2,\ldots$.

If there are no omitted regressors, $Z_1\gamma_1=Z_2\gamma_2=0$, then

$$\text{bias}_{ON}(S_N^2) = -\phi\sigma_2^2(k+\mu)/(T_1+\mu) \ , \tag{6.3.28}$$

$$\text{bias}_{ON}(S_A^2) = \sigma_2^2\Big(v_2(1-\phi)-2\phi(k+\mu)\Big)/(T+2\mu) \ , \tag{6.3.29}$$

---

[5] Note that if $e\sim N(0,\sigma_2^2\Sigma)$ then $\lambda_1=\theta_1/\sigma_2^2=\gamma_1'Z_1'M_1Z_1\gamma_1/2\sigma_1^2$.

$$\mathrm{bias}_{0N}(S_P^2) = \sigma_2^2\Big(-\phi(T+2\mu)(k+\mu)+v_2(T_1+\mu)Q_{20}-v_1\phi(T_2+\mu)Q_{02}\Big)\Big/$$

$$\Big((T_1+\mu)(T+2\mu)\Big) \ . \qquad (6.3.30)$$

Proof.

We establish this corollary using Corollary 6.3.2 as $e\sim N(0,\sigma_2^2\Sigma)$ when $\nu=\infty$. We use an identical procedure to that outlined in the proof of Corollary 4.3.3 to show that $\lim_{\nu\to\infty} Q^d_{ijn}=Q^d_{ij}$, $i,j,n=0,1,2,\ldots$. #

These theorems and corollaries have derived the bias functions of general families of estimators. Three members of particular interest are the so-called L, ML, and M never-pool, always-pool, and pre-test estimators of the error variance in the first sample. Their bias functions are easily obtained from the theorems and the corollaries given here, and we detail some of them in Appendix 6.1.

We have numerically evaluated the expressions given in the special cases of Corollaries 6.3.2 and 6.3.3 in Appendix 6.1, which consider Mt and normal regression disturbances respectively, for various choices of ν, α, v_1, v_2, k (and hence, T_1 and T_2) as functions of λ_1 and λ_2 and ϕ.[6] We have numerically evaluated the expressions for a wide range of these arguments: $v_1=v_2=20$; $v_1=16$, $v_2=8$; $v_1=25$, $v_2=10$; k=3,4,5; α=0.01, 0.05, 0.30, 0.50, 0.75 and those values of α associated with critical values of unity (c^*_L), of $(v_1T_2)/(v_2T_1)$ (c^*_{ML}) and of $\Big(v_1(v_2+2)\Big)\Big/\Big(v_2(v_1+2)\Big)$ (c^*_M);[7] ν=5, 10, 100, ∞;

---

[6] We recall that the notation s^2_{Pi} and s^2_{Ai}, i=L,ML,M, identifies that the corresponding never-pool estimators, s^2_{Ni}, are the L, ML, and M estimators of $\sigma^2_{e_1}$ when there are no excluded regressors from the design matrix and when the regression disturbances are normally distributed.

[7] We will show that the critical values c^*_L, c^*_{ML}, and c^*_M result in a minimum of the bias functions of s^2_{PL}, s^2_{PML}, and s^2_{PM}, respectively.

$\lambda_1 \in [0,5(0.5);5,10(1.0);10,20(2.0)]$; $\lambda_2 \in [0,5(0.5);5,10(1.0);10,20(2.0)]$; $\phi \in [0.05,1.0(0.05)]$. We have used Davies' (1980) algorithm and the subroutines GAMMLN and BETAI from Press *et al.* (1986) to assist with the evaluations of Q^d_{ij} and Q^d_{ijn}, respectively, i,j,n,=0, 1, 2,... . The computer programs were executed on a VAX 6230 computer.

The case of $v_1=16$, $v_2=8$, $k=3$ is illustrated in Figures 6.3.1 to 6.3.24 . These diagrams depict the relative bias functions[8] of the estimators as functions of ϕ for given values of λ_1 and λ_2. The values of λ_1 and λ_2 considered are 0 and 3, and $\phi \in [0.05,1.0]$.

We illustrate the relative bias functions of s^2_{NL}, s^2_{AL}, and s^2_{PL} in Figures 6.3.1 to 6.3.8; s^2_{NML}, s^2_{AML}, and s^2_{PML} in Figures 6.3.9 to 6.3.16; and s^2_{NM}, s^2_{AM}, and s^2_{PM} in Figures 6.3.17 to 6.3.24. In each of these sets of figures we present the relative bias functions when there are no omitted variables from either model ($\lambda_1=\lambda_2=0$); when the model for sample one is mis-specified but that for sample two is not ($\lambda_1=3$, $\lambda_2=0$); when the design matrix for sample one is correctly specified but that for sample two is mis-specified ($\lambda_1=0$, $\lambda_2=3$); and finally, when both models are mis-specified to the same degree ($\lambda_1=\lambda_2=3$). We illustrate the results for $\nu=5$ and $\nu=\infty$, and for the aforementioned values of α. We have not maintained the same scales on each of the diagrams so that the features of the relative bias functions are discernible. Further, note that negative relative bias values are in parentheses and the legend of the line types associated with each of the estimators follows. There are two relative bias functions with the same

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[8] We define the relative bias of an estimator $\bar{s}^2$ of $\sigma^2_{e_1}$ as $\mathrm{Rbias}(\bar{s}^2)=\mathrm{bias}(\bar{s}^2)/\sigma^2_2$. The scaling is undertaken so as to eliminate the scale parameter σ^2_2. It is for this reason that we utilise λ_1 and λ_2 rather than θ_1 and θ_2 in the diagrams. Of course, the figures can be interpreted as depicting the bias functions when $\sigma^2_2=1$.

line type. So, we distinguish the unrestricted estimator by an appropriate label and arrow.

Legend for Figures 6.3.1 to 6.3.24

Rbias(s^2_{Ni})	Rbias(s^2_{Ai})	Rbias(s^2_{Pi}) $\alpha = 0.01$
Rbias(s^2_{Pi}) $\alpha = 0.05$	Rbias(s^2_{Pi}) $\alpha = 0.30$	Rbias(s^2_{Pi}) $\alpha = 0.75$
	Rbias(s^2_{Pi}) $c = c^*_i$	

We now present some features of the bias functions.

(a) The bias expressions derived by Bancroft (1944) are easily obtained by setting $\lambda_1=\lambda_2=0$ in (A6.19), (A6.20), and (A6.21) of Appendix 6.1 .[9]

(b) If α=1, that is, we always reject H_0, then c=0, $Q^{d\tau}_{ij}=0$ for all i,j, and the bias function of the pre-test estimator equals that of the never-pool estimator. Conversely, if α=0, that is, we never reject H_0, then c=∞, $Q^{d\tau}_{ij}=1$ for all i,j, and the bias function of the pre-test estimator equals that of the always-pool estimator.

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[9] The apparent difference is merely due to the fact that Bancroft considers the alternative hypothesis that $\sigma^2_2 < \sigma^2_1$ whereas we postulate the converse. See also Toyoda and Wallace (1975), and Bancroft and Han (1983).

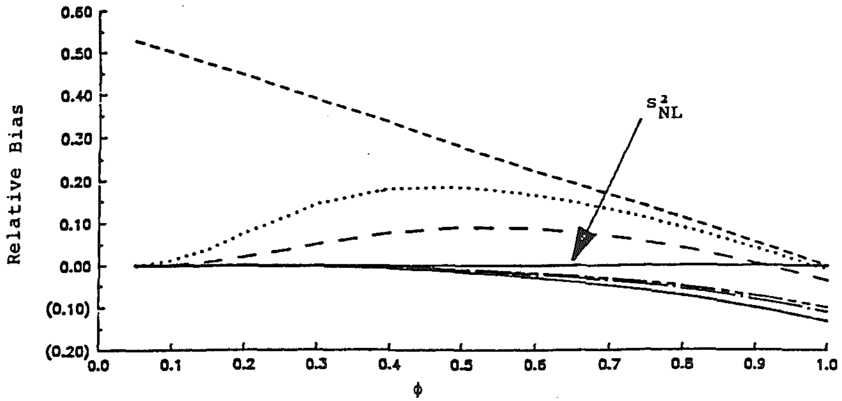

FIGURE 6.3.1: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = 5$.

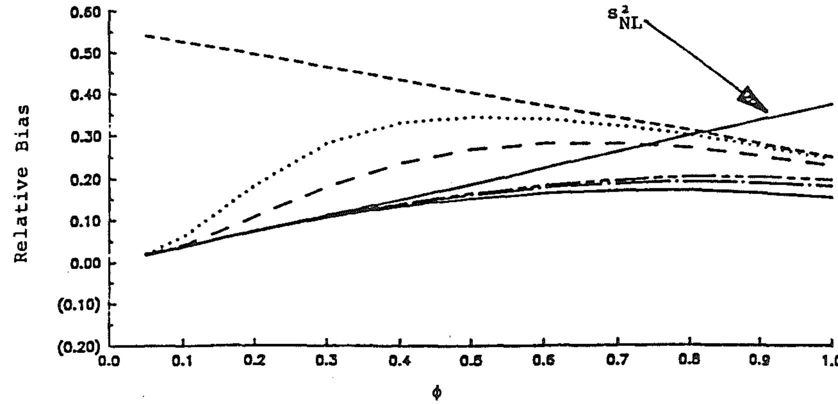

FIGURE 6.3.3: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = 5$.

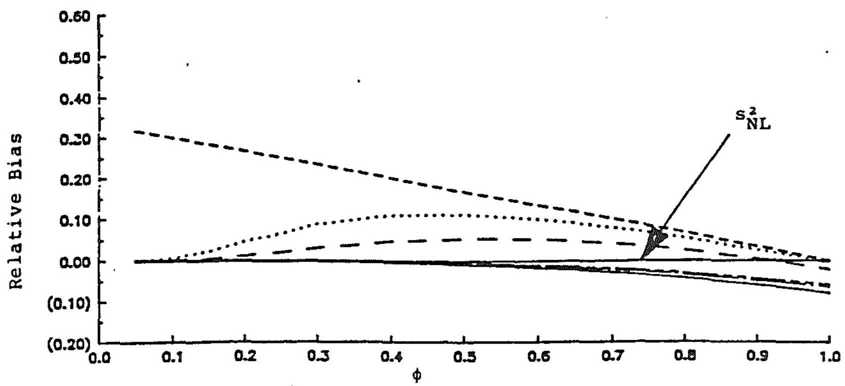

FIGURE 6.3.2: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = \infty$.

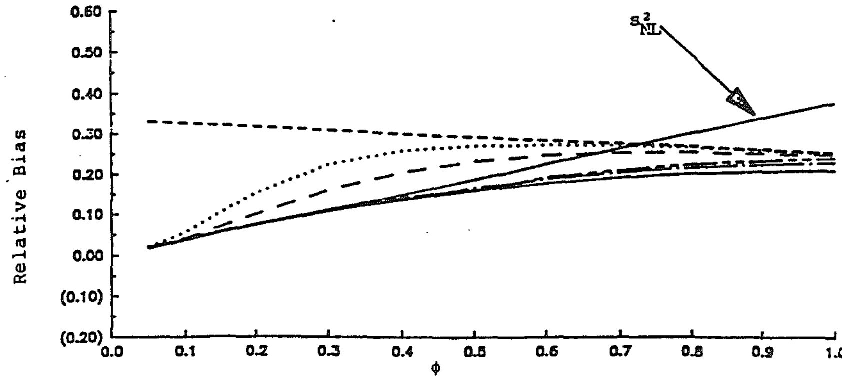

FIGURE 6.3.4: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = \infty$.

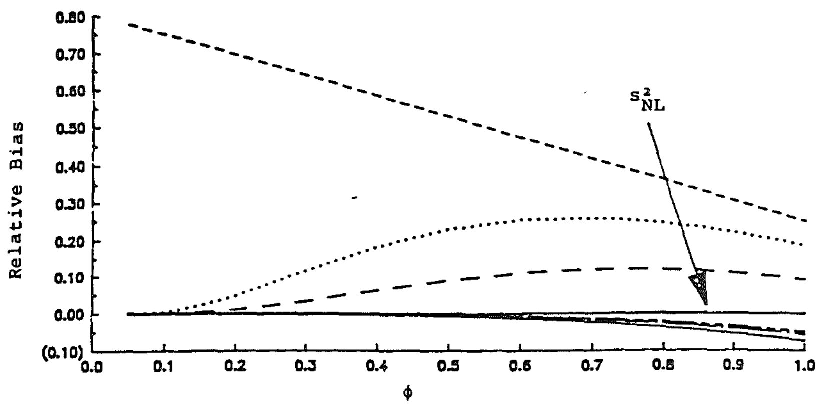

FIGURE 6.3.5: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = 5$.

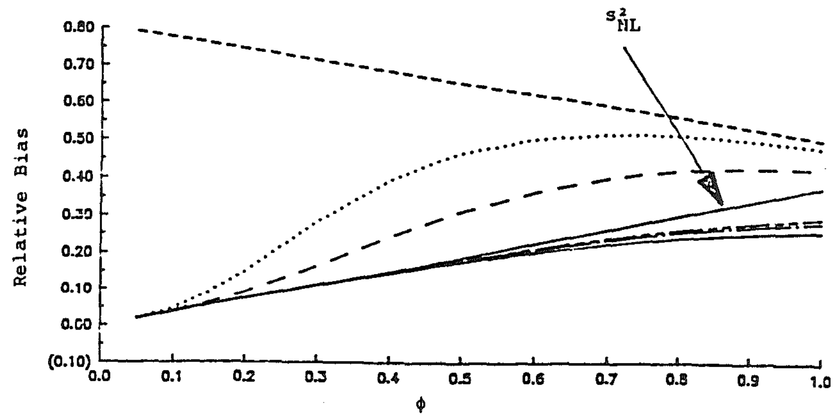

FIGURE 6.3.7: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = 5$.

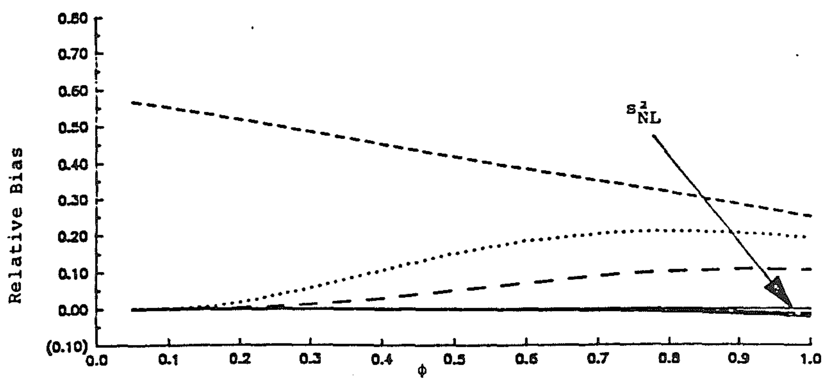

FIGURE 6.3.6: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = \infty$.

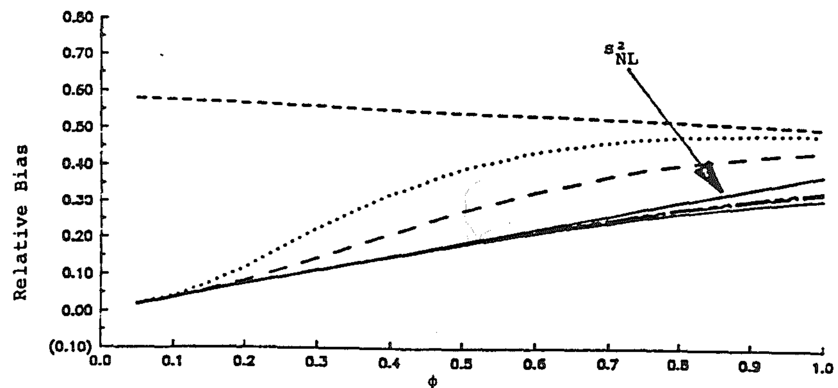

FIGURE 6.3.8: Relative bias functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = \infty$.

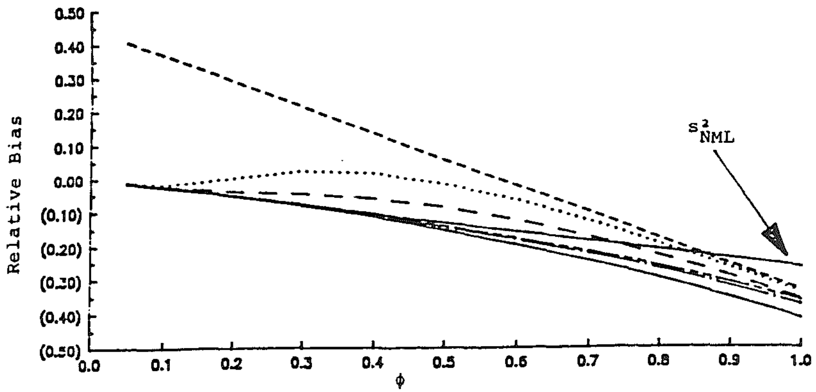

FIGURE 6.3.9: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = 5$.

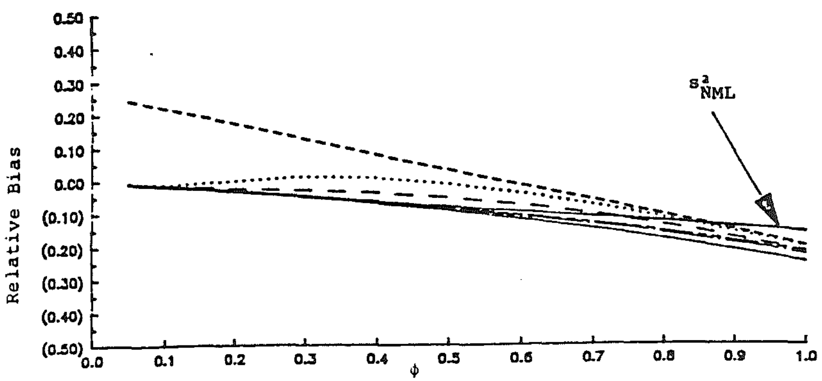

FIGURE 6.3.10: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = \infty$.

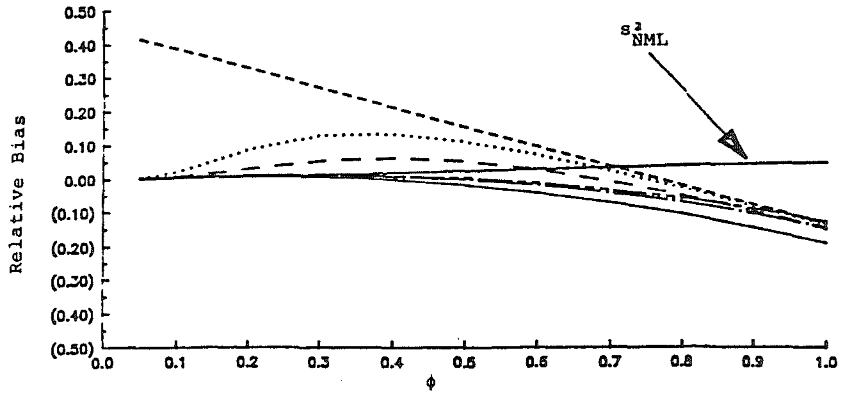

FIGURE 6.3.11: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = 5$.

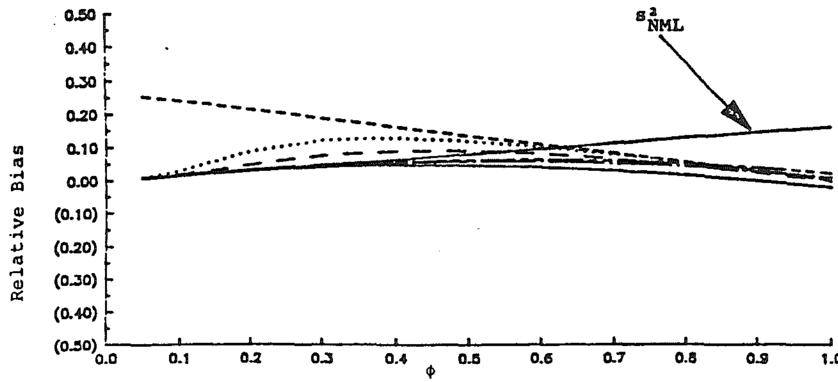

FIGURE 6.3.12: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = \infty$.

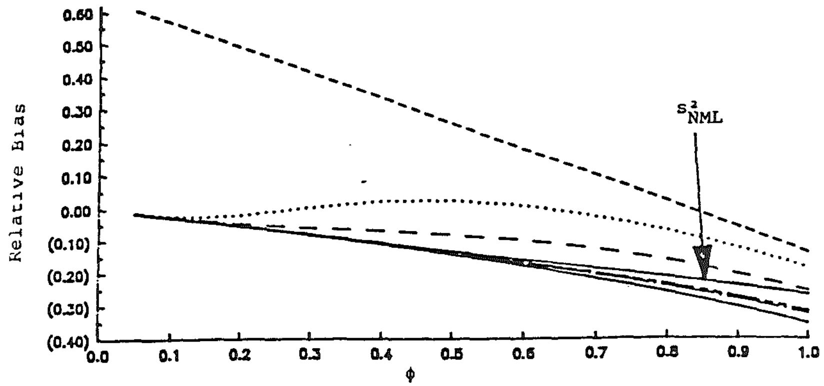

FIGURE 6.3.13: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = 5$.

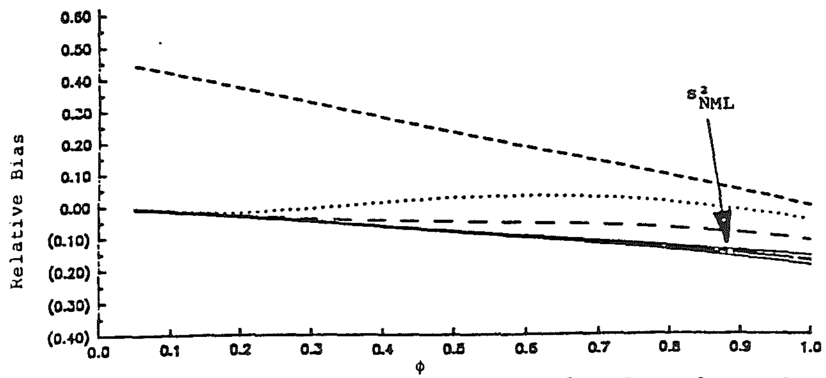

FIGURE 6.3.14: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0,\sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = \infty$.

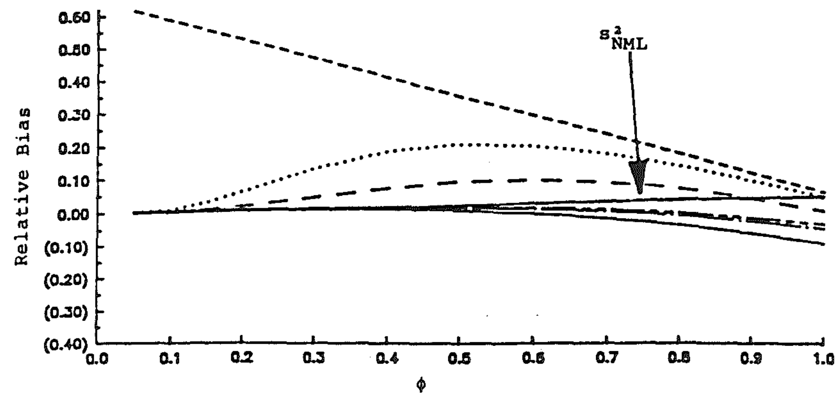

FIGURE 6.3.15: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = 5$.

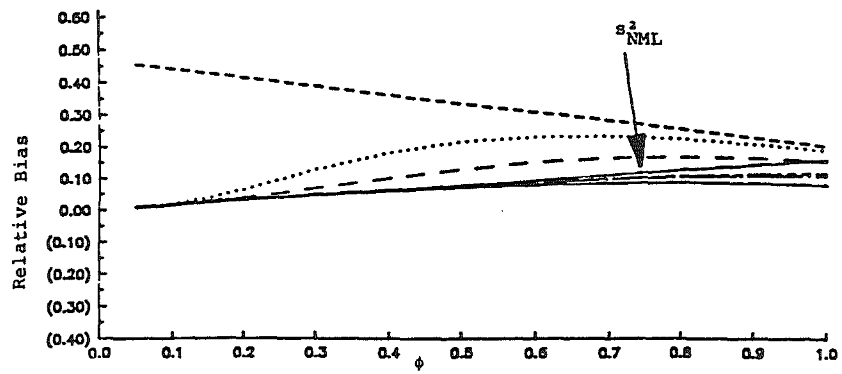

FIGURE 6.3.16: Relative bias functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0,\sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = \infty$.

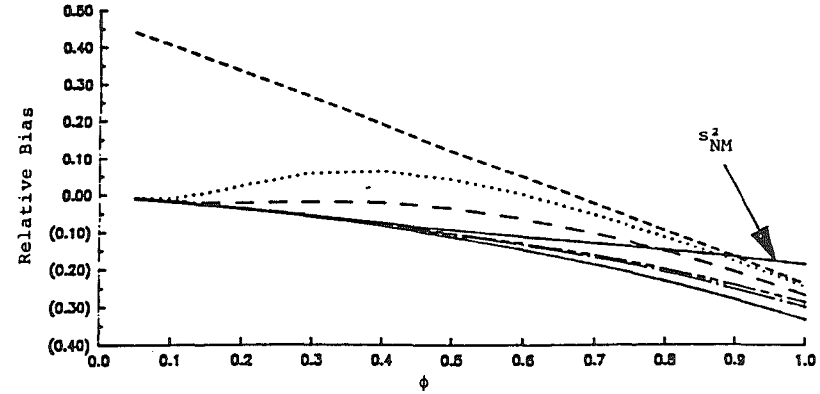

FIGURE 6.3.17: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = 5$.

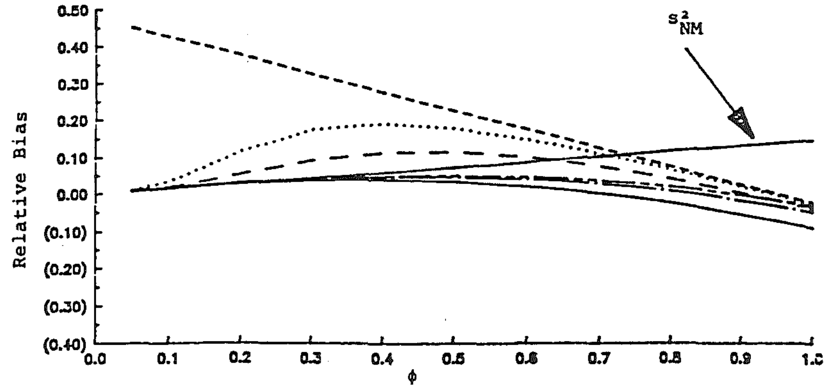

FIGURE 6.3.19: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = 5$.

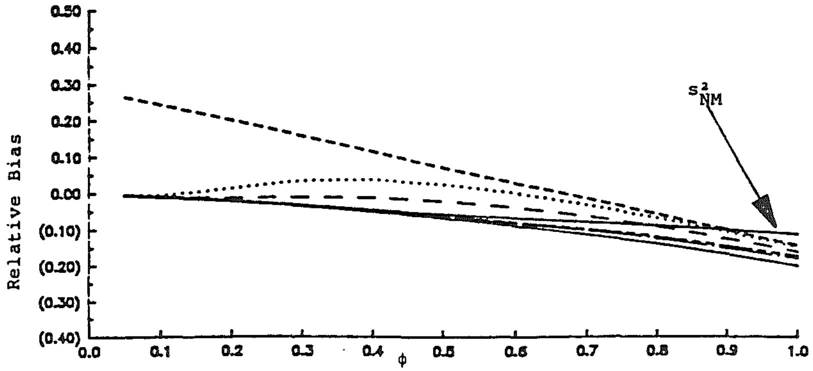

FIGURE 6.3.18: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 0$, $\nu = \infty$.

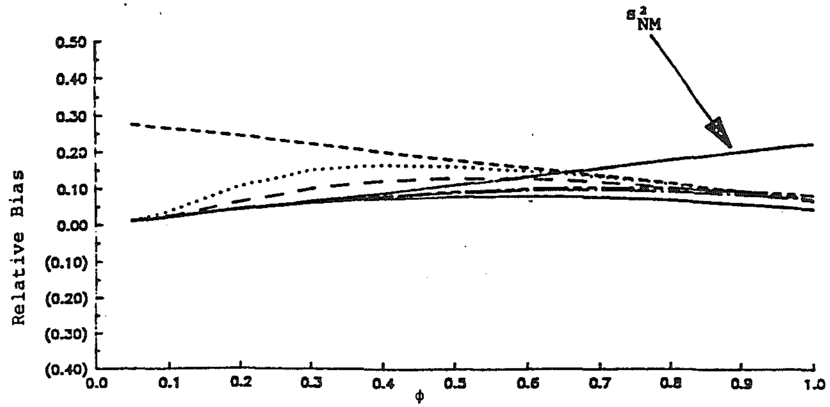

FIGURE 6.3.20: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 3$, $\lambda_2 = 0$, $\nu = \infty$.

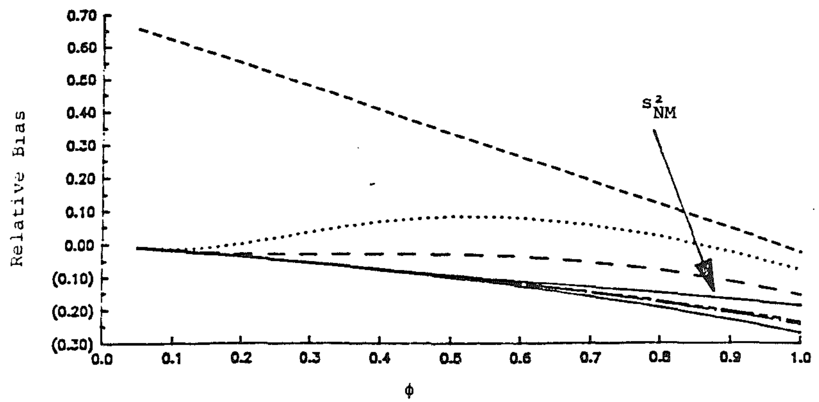

FIGURE 6.3.21: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = 5$.

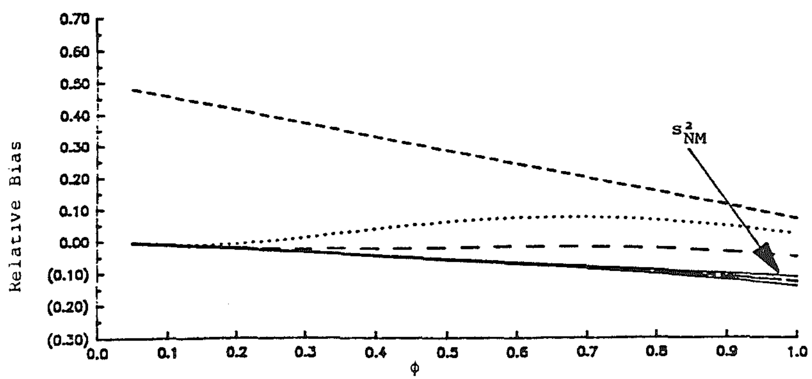

FIGURE 6.3.22: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = 0$, $\lambda_2 = 3$, $\nu = \infty$.

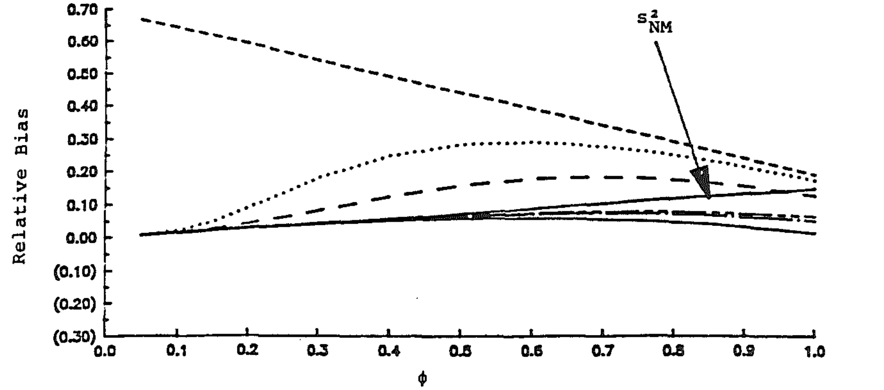

FIGURE 6.3.23: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = 5$.

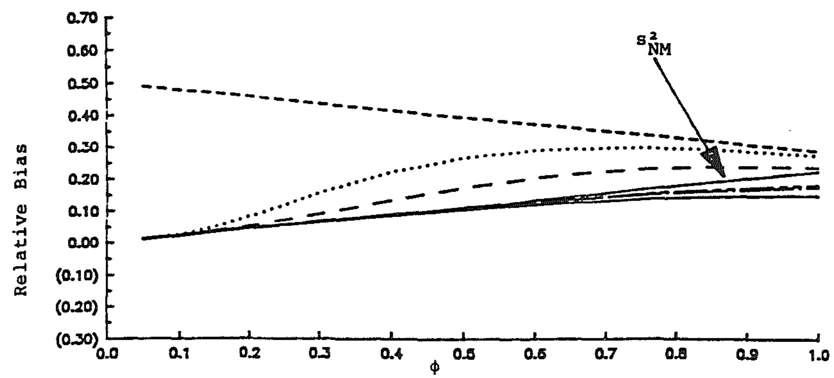

FIGURE 6.3.24: Relative bias functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\lambda_1 = \lambda_2 = 3$, $\nu = \infty$.

(c) bias(S_N^2) increases monotonically with θ_1, but is independent of θ_2. On the other hand, bias(S_A^2) monotonically increases with both θ_1 and θ_2, while bias(S_P^2), is a second-order function in both θ_1 and θ_2.

For non-zero ϕ, bias(S_A^2) is unbounded either as $\theta_1\to\infty$ (given θ_2) or as $\theta_2\to\infty$ (given θ_1). bias(S_N^2) is bounded as $\theta_2\to\infty$ (given θ_1) but is unbounded as $\theta_1\to\infty$, while bias(S_P^2) is unbounded as $\theta_1\to\infty$ (given θ_2) but it is bounded (by bias(S_N^2)) as $\theta_2\to\infty$ (given θ_1), as $Q_{ij}^{d\tau}\to 0$ as $\theta_2\to\infty$.. If $\phi\to 0$ then the bias of the always-pool estimator approaches $\left(v_2E(\tau^2)+2\theta_2\right)/(T+2\mu)$ which is bounded for finite θ_2 but it is unbounded as $\theta_2\to\infty$. Conversely, the bias of the pre-test estimator approaches the bias of the never-pool estimator, which approaches zero. Intuitively, for very small values of ϕ, pre-testing leads us to follow the correct path of rejecting H_0.

(d) If H_0 is true, $\phi=1$, $\theta_1=\theta_{10}=\gamma_1'Z_1'M_1Z_1\gamma_1/2$, and

$$\text{bias}_1(S_N^2) = \left(2\theta_{10}-(k+\mu)E(\tau^2)\right)/(T_1+\mu)\ , \tag{6.3.31}$$

$$\text{bias}_1(S_A^2) = \left(-2E(\tau^2)(k+\mu)+2(\theta_{10}+\theta_2)\right)/(T+2\mu)\ , \tag{6.3.32}$$

$$\begin{aligned}\text{bias}_1(S_P^2) = \Bigg[&(T+2\mu)\left(2\theta_{10}-(k+\mu)E(\tau^2)\right)+\int_0^\infty \tau^2\Big(v_2(T_1+\mu)\,{}_1Q_{20}^{d\tau}\\ &-v_1(T_2+\mu)\,{}_1Q_{02}^{d\tau}\Big)f(\tau)d\tau+2\int_0^\infty\Big(\theta_2(T_1+\mu)\,{}_1Q_{40}^{d\tau}\\ &-\theta_{10}(T_2+\mu)\,{}_1Q_{04}^{d\tau}\Big)f(\tau)d\tau\Bigg]\Big/\left((T_1+\mu)(T+2\mu)\right)\ ,\end{aligned} \tag{6.3.33}$$

where

$${}_1Q_{ij}^{d\tau} = \Pr.\left[F''_{(v_2+i,v_1+j;\lambda_{10\tau},\lambda_{2\tau})}\le\left(v_2(v_1+j)c\right)/\left(v_1(v_2+i)\right)\right]\ ,$$

$\lambda_{10\tau}=\theta_{10}/\tau^2$, $\lambda_{2\tau}=\theta_2/\tau^2$, $i,j=0,1,2,\ldots$.

So, the sign of $\left(\text{bias}_1(S_N^2)-\text{bias}_1(S_A^2)\right)$ will depend on the values of the arguments. The difference is unbounded either as $\theta_{10}\to\infty$ (given θ_2) or as $\theta_2\to\infty$ (given θ_1). If $\theta_{10}=\theta_2=0$, that is, neither model omits relevant regressors and $\mu=-k$, then the difference is equal to zero. This corresponds to the L component estimators case, for which, irrespective of $f(\tau)$, s^2_{NL} is unbiased for all ϕ and s^2_{AL} is unbiased for $\phi=1$. See, for example, Figures 6.3.1 and 6.3.2 . This unbiasedness need no longer hold if the models are mis-specified, and in such cases it is difficult to generalize about the sign of $\left(\text{bias}_1(S^2_N)-\text{bias}_1(S^2_A)\right)$. If, however, $\mu\neq-k$ and $\theta_{10}=\theta_2=0$, then the difference is $\gtreqless$ for $T_2 \lesseqgtr T_1$. Figures 6.3.9 and 6.3.10, and Figures 6.3.17 and 6.3.18, illustrate this result for the ML and the M components respectively. In these examples $T_2<T_1$ and so, the bias difference is greater than zero when $\phi=1$. Whether S^2_N has higher bias than S^2_A, when we have omitted regressors, and $\mu\neq-k$, depends on the degrees of mis-specification, on the magnitude of T_1 relative to T_2, and on the values of μ and $E(\tau^2)$.

$\left(\text{bias}_1(S^2_A)-\text{bias}_1(S^2_P)\right)$ also cannot be signed without knowledge of the arguments. For instance, if $\mu=-k$ (i.e. the L components) and $\theta_{10}=\theta_2=0$, then this difference is equal to $\left(-v_2(Q_{20}-Q_{02})/(v_1+v_2)\right)$ which is positive for all v_1 and v_2. Thus, there is a bias penalty for pre-testing when the prior information is valid and when we have not excluded relevant regressors. We note that the pre-test estimator is biased even though both of its components are unbiased in this situation.[10]

Our results suggest that if $\mu>-k$ (as it is for the ML and the M components) and $T_1>T_2$ then the pre-test estimator has higher absolute bias than the always-pool estimator under H_0 and assuming no specification error. That is, it is always preferable in terms of bias to impose valid prior information. This conclusion, however, no longer follows if there are

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[10] We recall our previous discussion regarding the distinction between the moments of a combined estimator and the combination of the moments of estimators.

omitted regressors. Then, the pre-test estimator may have smaller bias than the always-pool estimator, even if the error variances are equal. See, for example, Figures 6.3.3 to 6.3.8 .

Similarily, $\left(\text{bias}_1(S_P^2)-\text{bias}_1(S_N^2)\right)$ depends on the arguments of the problem. If the model is correctly specified then the difference is equal to $\left(v_2(T_1+\mu)Q_{20}-v_1(T_2+\mu)Q_{02}\right)/\left((T_1+\mu)(T+2\mu)\right)$ which is negative if μ=-k. This verifies our earlier result that s_{PL}^2 is negatively biased under H_0 whereas s_{NL}^2 is unbiased.

Whether or not $|\text{bias}_1(S_P^2)|>|\text{bias}_1(S_N^2)|$ when $\mu\neq$-k depends on the relative values of T_1 and T_2. In our example in which T_1=19 and T_2=11 we find, for the correctly specified model, that $|\text{bias}_1(s_{PML}^2)|>|\text{bias}_1(s_{NML}^2)|$ and $|\text{bias}_1(s_{PM}^2)|>|\text{bias}_1(s_{NM}^2)|$ irrespective of the value of ν. So, it would seem that in terms of minimising absolute bias it is preferable to ignore the prior information, even if it is valid, than to pre-test.

However, if we have omitted regressors then, pre-testing may result in the smallest absolute bias, regardless of the value of ϕ. In such situations the appropriate choice of the critical value is $c^*=\left(v_1(T_2+\mu)\right)/\left(v_2(T_1+\mu)\right)$. The following proposition proves that this critical value results in a minimum of the pre-test bias function for all possible ϕ. This proposition is valid for all feasible members of the family of error distributions that we are considering, and the result holds irrespective of whether or not there are excluded regressors. We include only a short proof here as its form follows the same lines as that for Proposition 5.2.1 .

Proposition 6.3.1

The pre-test bias function has a minimum when $c^*=\left(v_1(T_2+\mu)\right)/\left(v_2(T_1+\mu)\right)$.

Proof.

$$\text{bias}(S_P^2)=E\left(S_P^2-\sigma_{e_1}^2\right)=E\left[\left(S_A^2-\phi E(\tau^2)\right)I_{[0,c]}(J)+\left(S_N^2-\phi E(\tau^2)\right)I_{(c,\infty)}(J)\right]$$

$$=\int_0^{\infty}\tau^2 E_N\Bigg\{\Big(\phi(e^{*\prime}M_1^*e^*/\tau^2)/(T_1+\mu)-\phi E(\tau^2)/\tau^2\Big)+\Big[\Big((e^{*\prime}M_2^*e^*/\tau^2)(T_1+\mu)-$$

$$\phi(e^{*\prime}M_1^*e^*/\tau^2)(T_2+\mu)\Big)/\Big((T_1+\mu)(T+2\mu)\Big)I\Big((e^{*\prime}M_2^*e^*/\tau^2)$$

$$\le c\phi v_2(e^{*\prime}M_1^*e^*/\tau^2)/v_1\Big)\Big]\Bigg\}f(\tau)d\tau, \qquad (6.3.34)$$

where $E_N\{.\}=E\{.\}$ when $e\sim N(0,\tau^2\Sigma)$. Under this normality assumption it is straightforward to show that the quadratic forms $(e^{*\prime}M_2^*e^*/\tau^2)$ and $(e^{*\prime}M_1^*e^*/\tau^2)$ are independent, and that they are, respectively, non-central Chi square random variates with v_2 and v_1 degrees of freedom and non-centrality parameters $\lambda_{2\tau}$ and $\lambda_{1\tau}$. So, we can write (6.3.34) as

$$\text{bias}(S_P^2) = \int_0^{\infty}\tau^2 E_N^1\Bigg\{\Big((\phi e^{*\prime}M_1^*e^*/\tau^2)/(T_1+\mu)-\phi E(\tau^2)/\tau^2\Big)+\int_0^{g}\Big[\Big((e^{*\prime}M_2^*e^*/\tau^2)$$

$$.(T_1+\mu)-(\phi e^{*\prime}M_1^*e^*/\tau^2)(T_2+\mu)\Big)/\Big((T_1+\mu)(T+2\mu)\Big)\Big]$$

$$.f_N(e^{*\prime}M_2^*e^*/\tau^2)d(e^{*\prime}M_2^*e^*/\tau^2)\Bigg\}f(\tau)d\tau\ ,$$

where $g=c\phi v_2(e^{*\prime}M_1^*e^*/\tau^2)/v_1$, $E_N^1\{.\}$ and $E_N^2\{.\}$ are the $E_N\{.\}$ with respect to $(e^{*\prime}M_1^*e^*/\tau^2)$ and $(e^{*\prime}M_2^*e^*/\tau^2)$ respectively, and $f_N(e^{*\prime}M_2^*e^*/\tau^2)$ is the density function of $(e^{*\prime}M_2^*e^*/\tau^2)$ under the normality assumption. So,

$$\frac{\partial\,\text{bias}(S_P^2)}{\partial c}=\int_0^{\infty}\tau^2E_N^1\Bigg\{\left(\frac{c\phi v_2 e^{*\prime}M_1^*e^*}{\tau^2 v_1}\right)\left(\frac{\phi e^{*\prime}M_1^*e^*}{\tau^2}\right)f_N\left(\frac{c\phi v_2 e^{*\prime}M_1^*e^*}{\tau^2 v_1}\right)$$

$$.\Big((cv_2/v_1)(T_1+\mu)-(T_2+\mu)\Big)/\Big((T_1+\mu)(T+2\mu)\Big)\Bigg\}f(\tau)d\tau\ . \qquad (6.3.35)$$

A sufficient condition for (6.3.35) to be zero is for $cv_2(T_1+\mu)=v_1(T_2+\mu)$, that is, $c^*=\Big(v_1(T_2+\mu)\Big)/\Big(v_2(T_1+\mu)\Big)$. We can check that c^* results in a minimum, not a maximum, of the bias function. #

So, $c^*_L=1$, $c^*_{ML}=(v_1T_2)/(v_2T_1)$, and $c^*_M=\left[v_1(v_2+2)\right]/\left[v_2(v_1+2)\right]$. We have included the pre-test estimator with these critical values in our evaluations. For the case illustrated in the figures a critical value of unity corresponds to a nominal size of 47.3% while nominal sizes of 38% and of 40.6% result in critical values of $(v_1T_2)/(v_2T_1)$ and of $\left[v_1(v_2+2)\right]/\left[v_2(v_1+2)\right]$ respectively.

Our results suggest, first, that if $\theta_1>0$ and $\theta_2\geq0$, and we employ the L components, then the smallest absolute bias occurs when we pre-test using a critical value of unity. However, if $\theta_1=0$, regardless of the value of θ_2, then s^2_{NL} is unbiased. Secondly, the results suggest, when using the ML or the M components in a mis-specified model, that there is no definitive strategy. In this situation we need to specify some optimality criterion in order to choose an appropriate estimator.

(e) The bias function of s^2_{NL} is independent of $f(\tau)$ while those of s^2_{AL} and s^2_{PL} depend on the specific distribution of the regression disturbances. When $e\sim Mt(0,\nu\sigma^2_2/(\nu-2)\Sigma)$ we find that $bias_{Mt}(s^2_{AL})$, for given values of λ_1 and λ_2,[11] increases for $\phi\in(0,1)$ as ν decreases. If $\phi=1$ (that is, the error variances are equal), then $bias_{Mt}(s^2_{AL}|\phi=1)=2\sigma^2_2(\lambda_1+\lambda_2)/(v_2+v_2)$, which is independent of ν.

For those pre-test estimators which have smaller bias than s^2_{NL} for all ϕ the effect of a decrease in ν is to shift the bias functions downwards. This increases the difference between the bias function of s^2_{NL} and that of s^2_{PL}, particularly for ϕ values in the neighbourhood of unity. For the remaining pre-test estimators (which will be those with a relatively

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[11] We recall that $\lambda_i=\theta_i/\sigma^2_2$, and that we parameterise with respect to λ_1 and λ_2 rather than θ_1 and θ_2 so as to eliminate the need to specify the scale parameter σ^2_2.

small nominal test size), a decrease in ν shifts the bias functions upwards except for some ϕ around the neighbourhood of $\phi=1$. This increases the possible maximum difference between $\text{bias}_{Mt}(s^2_{NL})$ and $\text{bias}_{Mt}(s^2_{PL})$.

The bias functions of s^2_{NML}, s^2_{AML}, and s^2_{PML} depend on $f(\tau)$. For given values of λ_1 and λ_2, we find that the $\text{bias}_{Mt}(s^2_{NML})$ shifts upwards as ν increases. If there are no omitted regressors in sample one, regardless of the degree of mis-specification in the model for sample two, then this results in $|\text{bias}_N(s^2_{NML})|<|\text{bias}_{Mt}(s^2_{NML})|\nu<\infty|$. However, if θ_1 is sufficiently different from zero then the inequality is reversed. Similar changes occur for the bias functions of the never-pool M estimators as ν varies. The $\text{bias}_{Mt}(s^2_{AML})$ is higher for small ϕ, and decreases as ϕ increases at a faster rate, when ν is small than when it is relatively large. So, for relatively small values of ϕ (say, $\phi<0.5$) $|\text{bias}_{Mt}(s^2_{AL})|\nu<\infty|>|\text{bias}_N(s^2_{AL})|$ while the inequality may be reversed for relatively large values of ϕ. Our results suggest that the bias functions of s^2_{AM} as ν varies behave in a similar fashion to that described here for s^2_{AML}. Further, the aforementioned changes in the bias functions of s^2_{PL} as ν varies also appear to hold for s^2_{PML} and s^2_{PM}.

(f) The numerical evaluations suggest that if one adopts a pre-test strategy of minimizing the maximum absolute bias then, of the three components considered, it is generally preferable to use the L component estimators, if we have not mis-specified the design matrix of the model for sample one, regardless of the specification error of the model for sample two. However, if the models for both samples are sufficiently mis-specified in terms of omitted regressors then the optimal strategy appears to be to use the so-called ML component estimators.

We now consider the risk functions of the estimators, where we define the risk of an estimator $\bar{s}^2$ of $\sigma^2_{e_1}$ as $\rho\left(\sigma^2_{e_1},\bar{s}^2\right)=E\left(\bar{s}^2-\sigma^2_{e_1}\right)^2=E\left(\bar{s}^2-\phi E(\tau^2)\right)^2$.

Theorem 6.3.2

Under the assumptions of Theorem 6.3.1

$$\rho\left(\sigma^2_{e_1},S^2_N\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T_1+\mu)(T_1+\mu-2v_1)\right.$$
$$\left.+4\theta_1E(\tau^2)(2-k-\mu)+4\theta_1^2\right)/(T_1+\mu)^2 , \tag{6.3.36}$$

$$\rho\left(\sigma^2_{e_1},S^2_A\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1E(\tau^2)-2v_1(T+2\mu)\left(E(\tau^2)\right)^2\right.\right.$$
$$\left.+\left(2\theta_1-E(\tau^2)(T+2\mu)\right)^2\right]+2\phi\left[v_1v_2E(\tau^4)-v_2(T+2\mu)\left(E(\tau^2)\right)^2\right.$$
$$\left.-2\theta_2E(\tau^2)\left(v_2+2(k+\mu)\right)+2v_2\theta_1E(\tau^2)+4\theta_1\theta_2\right]+v_2(v_2+2)E(\tau^4)+$$
$$\left.4(v_2+2)\theta_2E(\tau^2)+4\theta_2^2\right)/(T+2\mu)^2 , \tag{6.3.37}$$

$$\rho\left(\sigma^2_{e_1},S^2_P\right) = \left(\phi^2(T+2\mu)^2\left[v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T_1+\mu)(T_1+\mu-2v_1)\right.\right.$$
$$\left.+4\theta_1E(\tau^2)(2-k-\mu)+4\theta_1^2\right]+\int_0^\infty\left(\phi^2(T_2+\mu)\left[-(2T_1+T_2+3\mu)\left(v_1(v_1+2)\tau^4Q^{d\tau}_{04}+\right.\right.\right.$$
$$\left.\left.4(v_1+2)\theta_1\tau^2Q^{d\tau}_{06}+4\theta_1^2Q^{d\tau}_{08}\right)+2E(\tau^2)(T_1+\mu)(T+2\mu)\left(v_1\tau^2Q^{d\tau}_{02}+2\theta_1Q^{d\tau}_{04}\right)\right]$$
$$+2(T_1+\mu)^2\phi\left[v_1v_2\tau^4Q^{d\tau}_{22}-v_2(T+2\mu)E(\tau^2)\tau^2Q^{d\tau}_{20}+2v_1\theta_2\tau^2Q^{d\tau}_{42}-\right.$$
$$\left.2(T+2\mu)\theta_2E(\tau^2)Q^{d\tau}_{40}+2v_2\theta_1\tau^2Q^{d\tau}_{24}+4\theta_1\theta_2Q^{d\tau}_{44}\right]+(T_1+\mu)^2\left[v_2(v_2+2)\tau^4Q^{d\tau}_{40}\right.$$
$$\left.\left.\left.+4(v_2+2)\tau^2\theta_2Q^{d\tau}_{60}+4\theta_2^2Q^{d\tau}_{80}\right]\right)f(\tau)d\tau\right)/\left((T_1+\mu)(T+2\mu)\right)^2 . \tag{6.3.38}$$

Proof.

$$\rho\left(\sigma_{e_1}^2,S_N^2\right) = E\left(S_N^4\right)-2\phi E(\tau^2)E\left(S_N^2\right)+\phi^2\left(E(\tau^2)\right)^2 , \tag{6.3.39}$$

and, using the relevant details of the proof to Theorem 6.3.1 ,

$$E\left(S_N^2\right) = \phi\left(v_1E(\tau^2)+2\theta_1\right)/(T_1+\mu) , \tag{6.3.40}$$

so we only need $E\left(S_N^4\right)=\int_0^\infty E_N\left(S_N^4\right)f(\tau)d\tau=\int_0^\infty E_N\left(\frac{\phi^2\tau^4}{(T_1+\mu)^2}\left[\frac{e^{*\prime}M_1^*e^*}{\tau^2}\right]^2\right)f(\tau)d\tau$. If $e\sim N(0,\tau^2\Sigma)$ then $e^{*\prime}M_1^*e^*/\tau^2\sim\chi^2_{v_1;\lambda_{1\tau}}$ and $E_N(e^{*\prime}M_1^*e^*/\tau^2)^2=v_1(v_1+2)+4(v_1+2)\lambda_{1\tau}+4\lambda_{1\tau}^2$. So, $E\left(S_N^4\right)=\phi^2\left(v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1E(\tau^2)+4\theta_1^2\right)/(T_1+\mu)^2$ which we substitute into (6.3.39), along with (6.3.40).

To establish $\rho\left(\sigma_{e_1}^2,S_A^2\right)$ we write

$$\rho\left(\sigma_{e_1}^2,S_A^2\right) = E\left(S_A^4\right)-2\phi E(\tau^2)E\left(S_A^2\right)+\phi^2\left(E(\tau^2)\right)^2 , \tag{6.3.41}$$

in which we need to determine $E\left(S_A^4\right)=\int_0^\infty E_N\left(S_A^4\right)f(\tau)d\tau=\int_0^\infty\left[\tau^4\left(\phi^2E_N(e^{*\prime}M_1^*e^*/\tau^2)^2\right.\right.$ $\left.\left.+2\phi E_N(e^{*\prime}M_1^*e^*/\tau^2)E_N(e^{*\prime}M_2^*e^*/\tau^2)+E_N(e^{*\prime}M_2^*e^*/\tau^2)^2\right)/(T+2\mu)^2\right]f(\tau)d\tau$ as $S_A^2=(\phi e^{*\prime}M_1^*e^*+e^{*\prime}M_2^*e^*)/(T+2\mu)$, and $E_N(.)=E(.)$ when $e\sim N(0,\tau^2\Sigma)$. Under this normality assumption the quadratic forms $(e^{*\prime}M_1^*e^*/\tau^2)$ and $(e^{*\prime}M_2^*e^*/\tau^2)$ are independent and they are non-central χ^2 random variates with v_1 and v_2 degrees of freedom and non-centrality parameters $\lambda_{1\tau}$ and $\lambda_{2\tau}$. So, $E_N\left(S_A^4\right)=$ $\tau^4\left(\phi^2\left[v_1(v_1+2)+4(v_1+2)\lambda_{1\tau}+4\lambda_{1\tau}^2\right]+2\phi\left[v_1v_2+2v_1\lambda_{2\tau}+2v_2\lambda_{1\tau}+4\lambda_{1\tau}\lambda_{2\tau}\right]+v_2(v_2+2)+\right.$ $\left.4(v_2+2)\lambda_{2\tau}+4\lambda_{2\tau}^2\right)/(T+2\mu)^2$, and then

$$\begin{aligned}E\left(S_A^4\right)=\Big(\phi^2\left[v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1E(\tau^2)+4\theta_1^2\right]+2\phi\Big[v_1v_2E(\tau^4)+\\ 2v_1\theta_2E(\tau^2)+2v_2\theta_1E(\tau^2)+4\theta_1\theta_2\Big]+v_2(v_2+2)E(\tau^4)+\\ 4(v_2+2)\theta_2E(\tau^2)+4\theta_2^2\Big)/(T+2\mu)^2 .\end{aligned} \tag{6.3.42}$$

The proof is completed by substituting (6.3.42) into (6.3.41), along with

$E\left(S_A^2\right)$ which we derived in the proof to Theorem 6.3.1.

Finally, to establish $\rho\left(\sigma_{e_1}^2, S_P^2\right)$ we have

$$\rho\left(\sigma_{e_1}^2, S_P^2\right) = E\left(S_P^4\right) - 2\phi E(\tau^2)E\left(S_P^2\right) + \phi^2\left(E(\tau^2)\right)^2 \qquad (6.3.43)$$

in which we need to determine $E\left(S_P^4\right) = \int_0^\infty E_N\left(S_P^4\right) f(\tau)d\tau$. Using (6.3.9)

$$S_P^4 = \tau^4\left(\phi^2(T+2\mu)^2(e^{*\prime}M_1^*e^*/\tau^2)^2 + \left[-\phi^2(T_2+\mu)(2T_1+T_2+3\mu)(e^{*\prime}M_1^*e^*/\tau^2)^2\right.\right.$$

$$\left. + (T_1+\mu)^2(e^{*\prime}M_2^*e^*/\tau^2)^2 + 2\phi(T_1+\mu)^2(e^{*\prime}M_1^*e^*/\tau^2)(e^{*\prime}M_2^*e^*/\tau^2)\right]$$

$$\left.\cdot I_{[0,c\phi]}\left[(v_1e^{*\prime}M_2^*e^*/\tau^2)/(v_2e^{*\prime}M_1^*e^*/\tau^2)\right]\right)\Big/\left((T_1+\mu)(T+2\mu)\right)^2 ,$$

and using Lemma 1 of Clarke *et al.* (1987a)

$$E_N\left(S_P^4\right) = \left(\phi^2(T+2\mu)^2\left[v_1(v_1+2)\tau^4 + 4(v_1+2)\theta_1\tau^2 + 4\theta_1^2\right] - \phi^2(T_2+\mu)(2T_1+T_2+3\mu)\right.$$

$$\cdot\left[v_1(v_1+2)\tau^4Q_{04}^{d\tau} + 4(v_1+2)\theta_1\tau^2Q_{06}^{d\tau} + 4\theta_1^2Q_{08}^{d\tau}\right] + (T_1+\mu)^2$$

$$\cdot\left[v_2(v_2+2)\tau^4Q_{40}^{d\tau} + 4(v_2+2)\theta_2Q_{60}^{d\tau} + 4\theta_2^2Q_{80}^{d\tau}\right] + 2\phi(T_1+\mu)^2\left[v_1v_2\tau^4Q_{22}^{d\tau} + \right.$$

$$\left.\left. 2v_1\theta_2\tau^2Q_{42}^{d\tau} + 2v_2\theta_1Q_{24}^{d\tau} + 4\theta_1\theta_2Q_{44}^{d\tau}\right]\right)\Big/\left((T_1+\mu)(T+2\mu)\right)^2 .$$

To complete the proof we integrate this last equation with respect to τ to give $E\left(S_P^4\right)$, then substitute $E\left(S_P^4\right)$ into (6.3.44), along with the expression for $E\left(S_P^2\right)$ which we derived in the proof of Theorem 6.3.1 . #

Aside from depending on the arguments of the model T_1, T_2 and k, the risk functions depend on first, the true error variances $\sigma_{e_1}^2$ and $\sigma_{e_2}^2$ via ϕ; secondly they depend on $f(\tau)$; thirdly, $\rho\left(\sigma_{e_1}^2, S_P^2\right)$ depends on the nominal significance level of the pre-test; and finally, the risk functions depend on the degree of mis-specification in each sample, via θ_1 and θ_2. We note that the data only enter the problem via θ_1 and θ_2. If there are no omitted regressors then $Z_1\gamma_1=Z_2\gamma_2=0$ and $\theta_1=\theta_2=0$: the risk functions for this particular case are given in Corollary 6.3.4 . We follow this with Corollaries 6.3.5 and 6.3.6 which present the risk functions when $e\sim Mt\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$ and $e\sim N(0,\sigma_2^2\Sigma)$, respectively.

Corollary 6.3.4

If there are no omitted regressors ($Z_1\gamma_1=Z_2\gamma_2=0$), and $e\sim ESD_N(0,\Sigma)$, then

$$\rho_0\left(\sigma_{e_1}^2,S_N^2\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T_1+\mu)(T_1+\mu-2v_1)\right)/(T_1+\mu)^2, \qquad (6.3.44)$$

$$\rho_0\left(\sigma_{e_1}^2,S_A^2\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(T+2\mu)(T+2\mu-2v_1)\right]+2v_2\phi\left[v_1E(\tau^4)\right.\right.$$

$$\left.\left.-(T+2\mu)\left(E(\tau^2)\right)^2\right]+v_2(v_2+2)E(\tau^4)\right)/(T+2\mu)^2 , \qquad (6.3.45)$$

$$\rho_0\left(\sigma_{e_1}^2,S_P^2\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)\left((T+2\mu)^2-(T_2+\mu)(2T_1+T_2+3\mu)Q_{04}\right)\right.\right.$$

$$\left.+(T+2\mu)(T_1+\mu)\left(E(\tau^2)\right)^2\left((T_1+\mu-2v_1)(T+2\mu)+2(T_2+\mu)v_1Q_{02}\right)\right]$$

$$+2(T_1+\mu)^2\phi\left[v_1v_2E(\tau^4)Q_{22}-v_2(T+2\mu)\left(E(\tau^2)\right)^2Q_{20}\right]$$

$$\left.+v_2(v_2+2)(T_1+\mu)^2E(\tau^4)Q_{40}\right)/\left((T_1+\mu)(T+2\mu)\right)^2 . \qquad (6.3.46)$$

Proof.

If $Z_1\gamma_1=Z_2\gamma_2=0$ then $\theta_1=\theta_2=0$ and $Q_{ij}^{d\tau}=Q_{ij}$ which does not depend on τ. So, (6.3.44), (6.3.45) and (6.3.46) follow from Theorem 6.3.2 . #

Corollary 6.3.5

If we use the mis-specified model (6.2.5) rather than the true model (6.2.2) when $e\sim Mt\left(0,\nu\sigma_2^2/(\nu-2)\Sigma\right)$, and the pre-test is of H_0 in (6.2.6), then for $\nu>4$

$$\rho_{Mt}\left(\sigma_{e_1}^2,S_N^2\right) = \phi^2\sigma_2^4\left(2\nu^2v_1(v_1+\nu-2)+\nu^2(k+\mu)^2(\nu-4)+4\lambda_1\nu(\nu-2)(\nu-4)(2-k-\mu)\right.$$

$$+4\lambda_1^2(\nu-2)^2(\nu-4)\Big)\Big/\Big((T_1+\mu)^2(\nu-2)^2(\nu-4)\Big)\ , \tag{6.3.47}$$

$$\rho_{Mt}\left(\sigma^2_{e_1},S^2_A\right) = \sigma^4_2\Big(\phi^2\Big[\nu^2(\nu-4)\Big(v_2+2(k+\mu)\Big)^2+2\nu^2v_1(v_1+\nu-2)-$$

$$4\lambda_1\nu(\nu-2)(\nu-4)\Big(v_2+2(k+\mu-1)\Big)+4\lambda_1^2(\nu-2)^2(\nu-4)\Big]+2\phi\Big[2v_1v_2\nu^2-$$

$$\nu^2(\nu-4)v_2\Big(v_2+2(k+\mu)\Big)+2\nu(\nu-2)(\nu-4)\Big\{v_2\lambda_1-\lambda_2\Big(v_2+2(k+\mu)\Big)\Big\}$$

$$+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)\Big]+v_2(v_2+2)\nu^2(\nu-2)+4(v_2+2)$$

$$.\lambda_2\nu(\nu-2)(\nu-4)+4\lambda_2^2(\nu-2)^2(\nu-4)\Big)\Big/\Big((T+2\mu)^2(\nu-2)^2(\nu-4)\Big)\ , \tag{6.3.48}$$

$$\rho_{Mt}\left(\sigma^2_{e_1},S^2_P\right) = \sigma^4_2\Big\{\phi^2(T+2\mu)^2\Big(\nu^2(\nu-4)(k+\mu)^2+2v_1\nu^2(v_1+\nu-2)$$

$$+4\lambda_1\nu(\nu-2)(\nu-4)(2-k-\mu)+4\lambda_1^2(\nu-2)^2(\nu-4)\Big)+\phi^2(T_2+\mu)\Big[-(\nu-2)$$

$$.(2T_1+T_2+3\mu)\Big(v_1(v_1+2)\nu^2Q^d_{040}+4(v_1+2)\lambda_1\nu(\nu-4)Q^d_{061}+4\lambda_1^2(\nu-2)$$

$$.(\nu-4)Q^d_{082}\Big)+2\nu(\nu-4)(T_1+\mu)(T+2\mu)\Big(v_1\nu Q^d_{021}+2\lambda_1(\nu-2)Q^d_{042}\Big)\Big]$$

$$+2(T_1+\mu)^2\phi\Big[v_1v_2\nu^2(\nu-2)Q^d_{220}-v_2(T+2\mu)\nu^2(\nu-4)Q^d_{201}+2v_1\lambda_2\nu(\nu-2)$$

$$.(\nu-4)Q^d_{421}-2(T+2\mu)\lambda_2\nu(\nu-2)(\nu-4)Q^d_{402}+2v_2\lambda_1\nu(\nu-2)(\nu-4)Q^d_{241}$$

$$+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)Q^d_{442}\Big]+(T_1+\mu)^2(\nu-2)\Big[v_2(v_2+2)\nu^2Q^d_{400}$$

$$+4(v_2+2)\lambda_2\nu(\nu-4)Q^d_{601}+4\lambda_2^2(\nu-2)(\nu-4)Q^d_{802}\Big]\Big\}$$

$$\Big/\Big((T_1+\mu)^2(T+2\mu)^2(\nu-2)^2(\nu-4)\Big)\ . \tag{6.3.49}$$

If there are no omitted regressors, $Z_1\gamma_1=Z_2\gamma_2=0$, then

$$\rho_{OMt}\left(\sigma^2_{e_1},S^2_N\right) = \phi^2\nu^2\sigma^4_2\left(2v_1(v_1+\nu-2)+(k+\mu)^2(\nu-4)\right)/$$

$$\left((T_1+\mu)^2(\nu-4)(\nu-2)^2\right), \tag{6.3.50}$$

$$\rho_{OMt}\left(\sigma^2_{e_1},S^2_A\right) = \nu^2\sigma^4_2\left(\phi^2\left[(\nu-4)\left(v_2+2(k+\mu)\right)^2+2v_1(v_1+\nu-2)\right]+2v_2\phi\left[2v_1-(\nu-4)\right.\right.$$

$$\left.\left.\cdot\left(v_2+2(k+\mu)\right)\right]+v_2(v_2+2)(\nu-2)\right)/\left((T+2\mu)^2(\nu-2)^2(\nu-4)\right), \tag{6.3.51}$$

$$\rho_{OMt}\left(\sigma^2_{e_1},S^2_P\right) = \nu^2\sigma^4_2\left(\phi^2\left[(T+2\mu)^2\left((\nu-4)(k+\mu)^2+2v_1(v_1+\nu-2)\right)-v_1(v_1+2)\right.\right.$$

$$\left.\cdot(\nu-2)(2T_1+T_2+3\mu)(T_2+\mu)Q_{04}+2v_1(T_1+\mu)(T+2\mu)(T_2+\mu)(\nu-4)Q_{02}\right]$$

$$+2(T_1+\mu)^2\phi v_2\left[v_1(\nu-2)Q_{22}-(T+2\mu)(\nu-4)Q_{20}\right]+v_2(v_2+2)(T_1+\mu)^2$$

$$\left.\cdot(\nu-2)Q_{40}\right)/\left((T_1+\mu)^2(T+2\mu)^2(\nu-4)(\nu-2)^2\right). \tag{6.3.52}$$

Proof.

$e\sim Mt\left(0,\nu\sigma^2_2/(\nu-2)\Sigma\right)$ when $\tau\sim IG$ with scale parameter σ^2_2 and degrees of freedom parameter ν. So, $\sigma^2_{e_2}=\nu\sigma^2_2/(\nu-2)$, $\sigma^2_{e_1}=\phi\sigma^2_{e_2}=\phi\nu\sigma^2_2/(\nu-2)$, $E(\tau^2)$ is given by (6.3.21),

$$E(\tau^4) = \nu^2\sigma^4_2/\left((\nu-2)(\nu-4)\right), \tag{6.3.53}$$

and we then establish (6.3.50) and (6.3.51) from (6.3.36) and (6.3.37) of Theorem 6.3.2 , respectively.

Now, $\int_0^\infty\tau^4Q^{d\tau}_{ij}f(\tau)d\tau=\nu^2\sigma^4_2Q^d_{ij0}/\left((\nu-2)(\nu-4)\right)$ using (5.2.39), i,j=0, 1, 2,... . This, along with (6.3.21), (6.3.22), (6.3.23), (6.3.53), and (6.3.38) of Theorem 6.3.2 establishes $\rho_{Mt}\left(\sigma^2_{e_1},S^2_P\right)$.

To complete the proof we need only note that if $Z_1\gamma_1=Z_2\gamma_2=0$ then $\lambda_1=\lambda_2=0$ and $Q^d_{ijn}=Q_{ij}$. (6.3.50), (6.3.51) and (6.3.52) then follow from (6.3.47), (6.3.48) and (6.3.49). We could alternatively obtain these expressions by using Corollary 6.3.4 and (6.3.21) and (6.3.53). #

Corollary 6.3.6

If we use the mis-specified model (6.2.5) rather than the true model (6.2.2) when $e \sim N(0,\sigma_2^2\Sigma)$, and the pre-test is of H_0 in (6.2.6), then $\sigma_{e_2}^2=\sigma_2^2$, $\sigma_{e_1}^2=\sigma_1^2$ (say), and

$$\rho_N\left(\sigma_1^2,S_N^2\right) = \phi^2\sigma_2^4\left(2(v_1+4\lambda_1)+\left(2\lambda_1-(k+\mu)\right)^2\right)/(T_1+\mu)^2 , \tag{6.3.54}$$

$$\rho_N\left(\sigma_1^2,S_A^2\right) = \sigma_2^4\left(\phi^2\left[\left(2\lambda_1-v_2-2(k+\mu)\right)^2+2(v_1+4\lambda_1)\right]+2\phi(v_2+2\lambda_2)\right.$$

$$\left.\cdot\left(2\lambda_1-v_2-2(k+\mu)\right)+v_2(v_2+2)+4(v_2+2)\lambda_2+4\lambda_2^2\right)/(T+2\mu)^2 , \tag{6.3.55}$$

$$\rho_N\left(\sigma_1^2,S_P^2\right) = \sigma_2^4\left(\phi^2(T+2\mu)^2\left[(k+\mu)^2+2v_1+4\lambda_1(2-k-\mu)+4\lambda_1^2\right]+\phi^2(T_2+\mu)\right.$$

$$\cdot\left[-(2T_1+T_2+3\mu)\left(v_1(v_1+2)Q_{04}^d+4(v_1+2)\lambda_1Q_{06}^d+4\lambda_1^2Q_{08}^d\right)+2(T_1+\mu)(T+2\mu)\right.$$

$$\left.\cdot\left(v_1Q_{02}^d+2\lambda_1Q_{04}^d\right)\right]+2(T_1+\mu)^2\phi\left[v_1v_2Q_{22}^d-v_2(T+2\mu)Q_{20}^d+2v_1\lambda_2Q_{42}^d\right.$$

$$\left.-2(T+2\mu)\lambda_2Q_{40}^d+2v_2\lambda_1Q_{24}^d+4\lambda_1\lambda_2Q_{44}^d\right]+(T_1+\mu)^2\left[v_2(v_2+2)Q_{40}^d\right.$$

$$\left.\left.+4(v_2+2)\lambda_2Q_{60}^d+4\lambda_2^2Q_{80}^d\right]\right)/\left((T_1+\mu)^2(T+2\mu)^2\right). \tag{6.3.56}$$

If there are no omitted regressors, $Z_1\gamma_1=Z_2\gamma_2= 0$, then

$$\rho_{ON}\left(\sigma_1^2,S_N^2\right) = \phi^2\sigma_2^4\left(2v_1+(k+\mu)^2\right)/(T_1+\mu)^2 , \tag{6.3.57}$$

$$\rho_{ON}\left(\sigma_1^2,S_A^2\right) = \sigma_2^4\left\{\phi^2\left[\left(v_2+2(k+\mu)\right)^2+2v_1\right]-2\phi v_2\left(v_2+2(k+\mu)\right)\right.$$

$$\left.+v_2(v_2+2)\right\}/(T+2\mu)^2 , \tag{6.3.58}$$

$$\rho_{ON}\left(\sigma_1^2,S_P^2\right) = \sigma_2^4\left\{\phi^2\left[(T+2\mu)^2\left((k+\mu)^2+2v_1\right)-v_1(v_1+2)(T_2+\mu)(2T_1+T_2+3\mu)Q_{04}\right.\right.$$

$$+2v_1(T_1+\mu)(T_2+\mu)(T+2\mu)Q_{02}\Big]+2(T_1+\mu)^2\phi v_2\Big[v_1Q_{22}-(T+2\mu)Q_{20}\Big]$$

$$+v_2(v_2+2)(T_1+\mu)^2Q_{40}\Bigg)\Big/\Big((T_1+\mu)^2(T+2\mu)^2\Big) \ . \tag{6.3.59}$$

Proof.

We establish this corollary from Corollary 6.3.5 as $e\sim N(0,\sigma_2^2\Sigma)$ when $\nu=\infty$. In this case, $\lim_{\nu\to\infty} Q^d_{ijn}=Q^d_{ij}$, i,j,n=0,1,2,... . #

Remarks

(i) From (A6.55), (A6.66), and (A6.67) of Appendix 6.1 we can easily derive the expressions of Bancroft (1944) (allowing for the change in H_1) and of Toyoda and Wallace (1975).[12]

(ii) If $\alpha=0$, $c=\infty$, then $Q^{d\tau}_{ij}=1$ so then we never reject the hypothesis that the error variances are equal. Then, $\rho(\sigma^2_{e_1},S^2_P)=\rho(\sigma^2_{e_1},S^2_A)$. Conversely, if $\alpha=1$, $c=0$, then $Q^{d\tau}_{ij}=0$ so that we reject H_0. Then, $\rho(\sigma^2_{e_1},S^2_P)=\rho(\sigma^2_{e_1},S^2_N)$.

In this section we have derived the risk functions for a family of estimators, three members of which are our so-called, L, ML, and M component estimators. Appendix 6.1 presents some risk expressions for these special cases, which are obtained from the theorems and the corollaries given here by substituting in the appropriate value of μ. In the next section we discuss these risk functions, as well as the more general ones presented here, when the model does not exclude variables. We follow this in Section 6.5 with a discussion of the risk functions when we have omitted regressors. In each of Section 6.4 and Section 6.5 we use some numerical evaluations of the risk functions, given in Appendix 6.2, to illustrate many of the features that we examine.

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[12] See (2.3.5), (2.3.6), and (2.3.7) of Chapter Two.

6.4 Comparisons of the Risk Functions when the Regressors are Correctly Specified

In this section we compare the risk functions of the never-pool, the always-pool, and the pre-test estimators of the error variance of sample one, $\sigma^2_{e_1}$, when there are no omitted regressors; that is, when $Z\gamma=0$. Given the complexities of these risk expressions, it is useful to evaluate them numerically, which we have done , assuming Mt errors, for the L, the ML, and the M component estimators for various values of ν, α, v_1, v_2 and k, as functions of ϕ, the ratio of $\sigma^2_{e_1}$ to $\sigma^2_{e_2}$.[13] We examined the same wide range of values of the arguments which we considered in our discussion of the bias functions. A representative selection of the results appears in Tables A6.2.1 to A6.2.3 of Appendix 6.2,[14,15] and the associated Figures 6.4.1 to 6.4.12 . The relative risk functions of s^2_{NL}, s^2_{AL}, and s^2_{PL}, from Table A6.2.1, are shown in Figures 6.4.1 to 6.4.4; those for s^2_{NML}, s^2_{AML}, and s^2_{PML}, from Table A6.2.2, are presented in Figures 6.4.5 to 6.4.8; while Figures 6.4.9 to 6.4.12 graph the results for the relative risks of s^2_{NM}, s^2_{AM}, and s^2_{ML} from Table A6.2.3 . In each of these sets of figures the diagrams

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[13] In this chapter, as we did in Chapters Four and Five, we evaluate the risk expressions relative to the scale parameter, σ^2_2, to eliminate the need to specify its value. So, the relative risk of an estimator $\bar{s}^2$ of $\sigma^2_{e_1}$ is given by, $R(\sigma^2_{e_1},\bar{s}^2)=\rho(\sigma^2_{e_1},\bar{s}^2)/\sigma^4_2$. We lose no generality in considering relative risk, and the results could equally be interpreted as the risk functions when $\sigma^2_2 = 1$.

[14] The introduction to Appendix 6.2 fully details the contents of these tables. We note that these tables report the relative risks in terms of the specification error case. So, the pertinent results for this section are those for which $\lambda_1=\lambda_2=0$.

[15] The computer programs we employed to generate the risk functions were executed on a VAX 6230 computer and we used similar techniques to those described for the evaluations of the bias functions.

differ by the value of ν that is considered. We present four values of ν: ν=5, 10, 100 and ∞.

The scales on the diagrams are not equivalent so that their features are clearly evident, and the relevant legend of the line types associated with each of the estimators follows. There are two relative risk functions with the same line type. We again distinguish them by the use of an appropriate arrow and a label.

Legend for Figures 6.4.1 to 6.4.12

$R_0(\sigma^2_{e_1}, s^2_{Ni})$	$R_0(\sigma^2_{e_1}, s^2_{Ai})$	$R_0(\sigma^2_{e_1}, s^2_{Pi})$ $\alpha = 0.01$
$R_0(\sigma^2_{e_1}, s^2_{Pi})$ $\alpha = 0.05$	$R_0(\sigma^2_{e_1}, s^2_{Pi})$ $\alpha = 0.30$	$R_0(\sigma^2_{e_1}, s^2_{Pi})$ $\alpha = 0.75$
	$R_0(\sigma^2_{e_1}, s^2_{Pi})$ $c = c^*_i$	

We now report some features of the risk functions when there are no omitted regressors.

(a) $\lim_{\phi\to 0}\rho_0(\sigma^2_{e_1},S^2_P)=\lim_{\phi\to 0}\rho_0(\sigma^2_{e_1},S^2_N)=0$ while $\lim_{\phi\to 0}\rho_0(\sigma^2_{e_1},S^2_A)=\left(v_2(v_2+2)E(\tau^4)\right)/(T+2\mu)^2>0$. Intuitively, it is better to ignore the prior information when it is very false, and pre-testing leads us to follow the correct strategy of ignoring the second sample when estimating $\sigma^2_{e_1}$.

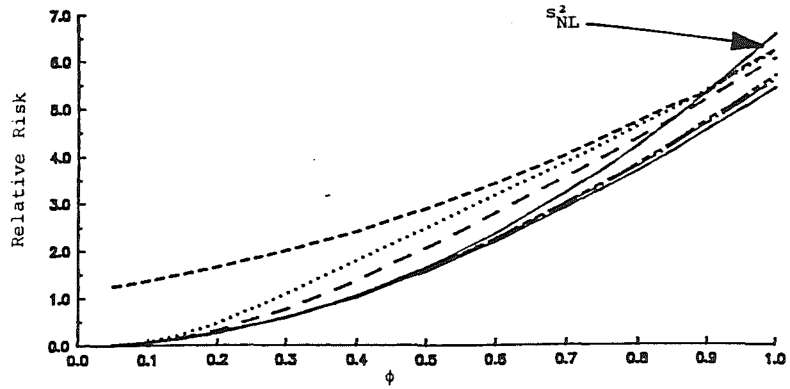

FIGURE 6.4.1: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 0$.

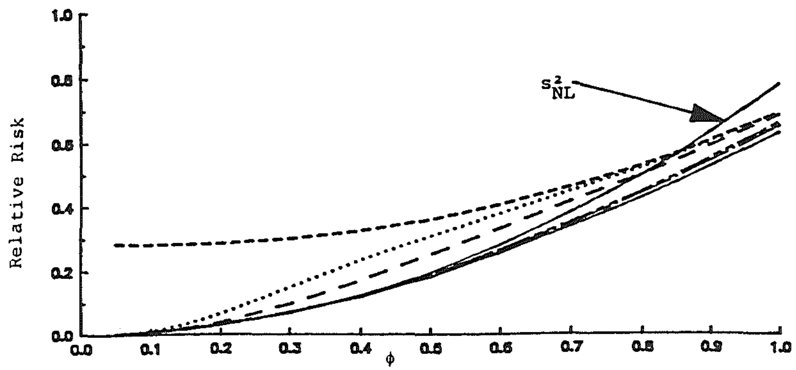

FIGURE 6.4.2: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 10$, $\lambda_1 = \lambda_2 = 0$.

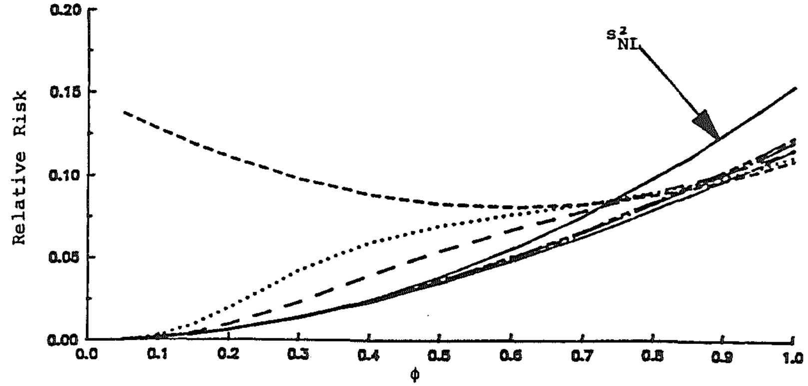

FIGURE 6.4.3: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 100$, $\lambda_1 = \lambda_2 = 0$.

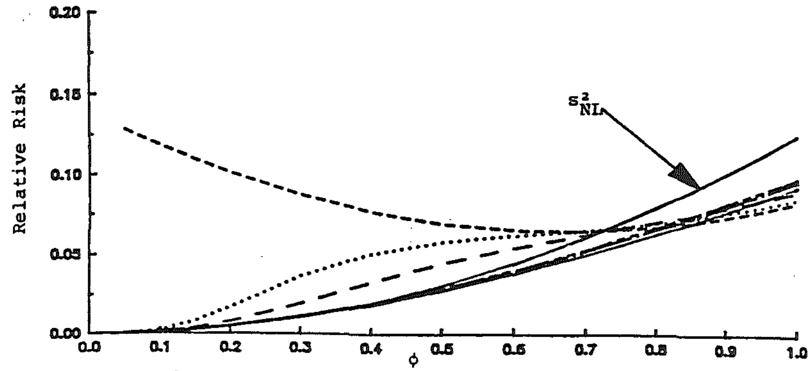

FIGURE 6.4.4: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma^2_2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 0$.

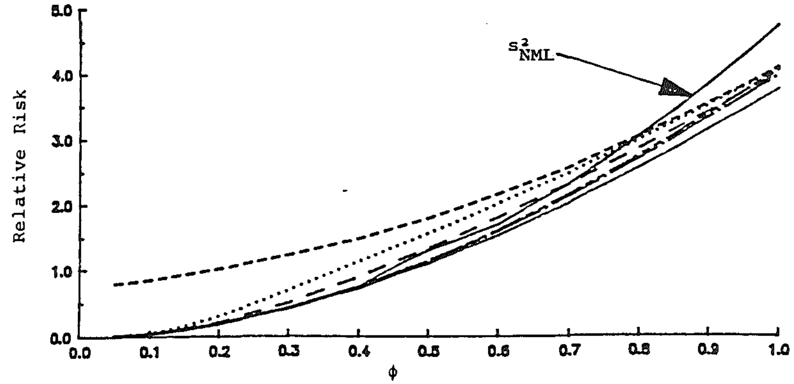

FIGURE 6.4.5: Pelative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 0$.

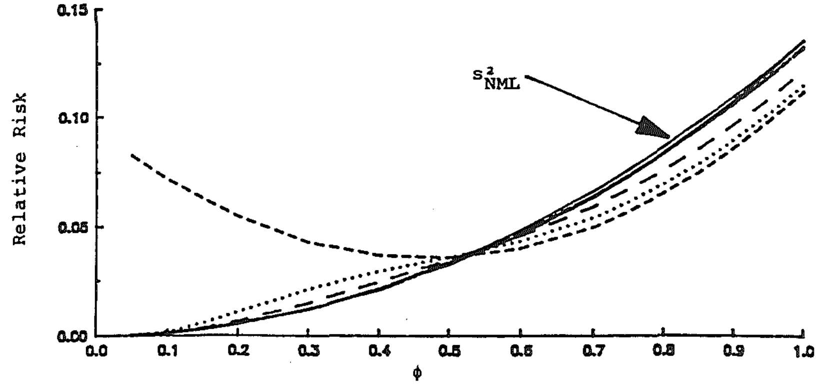

FIGURE 6.4.7: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 100$, $\lambda_1 = \lambda_2 = 0$.

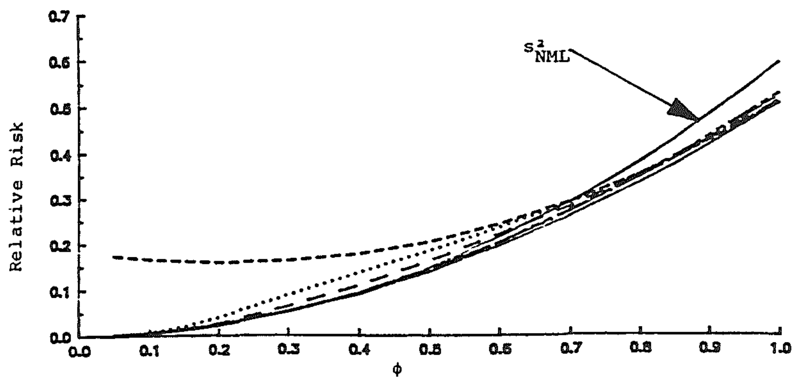

FIGURE 6.4.6: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 10$, $\lambda_1 = \lambda_2 = 0$.

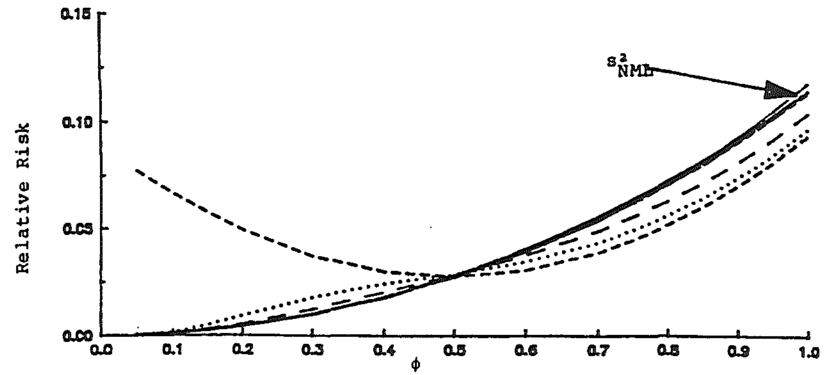

FIGURE 6.4.8: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 0$.

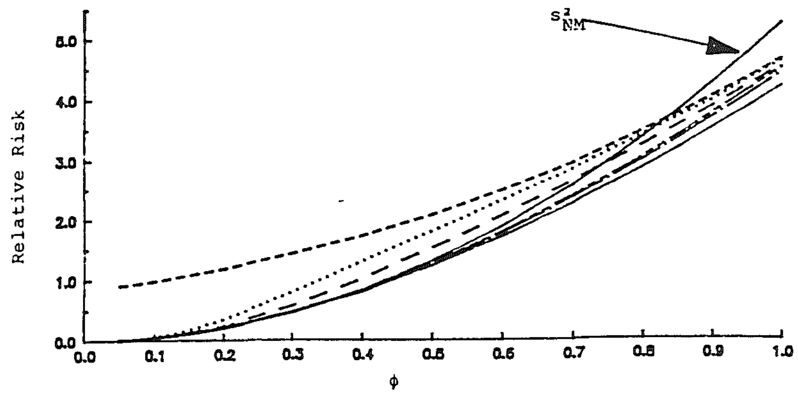

FIGURE 6.4.9: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 0$.

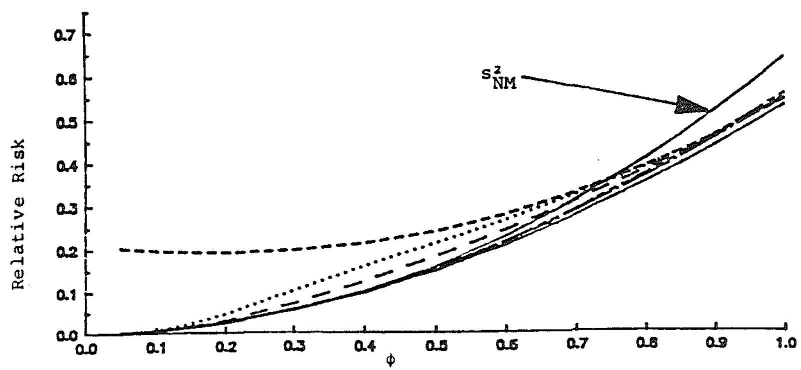

FIGURE 6.4.10: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 10$, $\lambda_1 = \lambda_2 = 0$.

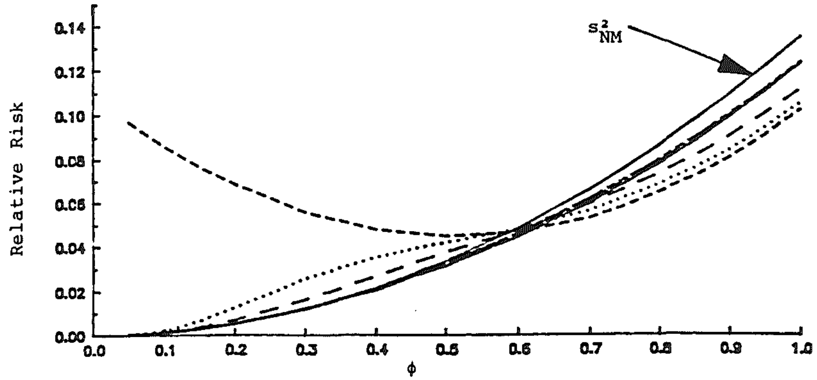

FIGURE 6.4.11: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma^2_2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 100$, $\lambda_1 = \lambda_2 = 0$.

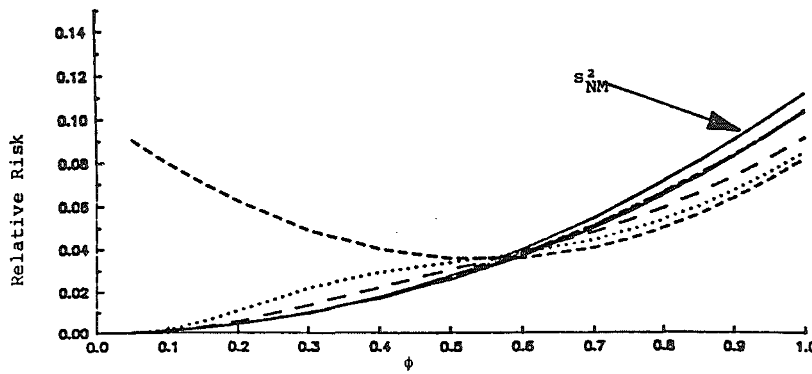

FIGURE 6.4.12: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma^2_2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 0$.

(b) If $\phi=1$, that is, the error variances are equal then the sign of

$$\rho_0(\sigma^2_{e_1}, S^2_A|\phi=1) - \rho_0(\sigma^2_{e_1}, S^2_N|\phi=1) = \Big\{E(\tau^4)\Big[(T_1+\mu)^2(v_1+v_2)(v_1+v_2+2)$$

$$-v_1(v_1+2)(T+2\mu)^2\Big] + \Big(E(\tau^2)\Big)^2(T+2\mu)(T_1+\mu)\Big[(T_2+\mu)\Big(v_1-(k+\mu)\Big)$$

$$-(T_1+\mu)\Big(v_2-(k+\mu)\Big)\Big]\Big\} \Big/ \Big((T+2\mu)(T_1+\mu)\Big)^2 \qquad (6.4.1)$$

is negative if $\Big\{.\Big\}<0$, so that imposing valid prior information produces a risk gain. The sign of (6.4.1) is not obvious. If we are employing the L components then (6.4.1) is equal to $\Big[-2E(\tau^4)v_2\Big/\Big(v_1(v_1+v_2)\Big)\Big]$ which is negative for all v_1, v_2. So, when using the L components it is always better to impose the valid prior information than to ignore it for all possible $f(\tau)$. However, if the ML components are being used then the sign of (6.4.1), which in this case is equal to

$$\Big\{2kT_1T(T_1-T_2)\Big(E(\tau^2)\Big)^2 + E(\tau^4)\Big[T_1^2(v_1+v_2)(v_1+v_2+2) - v_1(v_2+2)T^2\Big]\Big\} \Big/ (T_1^2T^2)$$

is still not clear. For our numerical evaluations, we found this difference to be negative for all possible values of ν. These results suggest that it is preferable to pool the samples, when the error distribution is Mt and when the error variances are equal, if using the ML components. We reach a similar conclusion if the M components are used, in which case (6.4.1) is equal to

$$\Big\{4\Big(E(\tau^2)\Big)^2(v_1+2)(v_1+v_2+4)(v_1-v_2) + E(\tau^4)(v_1+2)\Big[(v_1+2)(v_1+v_2)(v_1+v_2+2) -$$
$$v_1(v_1+v_2+4)^2\Big]\Big\} \Big/ \Big((v_1+2)(v_1+v_2+4)\Big)^2 .$$

Our results do suggest, however, that for small values of ν the pre-test estimator may have smaller risk than the always-pool estimator under H_0. We will return to this feature in point (d).

(c) The risk functions of S^2_N and S^2_A have two intersections with respect to ϕ. Let these be ϕ_1 and ϕ_2. Their values are

$$\phi_i = \left[v_2(T_1+\mu)^2\left[v_1E(\tau^4)-(T+2\mu)\left(E(\tau^2)\right)^2\right]\pm(T_1+\mu)\left\{v_1v_2\left(E(\tau^4)\right)^2\right.\right.$$

$$.\left[v_1v_2(T_1+\mu)^2+(v_1+2)(v_2+2)(T_2+\mu)(2T_1+T_2+3\mu)\right]+v_2^2(T_1+\mu)^2(T+2\mu)^2\left(E(\tau^2)\right)^4$$

$$\left.\left.-2v_1v_2(T_1+\mu)(T+2\mu)E(\tau^4)\left(E(\tau^2)\right)^2\left[v_2(T+2\mu)+2(T_2+\mu)\right]\right\}^{1/2}\right]/\left[v_1(v_1+2)\right.$$

$$\left..E(\tau^4)(T_2+\mu)(2T_1+T_2+3\mu)-2v_1(T_1+\mu)(T+2\mu)(T_2+\mu)\left(E(\tau^2)\right)^2\right], \qquad (6.4.2)$$

$$= \omega\pm\kappa$$

i=1,2. Let $\phi_1=\omega+\kappa$, and $\phi_2=\omega-\kappa$. So, if $e\sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$ and we are using the L components then

$$\phi_{iL} = \left[v_1\left(2v_1-v_2(\nu-4)\right)\pm\left\{v_1(\nu-2)^2\left(v_1^3+(v_1+2)(v_2+2)(v_2+2v_1)\right)\right.\right.$$

$$\left.\left.+v_1^2(v_1+v_2)(\nu-4)\left(8-\nu(v_1+v_2+4)\right)\right\}^{1/2}\right]/$$

$$\left((\nu-4)(4v_1+2v_2-v_1v_2)+2(v_1+2)(2v_1+v_2)\right), \qquad (6.4.3)$$

while

$$\phi_{iML} = \left[v_2T_1^2\left(2v_1-(\nu-4)(T_2+k)\right)\pm T_1\left\{v_1v_2(\nu-2)^2\left(v_1v_2T_1^2+(v_1+2)(v_2+2)T_2\right.\right.\right.$$

$$\left.\left.\left..(2T_1+T_2)\right)+v_2T_1T(\nu-4)\left(v_2T_1T(\nu-4)-2v_1(\nu-2)(v_2T+2T_2)\right)\right\}^{1/2}\right]$$

$$/\left(v_1(v_1+2)(\nu-2)T_2(2T_1+T_2)-2v_1T_1TT_2(\nu-4)\right),$$

and

$$\phi_{iM} = \left[v_2(v_1+2)\left(2v_1-(\nu-4)(v_2+4)\right)\pm\left\{v_1v_2(\nu-2)^2(v_1+2)\left(v_1v_2(v_1+2)\right.\right.\right.$$

$$\left.+(v_2+2)^2(2v_1+v_2+6)\right)+v_2(v_1+2)(v_1+v_2+4)(\nu-4)$$

$$\left.\left.\left((\nu-4)(-v_1^2v_2-v_1v_2^2-6v_1v_2+2v_2^2-8v_1+8v_2)-4v_1(v_1v_2+v_2^2+6v_2+4)\right)\right\}^{1/2}\right]$$

$$/\left[v_1(v_2+2)\left(2(2v_1+v_2+6)-(v_2+2)(\nu-4)\right)\right], \qquad i=1,2.$$

Our numerical evaluations suggest that there are two possibilities. First, $0<\phi_{1j}<1$, $\phi_{2j}<0$ and secondly, $0<\phi_{1j}<1$, $\phi_{2j}>1$, j=L, ML, M. Thus, there exists one feasible intersection, $\phi_{1j}\in(0,1)$. So, the never-pool estimator dominates the always-pool estimator when $0<\phi<\phi_{1j}$. Alternatively,, the always-pool estimator has smaller risk than the never-pool estimator when $\phi_{1j}<\phi\leq 1$. For this ϕ-range the gain in sampling variance from the extra degrees of freedom when pooling the samples outweighs the bias from pooling the (unequal) variances. These conclusions accord with those found by Toyoda and Wallace (1975), who consider $e\sim N(0,\sigma_2^2\Sigma)$ and s_{NL}^2, s_{AL}^2, and s_{PL}^2. In this situation (6.4.3) equals

$$\phi_{iL}^{N}=\left[-v_1v_2\pm\left\{2v_1(v_2^2+v_1v_2+2v_2+4v_1)\right\}^{1/2}\right]/(4v_1+2v_2-v_1v_2) ,$$

which are the intersections derived by Toyoda and Wallace, i=1,2.

Our numerical evaluations also suggest that $\phi_{1ML}<\phi_{1M}<\phi_{1L}$ if $v_2\leq v_1$, while the inequalities are reversed if $v_1<v_2$. Further, ϕ_{1j} decreases as ν increases, j=L, ML, M. This implies, if we assumed normal regression disturbances when in fact $e\sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$, $\nu<\infty$, that then there is a ϕ-range over which we would incorrectly choose to pool the samples.

(d) Bancroft (1944) and Toyoda and Wallace (1975) showed that there is a ϕ-range over which it is preferable to pre-test rather than to always-pool or to never-pool the two samples when $e\sim N(0,\sigma_2^2\Sigma)$, using the usual L components. They find that there is a family of pre-test estimators, with $c\in(0,2)$, which strictly dominate first, the never-pool estimator for all ϕ and secondly, the always-pool estimator for a wide range of ϕ. It is only within the neighbourhood of $\phi=1$ that the risk of s_{AL}^2 is smaller than that of s_{PL}^2. Ohtani and Toyoda (1978) prove that of this family of dominating estimators the pre-test estimator with c=1 has the smallest risk.

Our results show that these findings carry over to the broader distribution of errors that we are investigating. In particular, we further find, if ν is relatively small then, the pre-test estimator can strictly dominate the always-pool estimator for all possible ϕ. In these cases it is always preferable to pre-test. We first derive the minimum of the pre-test risk function, $\rho(\sigma^2_{e_1}, S^2_P)$.

Proposition 6.4.1

The pre-test risk function has a minimum when $c^* = \left(v_1(T_2+\mu)\right) / \left(v_2(T_1+\mu)\right)$.

Proof.[16]

This proof follows a similar approach to the proof of Proposition 6.3.1 . Using the notation introduced there

$$\rho(\sigma^2_{e_1}, S^2_P) = E\left(S^2_P - \phi E(\tau^2)\right)^2 = E\left[\left(S^2_A - \phi E(\tau^2)\right)^2 I_{[0,c]}(J) + \left(S^2_N - \phi E(\tau^2)\right)^2 I_{(c,\infty)}(J)\right]$$

$$= \int_0^\infty \tau^4 E_N\left\{\left[\left((\phi e^{*\prime} M^*_1 e^*/\tau^2) + (e^{*\prime} M^*_2 e^*/\tau^2)\right)/(T+2\mu) - \phi E(\tau^2)/\tau^2\right]^2\right.$$

$$I\left((e^{*\prime} M^*_2 e^*/\tau^2) \le c\phi v_2 (e^{*\prime} M^*_1 e^*/\tau^2)/v_1\right) + \left[(\phi e^{*\prime} M^*_1 e^*/\tau^2)/(T_1+\mu) - \phi E(\tau^2)/\tau^2\right]^2$$

$$\left.\cdot\left[1 - I\left((e^{*\prime} M^*_2 e^*/\tau^2) \le c\phi v_2 (e^{*\prime} M^*_1 e^*/\tau^2)/v_1\right)\right]\right\} f(\tau) d\tau ,$$

$$= \int_0^\infty \tau^4 E^1_N\left\{\int_0^g \left[\left((\phi e^{*\prime} M^*_1 e^*/\tau^2) + (e^{*\prime} M^*_2 e^*/\tau^2)\right)/(T+2\mu) - \phi E(\tau^2)/\tau^2\right]^2\right.$$

$$\cdot f_N(e^{*\prime} M^*_2 e^*/\tau^2) d(e^{*\prime} M^*_2 e^*/\tau^2) + \left[(\phi e^{*\prime} M^*_1 e^*/\tau^2)/(T_1+\mu) - \phi E(\tau^2)/\tau^2\right]^2$$

$$\left.\cdot\left[1 - \int_0^g f_N(e^{*\prime} M^*_2 e^*/\tau^2) d(e^{*\prime} M^*_2 e^*/\tau^2)\right]\right\} f(\tau) d\tau ,$$

where $g = c\phi v_2 (e^{*\prime} M^*_1 e^*/\tau^2)/v_1$. So,

$$\frac{\partial \rho(\sigma^2_{e_1}, S^2_P)}{\partial c} = \int_0^\infty \tau^4 E^1_N\left\{\left(\frac{\phi^3 v_2 e^{*\prime} M^*_1 e^*/\tau^2}{v_1}\right) f_N\left(\frac{c\phi v_2 e^{*\prime} M^*_1 e^*/\tau^2}{v_1}\right)\right.$$

---

[16] The form of this proof is not the same as that used by Ohtani and Toyoda (1978). Further, this proof can be easily extended to allow for omitted regressors. c* remains unchanged.

$$\cdot\left\{\left[\left((e^{*\prime}M_1^*e^*/\tau^2)cv_2(e^{*\prime}M_1^*e^*/\tau^2)/v_1\right)/(T+2\mu)-E(\tau^2)/\tau^2\right]^2\right.$$

$$\left.-\left((e^{*\prime}M_1^*e^*/\tau^2)/(T_1+\mu)-E(\tau^2)\right)^2\right\}f(\tau)d\tau\ ,$$

and a sufficient condition for this derivative to be zero is for

$$\left[\left((e^{*\prime}M_1^*e^*/\tau^2)+cv_2(e^{*\prime}M_1^*e^*/\tau^2)/v_1\right)/(T+2\mu)-E(\tau^2)/\tau^2\right]^2$$

$$-\left[(e^{*\prime}M_1^*e^*/\tau^2)/(T_1+\mu)-E(\tau^2)/\tau^2\right]^2=0\ ,$$

from which we obtain c^*.[17] #

So, $c_L^*=1$, $c_{ML}^*=(v_1T_2)/(v_2T_1)$, and $c_M^*=\left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right)$. We have included the pre-test estimators with these critical values on the appropriate L, ML and M diagrams.[18] These figures illustrate the strict dominance of a family of pre-test estimators over the never-pool estimators for all ϕ and ν, and that the pre-test estimator with $c=c^*$ has the smallest risk of this family of dominating estimators.

The diagrams also show that for small values of ν (for instance, $\nu=5$ and $\nu=10$) these pre-test estimators also strictly dominate the always-pool estimators, even when $\phi=1$. This will occur if

$$\nu < \left\{-2v_1\phi^2(T_2+\mu)\left(-(v_1+2)(2T_1+T_2+3\mu)(1-Q_{04})+4(T+2\mu)(T_1+\mu)(1-Q_{02})\right)\right.$$

$$\left.-4(T_1+\mu)^2\phi v_2\left(-2(T+2\mu)(1-Q_{20})+v_1(1-Q_{22})\right)-2v_2(v_2+2)(T_1+\mu)^2(1-Q_{40})\right\}$$

---

[17] Note that c^* equals the value of c which minimises bias(S_P^2).

[18] For the case illustrated a critical value of unity is equal to a nominal size of 47.3% while the critical values of $(v_1T_2)/(v_2T_1)$ and of $\left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right)$ correspond to nominal sizes of 38.0% and of 40.6% respectively.

$$/\left\{\phi^2 v_1(T_2+\mu)\left[(v_1+2)(2T_1+T_2+3\mu)(1-Q_{04})-2(T_1+\mu)(T+2\mu)(1-Q_{02})\right]\right.$$

$$\left.+2(T_1+\mu)^2\phi v_2\left[(T+2\mu)(1-Q_{20})-v_1(1-Q_{22})\right]-v_2(v_2+2)(T_1+\mu)^2(1-Q_{40})\right\}$$

$=\nu^*$, say.[19]

So, for $\nu<\nu^*$ we should pre-test, even if the error variances are equal, using $c=c^*$. For these values of ν the pre-test estimator has smaller variability than either of its component estimators. However, if $\nu>\nu^*$ then the pre-test estimator which uses $c=c^*$ has the smallest risk for $\phi\in[0,\phi^*)$, where ϕ^* is that value of ϕ for which $\rho_{OMt}(\sigma^2_{e_1},S^2_P|c=c^*)=\rho_{OMt}(\sigma^2_{e_1},S^2_A)$, but for $\phi\in[\phi^*,1]$ it is better to always pool the two samples. It is never preferable to ignore the fact that the error variances may be equal.

(e) Of the three component estimators we considered, the numerical results suggest, if one adopted a pre-test strategy and a crude minimax risk criterion that then, for normal disturbance terms the preferred estimator is s^2_{PM} for $\alpha=0.01$ and s^2_{PL} for $\alpha\geq0.05$. However, if ν is small (that is, if the marginal distribution of e has fatter tails than under normality) then it is preferable to use the ML component estimators. So, given our previous discussion, for small ν we should pre-test using the ML components and a critical value of $(v_1T_2)/(v_2T_1)$.

In this section we have compared the risk functions of S^2_N, S^2_A, and S^2_P when we have correctly specified the design matrices. We have paid particular attention to three special members, the L, the ML, and the M component estimators. We reiterate that we have used the L, the ML, and the M notation to identify that s^2_{NL}, s^2_{NML}, and s^2_{NM} are the never-pool least

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[19] ν^* is obtained by solving for those values of ν for which $\rho_{OMt}(\sigma^2_e,S^2_P)-\rho_{OMt}(\sigma^2_e,S^2_A)<0$.

squares, maximum likelihood and minimum mean squared error estimators respectively, of the error variance of the first sample under a normality assumption. The notation does not imply that these properties carry over to the pre-test and always-pool estimators, nor do they necessarily extend to the broader family of error distributions under investigation.

Some of the results which we observed occur regardless of the specific form of $f(\tau)$. For example, a minimum of the pre-test risk function results when $c=c^*$ for all possible $f(\tau)$. However, we find that no one of the L, the ML, or the M estimators is strictly preferred for all values of ν when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$.

6.5 Comparisons of the Risk Functions when Relevant Regressors are Omitted

In this section we consider the risk functions of the never-pool, the always-pool, and the pre-test estimators of the error variance of sample one, $\sigma_{e_1}^2$, when we omit regressors from the design matrices. We assume that these need not be the same regressors for each sample. As in the previous section, we have undertaken numerical evaluations of the risk expressions given in the special cases of Corollaries 6.3.5 and 6.3.6 in Appendix 6.1 for various choices of ν, α, v_1, v_2, and k, as functions of ϕ, the ratio of $\sigma_{e_1}^2$ to $\sigma_{e_2}^2$, and as functions of λ_1 and of λ_2, our specification error measures in the models for sample one and for sample two respectively.[20]

We examined the same range of values of the arguments which were investigated in our discussion of the bias functions. Tables A6.2.1 to

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[20] As in the previous section we consider risk relative to σ_2^4, and parameterise with respect to λ_1 and to λ_2, rather than with respect to θ_1 and to θ_2.

A6.2.12 of Appendix 6.2 of this chapter give a representative selection of the evaluations. Table A6.2.1 presents the relative risks of s^2_{NL}, s^2_{AL}, and of s^2_{PL} as functions of ϕ for given values of λ_1 and λ_2. Some of these results appear in Figures 6.5.3 to 6.5.6 . The corresponding results for s^2_{NML}, s^2_{AML}, and s^2_{PML} (s^2_{NM}, s^2_{AM}, and s^2_{PM}) are given in Table A6.2.2 (Table A6.2.3) and the associated Figures 6.5.7 to 6.5.12 (Figures 6.5.13 to 6.5.18).

Also of interest is the impact on the risk functions of varying degrees of mis-specification. Accordingly, Tables A6.2.4 to A6.2.6 (Tables A6.2.7 to A6.2.9) give, respectively, the relative risks of the L, the ML, and the M component estimators as functions of λ_1 (of λ_2) for given values of ϕ and of λ_2 (of λ_1). These tables consider the case of v_1=16, v_2=8, k=3, ϕ=1, 0.5, 0.1, $\lambda_2(\lambda_1)$=0, 3, and $\lambda_1(\lambda_2)$=[0,4(0.5);4,10(1.0)]. Figures 6.5.19 to 6.5.30 (Figures 6.5.31 to 6.5.42) show some of these results.

We have again used different scales in many of the diagrams and the legend for the figures follows.

Legend for Figures 6.5.1 to 6.5.42

$R(\sigma^2_{e_1}, s^2_{Ni})$

$R(\sigma^2_{e_1}, s^2_{Ai})$

$R(\sigma^2_{e_1}, s^2_{Pi})$
$\alpha = 0.01$

$R(\sigma^2_{e_1}, s^2_{Pi})$
$\alpha = 0.05$

$R(\sigma^2_{e_1}, s^2_{Pi})$
$\alpha = 0.30$

$R(\sigma^2_{e_1}, s^2_{Pi})$
$\alpha = 0.75$

$R(\sigma^2_{e_1}, s^2_{Pi})$
$c = c^*_i$

As the relative risk functions of s^2_{Nj} and $s^2_{Pj}|c=c^*_j$ have the same line type we have again identified the risk function of s^2_{Nj} with an appropriate label and an arrow; j=L, ML, and M.

We now comment on some of the features of the risk functions.

(a) The risks of S^2_N, S^2_A, and S^2_P depend on the specification errors $Z_1\gamma_1$ and $Z_2\gamma_2$ through θ_1 and θ_2 respectively. $\rho(\sigma^2_{e_1},S^2_N)$ is independent of θ_2 and so, given a fixed value of θ_1, this risk function is bounded as $\theta_2\to\infty$, but it is unbounded as $\theta_1\to\infty$. Similarly, $\rho(\sigma^2_{e_1},S^2_P)$ is unbounded as $\theta_1\to\infty$, given θ_2, but it is bounded (by $\rho(\sigma^2_{e_1},S^2_N)$) as $\theta_2\to\infty$, given θ_1. Intuitively, if the model of the second sample is badly mis-specified relative to the first then pre-testing will lead us to ignore the second sample, which is the appropriate strategy.

However, $\rho(\sigma^2_{e_1},S^2_A)$ is unbounded as $\theta_1\to\infty$, given θ_2 or as $\theta_2\to\infty$, given θ_1. Further, $\left(\rho(\sigma^2_{e_1},S^2_A)-\rho(\sigma^2_{e_1},S^2_N)\right)$ and $\left(\rho(\sigma^2_{e_1},S^2_A)-\rho(\sigma^2_{e_1},S^2_P)\right)$ are unbounded as θ_1 or θ_2 approaches infinity, while $\left(\rho(\sigma^2_{e_1},S^2_N)-\rho(\sigma^2_{e_1},S^2_P)\right)$ is bounded, and is equal to zero, as $\theta_2\to\infty$, given θ_1, but it is unbounded as $\theta_1\to\infty$, given θ_2. So, in particular, the risk of S^2_A can be infinitely higher than that of S^2_N or S^2_P, even if the error variances are equal. Imposing valid prior information does not guarantee a reduction in risk. This accords with our findings in Chapters Four and Five.[21]

(b) Proposition 6.4.1 is applicable to the mis-specified model. That is, $\rho(\sigma^2_{e_1},S^2_P)$ has a minimum when $c^*=\left(v_1(T_2+\mu)\right)/\left(v_2(T_1+\mu)\right)$, and so $c^*_L=1$, $c^*_{ML}=(v_1T_2)/(v_2T_1)$, and $c^*_M=\left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right)$. For any given degree of mis-specification there exists a family of pre-test estimators which

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[21] Figures 6.5.19 to 6.5.42 illustrate the features we have raised in this point.

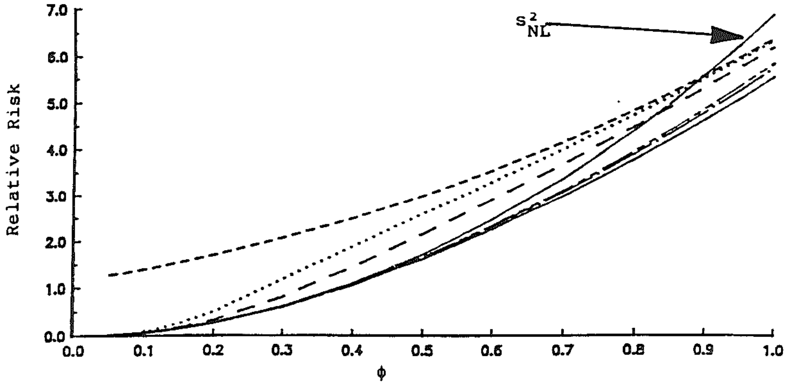

FIGURE 6.5.1: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 3$, $\lambda_2 = 0$.

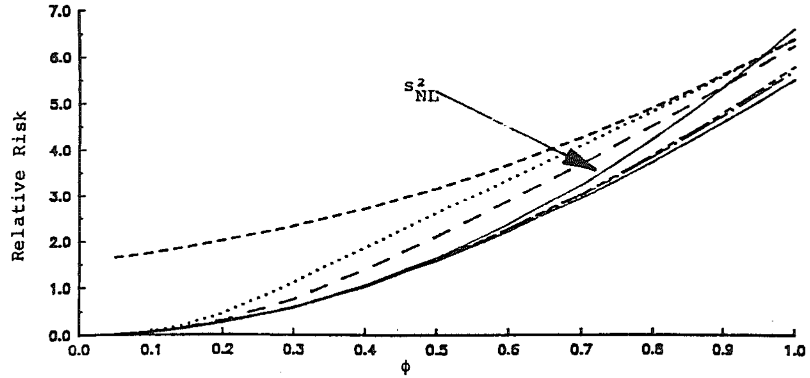

FIGURE 6.5.3: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 0$, $\lambda_2 = 3$.

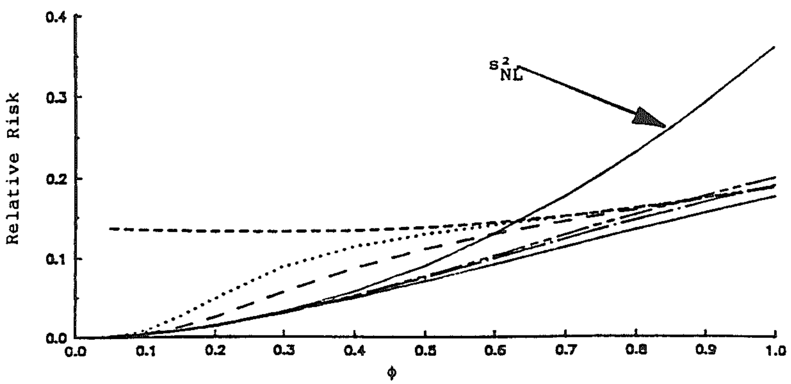

FIGURE 6.5.2: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 3$, $\lambda_2 = 0$.

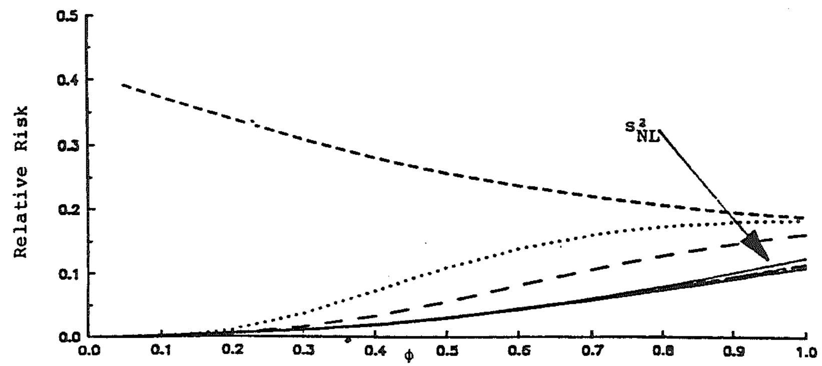

FIGURE 6.5.4: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 0$, $\lambda_2 = 3$.

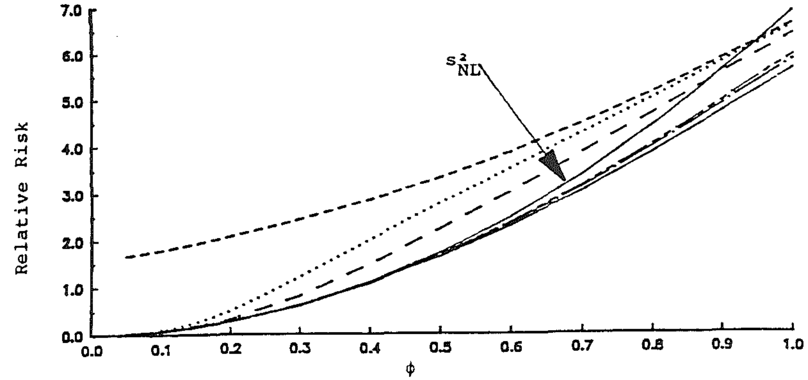

FIGURE 6.5.5: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 3$.

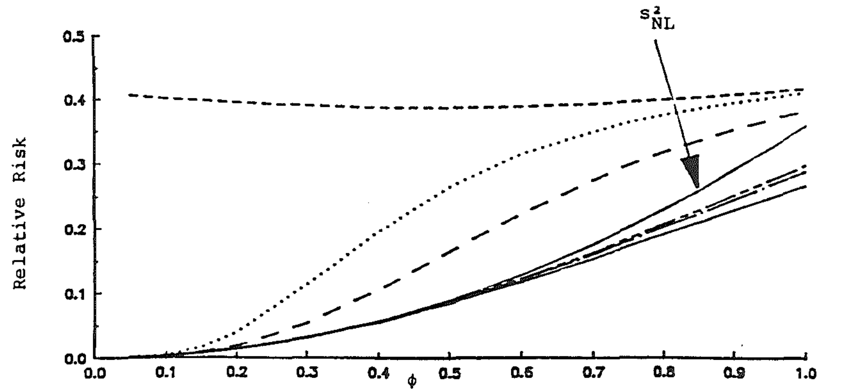

FIGURE 6.5.6: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 3$.

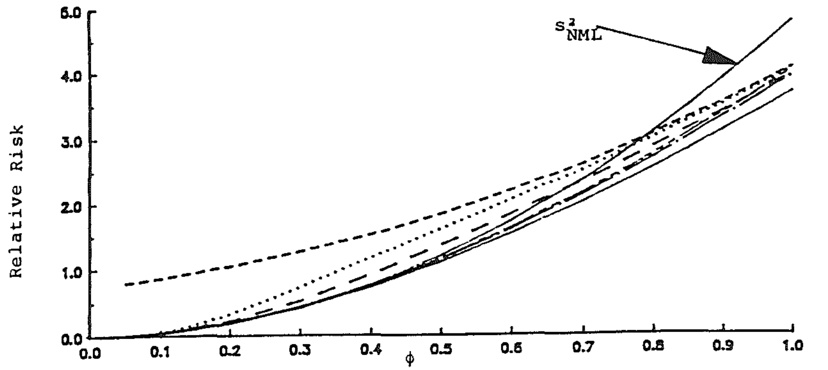

FIGURE 6.5.7: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 3$, $\lambda_2 = 0$.

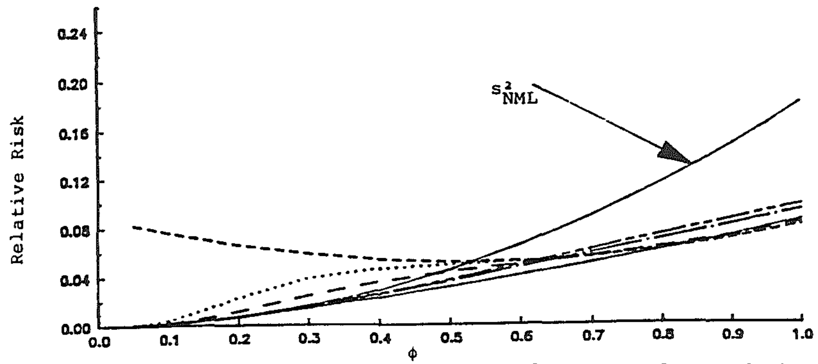

FIGURE 6.5.8: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 3$, $\lambda_2 = 0$.

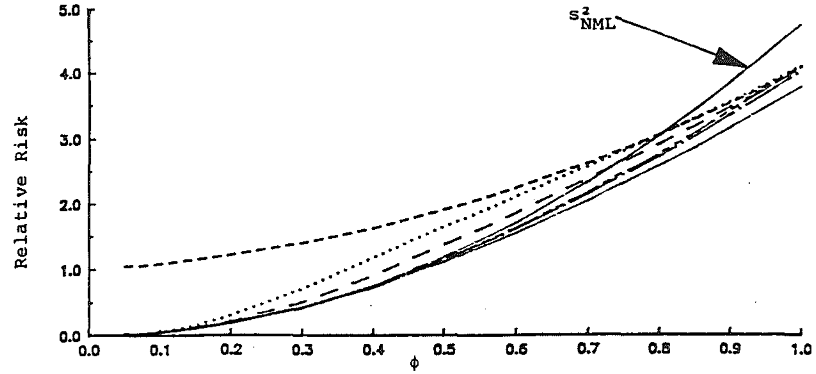

FIGURE 6.5.9: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 0$, $\lambda_2 = 3$.

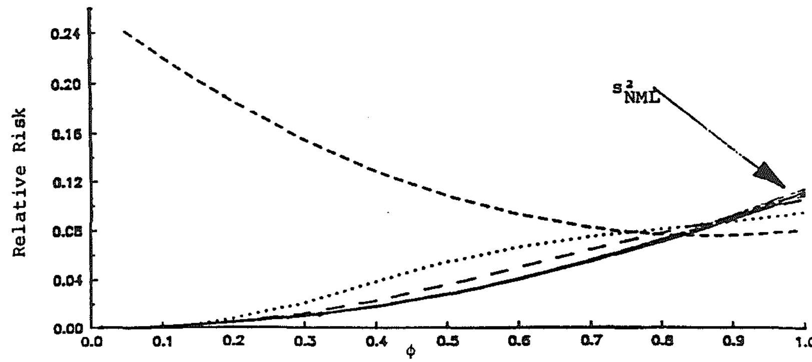

FIGURE 6.5.10: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 0$, $\lambda_2 = 3$.

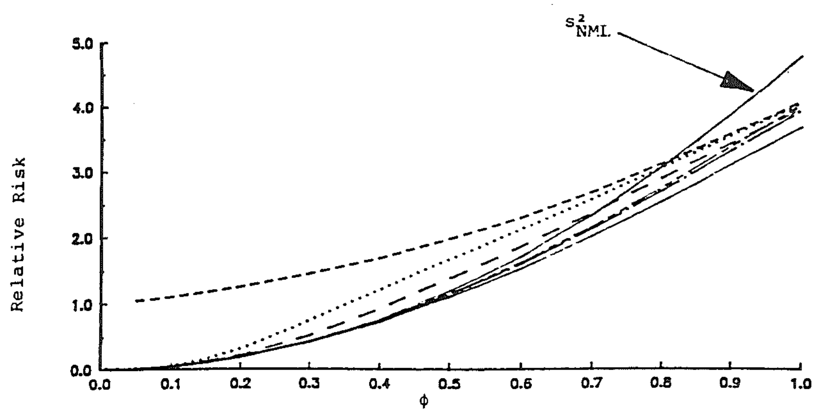

FIGURE 6.5.11: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 3$.

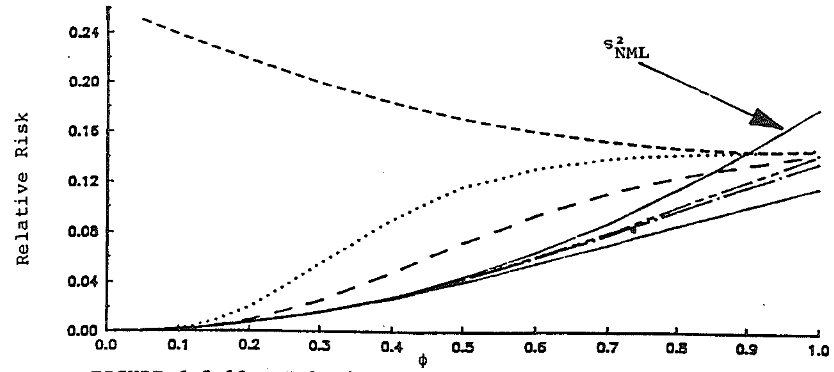

FIGURE 6.5.12: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0,\sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 3$.

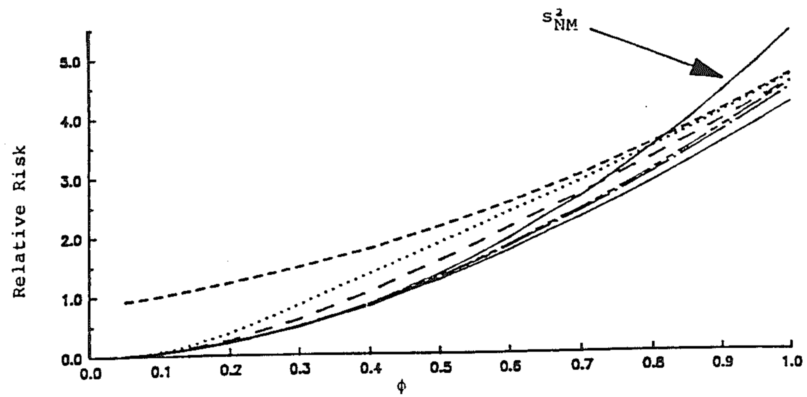

FIGURE 6.5.13: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 3$, $\lambda_2 = 0$.

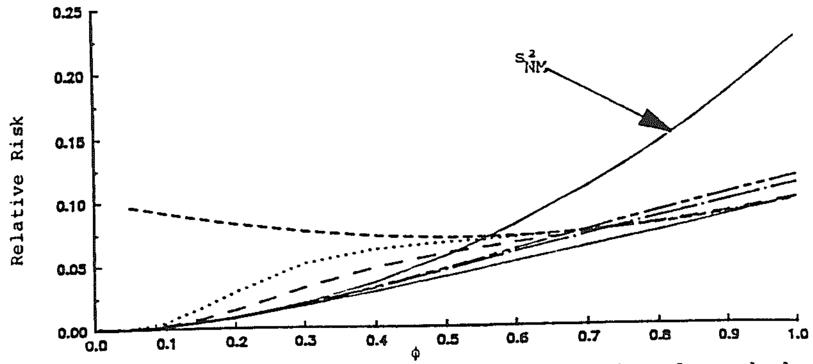

FIGURE 6.5.14: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 3$, $\lambda_2 = 0$.

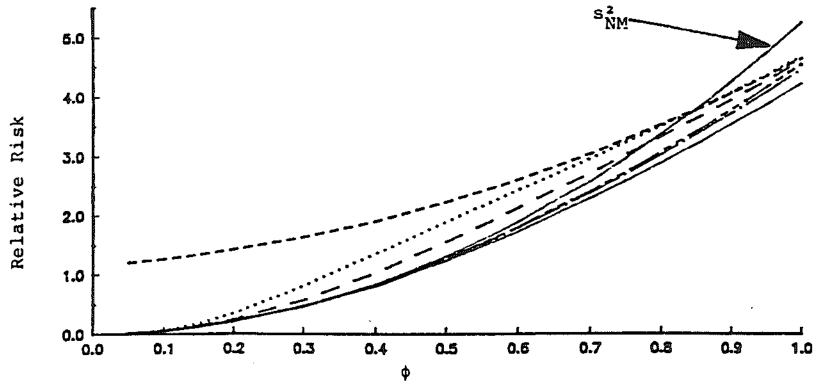

FIGURE 6.5.15: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = 0$, $\lambda_2 = 3$.

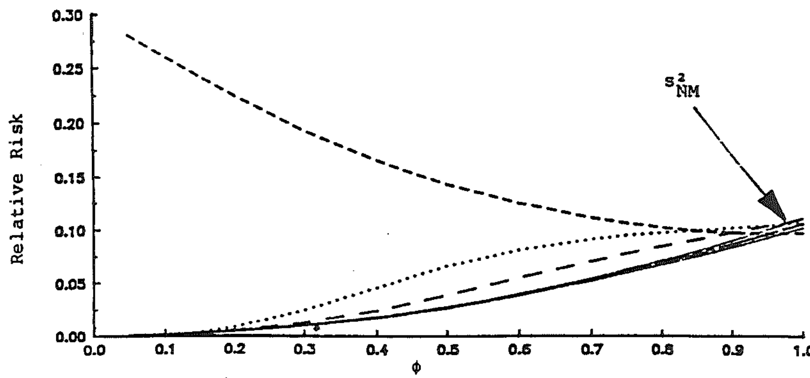

FIGURE 6.5.16: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = 0$, $\lambda_2 = 3$.

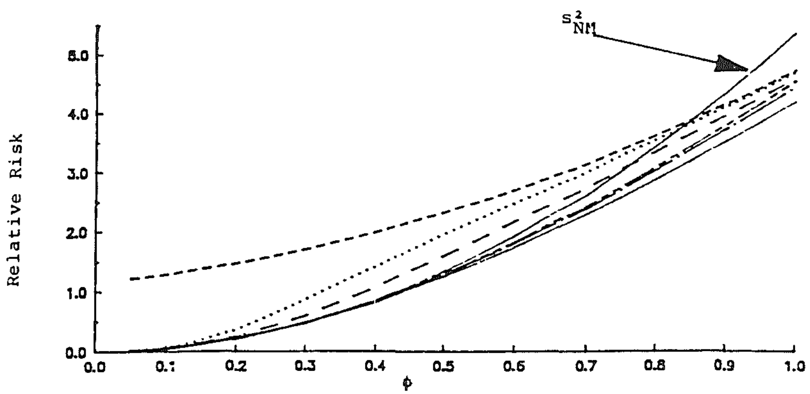

FIGURE 6.5.17: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\lambda_1 = \lambda_2 = 3$.

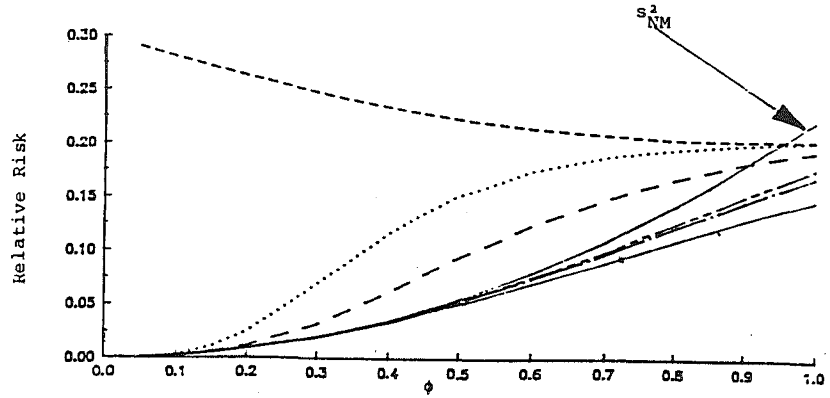

FIGURE 6.5.18: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0,\sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\lambda_1 = \lambda_2 = 3$.

strictly dominate the never-pool estimator, and of this family of dominating estimators the pre-test estimator which uses c=c* has the smallest risk. It is never preferable to ignore the prior information: pre-testing with an appropriately chosen critical value is always a better strategy. This result holds for all θ_1, and θ_2, and for all feasible SSD_N members.

Further, these pre-test estimators can also strictly dominate the always-pool estimator for all ϕ, given θ_1 and θ_2. This will occur when

$$\int_0^\infty \Bigg\{ \phi^2(T_2+\mu)\Big\{(2T_1+T_2+3\mu)\Big[v_1(v_1+2)\Big(E(\tau^4)-\tau^4Q_{04}^{d\tau}\Big)+4\theta_1(v_1+2)\Big(E(\tau^2)$$

$$-\tau^2Q_{06}^{d\tau}\Big)+4\theta_1^2(1-Q_{08}^{d\tau})\Big]-2E(\tau^2)(T_1+\mu)(T+2\mu)\Big[v_1\Big(E(\tau^2)-\tau^2Q_{02}^{d\tau}\Big)$$

$$+2\theta_1(1-Q_{04}^{d\tau})\Big]\Big\}+2(T_1+\mu)^2\phi\Big[-v_1v_2\Big(E(\tau^4)-\tau^4Q_{22}^{d\tau}\Big)+v_2(T+2\mu)E(\tau^2)$$

$$\cdot\Big(E(\tau^2)-\tau^2Q_{20}^{d\tau}\Big)-2\theta_2v_1\Big(E(\tau^2)-\tau^2Q_{42}^{d\tau}\Big)+2\theta_2E(\tau^2)(T+2\mu)(1-Q_{40}^{d\tau})$$

$$-2v_2\theta_1\Big(E(\tau^2)-\tau^2Q_{24}^{d\tau}\Big)-4\theta_1\theta_2(1-Q_{44}^{d\tau})\Big]-(T_1+\mu)^2\Big[v_2(v_2+2)\Big(E(\tau^4)$$

$$-\tau^4Q_{40}^{d\tau}\Big)+4(v_2+2)\theta_2\Big(E(\tau^2)-\tau^2Q_{60}^{d\tau}\Big)+4\theta_2^2(1-Q_{80}^{d\tau})\Big]\Bigg\}f(\tau)d\tau \qquad (6.5.1)$$

is negative. (6.5.1) depends on T_1, T_2, k, μ, as well as the degrees of mis-specification, θ_1 and θ_2, and the variance mixing distribution $f(\tau)$. When $e\sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$ (6.5.1) will be negative if

$$\phi^2(T_2+\mu)\Big[(2T_1+T_2+3\mu)(\nu-2)\Big(v_1(v_1+2)\nu^2(1-Q_{040}^{d})+4\lambda_1(v_1+2)\nu(\nu-4)(1-Q_{061}^{d})$$

$$+4\lambda_1^2(\nu-2)(\nu-4)(1-Q_{082}^{d})\Big)-2\nu(\nu-4)(T_1+\mu)(T+2\mu)\Big(v_1\nu(1-Q_{021}^{d})$$

$$+2\lambda_1(\nu-2)(1-Q_{042}^{d})\Big)\Big]+2(T_1+\mu)^2\phi\Big[-v_1v_2\nu^2(\nu-2)(1-Q_{220}^{d})$$

$$+v_2(T+2\mu)\nu^2(\nu-4)(1-Q^d_{201})-2\lambda_2 v_1\nu(\nu-2)(\nu-4)(1-Q^d_{421})$$

$$+2\lambda_2\nu(\nu-2)(\nu-4)(T+2\mu)(1-Q^d_{402})-2v_2\lambda_1\nu(\nu-2)(\nu-4)(1-Q^d_{241})$$

$$-4\lambda_1\lambda_2(\nu-2)^2(\nu-4)(1-Q^d_{442})\Big]-(T_1+\mu)^2(\nu-2)\Big[v_2(v_2+2)\nu^2(1-Q^d_{400})$$

$$+4(v_2+2)\lambda_2\nu(\nu-4)(1-Q^d_{601})+4\lambda_2^2(\nu-2)(\nu-4)(1-Q^d_{802})\Big] < 0. \qquad (6.5.2)$$

Our numerical evaluations show that if the models for either sample are, or for both samples is, sufficiently mis-specified then the optimal strategy is to pre-test using c=c*, even if the error variances are equal, when using the L, ML, or the M component estimators for $\nu>4$. The results suggest, though, that $\rho(\sigma^2_{e_1},s^2_{AL})$ is relatively less robust to the specification error than are $\rho(\sigma^2_{e_1},s^2_{AML})$ and $\rho(\sigma^2_{e_1},s^2_{AM})$. For example, for relatively large values of ν, for the case we consider in the diagrams, if $\lambda_1=\lambda_2=3$ then $s^2_{PL}|c=1$ strictly dominates s^2_{AL}, while for the ML and the M components there still exists a (small) ϕ-range, in the neighbourhood of ϕ=1, over which we prefer to impose the null hypothesis without first testing its validity. (Compare Figures 6.5.2, 6.5.8, and 6.5.14 .)

(c) $\rho(\sigma^2_{e_1},S^2_N)$ and $\rho(\sigma^2_{e_1},S^2_A)$ have two possible ϕ intersections:[1]

$$\phi_i = \Big((T_1+\mu)/(T_2+\mu)\Big)\Bigg\{(T_1+\mu)\Big[v_1v_2E(\tau^4)-v_2(T+2\mu)\Big(E(\tau^2)\Big)^2$$

$$-2\theta_2E(\tau^2)\Big(v_2+2(k+\mu)\Big)+2v_2\theta_1E(\tau^2)+4\theta_1\theta_2\Big]\pm\Bigg\{(T_1+\mu)^2\Big[v_1v_2E(\tau^4)$$

$$-v_2(T+2\mu)\Big(E(\tau^2)\Big)^2-2\theta_2E(\tau^2)\Big(v_2+2(k+\mu)\Big)+2v_2\theta_1E(\tau^2)+4\theta_1\theta_2\Big]^2$$

---

[1] We require $\Big[(2T_1+T_2+3\mu)\Big(v_1(v_2+2)E(\tau^4)+4\theta_1(v_1+2)E(\tau^2)+4\theta_1^2\Big)-2(T_1+\mu)(T+2\mu)E(\tau^2)\Big(v_1E(\tau^2)+2\theta_1\Big)\Big] \neq 0.$

$$+(T_2+\mu)\Big[(2T_1+T_2+3\mu)\Big(v_1(v_1+2)E(\tau^4)+4\theta_1(v_1+2)E(\tau^2)$$

$$+4\theta_1^2-2(T_1+\mu)(T+2\mu)E(\tau^2)\Big(v_1E(\tau^2)+2\theta_1\Big)\Big]\Big[v_2(v_2+2)E(\tau^4)+$$

$$4(v_2+2)\theta_2E(\tau^2)-4\theta_2^2\Big]\Big\}^{1/2}\Big\}\Big/\Big[(2T_1+T_2+3\mu)\Big(v_1(v_1+2)E(\tau^4)$$

$$+4\theta_1(v_1+2)E(\tau^2)+4\theta_1^2\Big)-2(T_1+\mu)(T+2\mu)E(\tau^2)\Big(v_1E(\tau^2)+2\theta_1\Big)\Big] \qquad (6.5.3)$$

$$= \omega\pm\kappa \ ,$$

$i=1,\ 2.$[2] Let $\phi_1=\omega+\kappa$, and $\phi_2=\omega-\kappa$. When $e\sim Mt(0,\nu\sigma_2^2/(\nu-2)\Sigma)$ and the model is correctly specified we found, in Section 6.4, that there always exists one feasible intersection. If we have omitted regressors, however, there are now four possibilities: (i) $0<\phi_1<1$, $\phi_2>1$; (ii) $\phi_1>1$, $\phi_2<0$; (iii) $0<\phi_1<1$, $\phi_2>1$; (iv) $\phi_1>1$, $\phi_2>1$. If v_1, v_2, λ_1, λ_2 and ν are such that cases (ii) or (iv) result then the always-pool estimator has higher risk than the never-pool estimator for all $\phi\in(0,1]$.

Our numerical evaluations suggest, for relatively large values of ν, if $v_1\geq v_2$ and if $\lambda_2\gg\lambda_1$ then, the never-pool estimator will usually strictly dominate the always-pool estimator. Intuitively, the gain in information from the second sample in terms of additional degrees of freedom is outweighed by the loss in the 'quality' of the information due to the relatively worse specification error in the second sample. For small values of ν we find there is still a small ϕ-range, in the neighbourhood of H_0, over which it is better to always-pool the samples than to never-pool them.

If, however, the mis-specification in the first sample is significantly higher than that in the second sample then there is a ϕ-range over which the always-pool estimator has smaller risk than the never-pool estimator.

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[2] (6.5.3) collapses to (6.4.2) when $\lambda_1=\lambda_2=0$.

Ceteris paribus, the width of this range increases with ν, and with λ_1. Our results suggest that these features apply equally to the L, the ML, or the M component estimators.

(d) Figures 6.5.19 to 6.5.30 illustrate the risks of the L, the ML, and the M estimators as functions of λ_1 for given values of λ_2, ϕ and ν. They consider $\lambda_1 \in [0,4]$, $\nu=5$, ∞, and $\lambda_2=0$, 3 when H_0 is true, that is $\phi=1$.[3] These results are a subset of those given in Tables A6.2.4, A6.2.5, and A6.2.6. The diagrams illustrate first, that $\rho(\sigma^2_{e_1}, s^2_{Nj})$, $\rho(\sigma^2_{e_1}, s^2_{Aj})$, and $\rho(\sigma^2_{e_1}, s^2_{Pj})$, (j=L, ML, and M) are increasing functions in λ_1. Secondly, they show the strict dominance of the pre-test estimator which uses $c=c^*_j$ for small values of ν, irrespective of the values of λ_1 and λ_2. If ν is relatively small then, regardless of the specification error, it is always preferable to pre-test using $c=c^*_j$, even when the error variances are equal.

Thirdly, they indicate that when $e \sim N(0, \sigma^2_2 \Sigma)$ there is usually a range of $\lambda_1 \in [0, \lambda^*_{1j})$ over which the always-pool estimator has the smallest risk, where λ^*_{1j} is that value of λ_1 for which $\rho(\sigma^2_{e_1}, s^2_{Aj}) = \rho(\sigma^2_{e_1}, s^2_{Pj}) | c=c^*_j)$. For $\lambda_1 > \lambda^*_{1j}$, $s^2_{Pj} | c=c^*_j$ dominates even though H_0 is true. λ^*_{1j} decreases with increases in λ_2, and our results suggest that $\lambda^*_{1L} > \lambda^*_{1M} > \lambda^*_{1ML}$ when $v_1 > v_2$.

(e) Tables 6.2.7, 6.2.8, and 6.2.9 give some examples of the risks of the L, the ML, and the M estimators as functions of λ_2 for fixed values of ϕ, ν, λ_1, v_1, and v_2. Figures 6.5.31 to 6.5.42 show the results when $\phi=1$ (H_0 true), $\nu=5$, ∞, and $\lambda_1=0$, 3. The diagrams illustrate first, that $\rho(\sigma^2_{e_1}, s^2_{Aj})$ is unbounded as $\lambda_2 \to \infty$; $\rho(\sigma^2_{e_1}, s^2_{Nj})$ is bounded, as this risk function does not depend on λ_2; while $\rho(\sigma^2_{e_1}, s^2_{Pj}) \to \rho(\sigma^2_{e_1}, s^2_{Nj})$ as $\lambda_2 \to \infty$: if the model of the

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[3] The results we note in this point apply to this particular ϕ value. The tables indicate how the risk functions compare for other ϕ values.

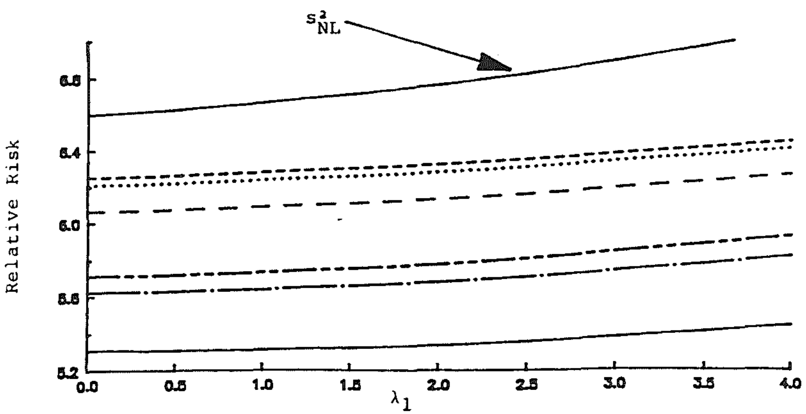

FIGURE 6.5.19: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 0$.

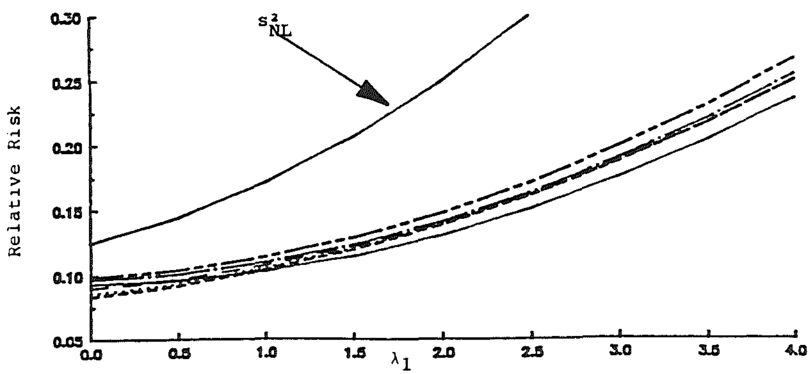

FIGURE 6.5.20: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 0$.

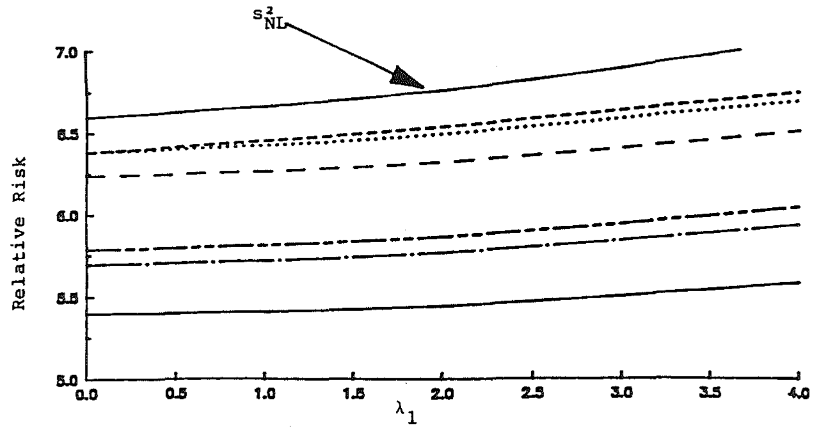

FIGURE 6.5.21: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 3$.

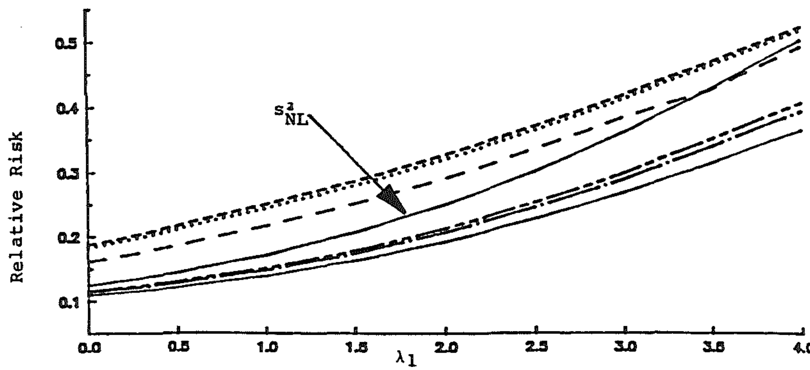

FIGURE 6.5.22: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 3$.

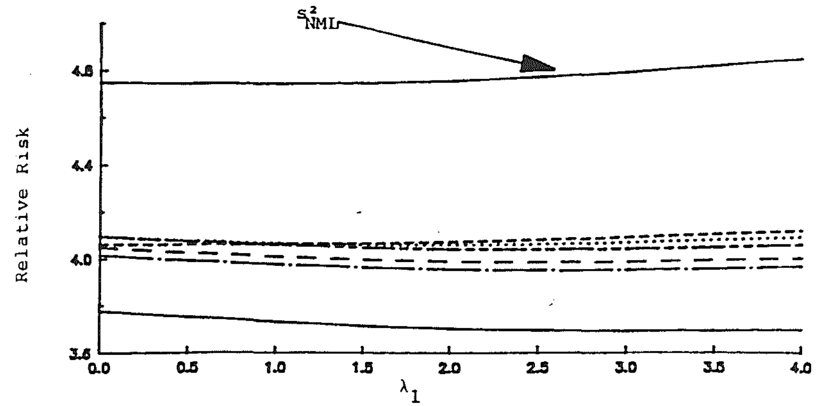

FIGURE 6.5.23: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 0$.

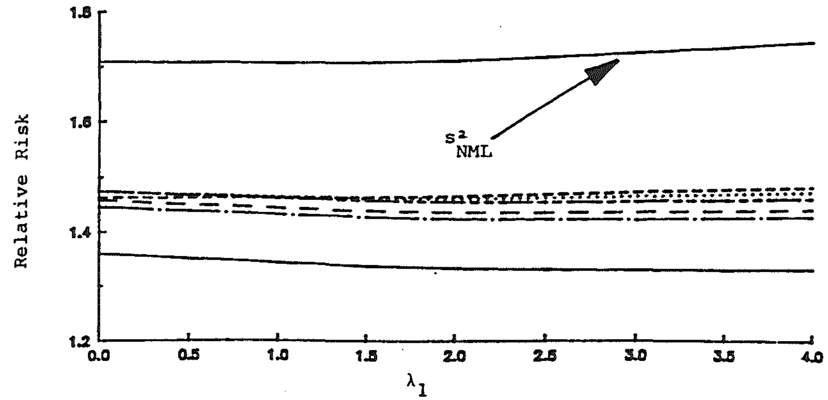

FIGURE 6.5.25: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 3$.

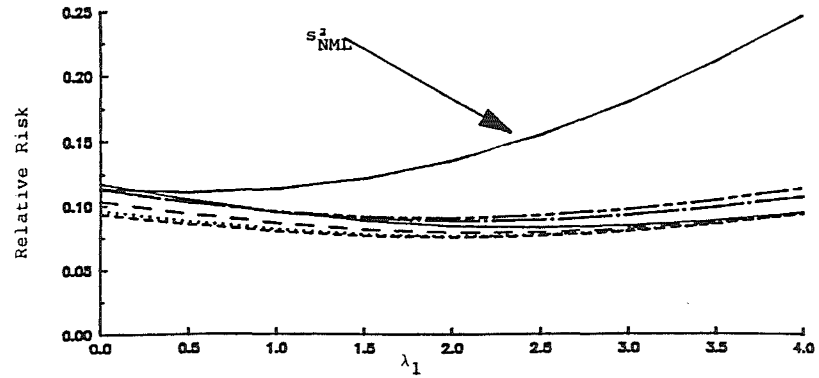

FIGURE 6.5.24: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 0$.

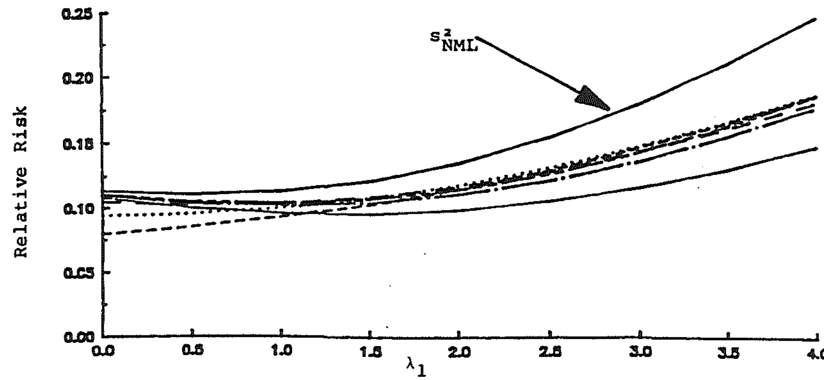

FIGURE 6.5.26: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 3$.

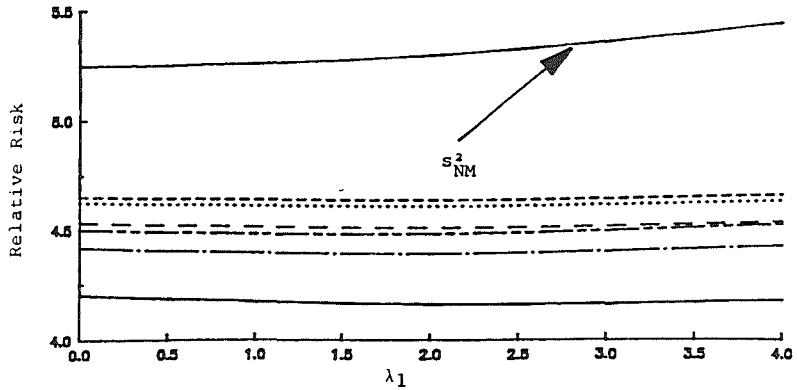

FIGURE 6.5.27: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 0$.

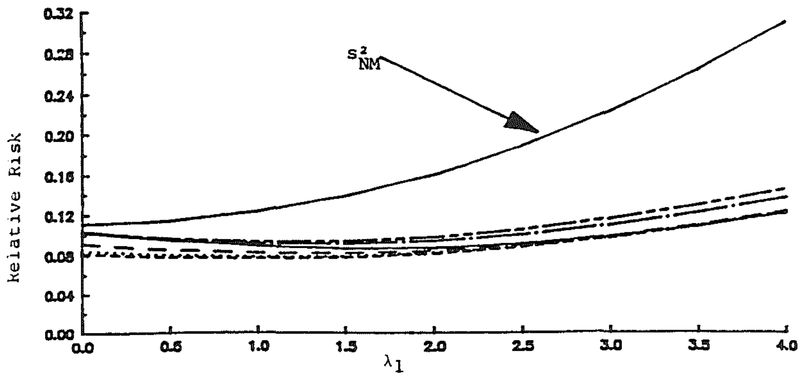

FIGURE 6.5.28: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 0$.

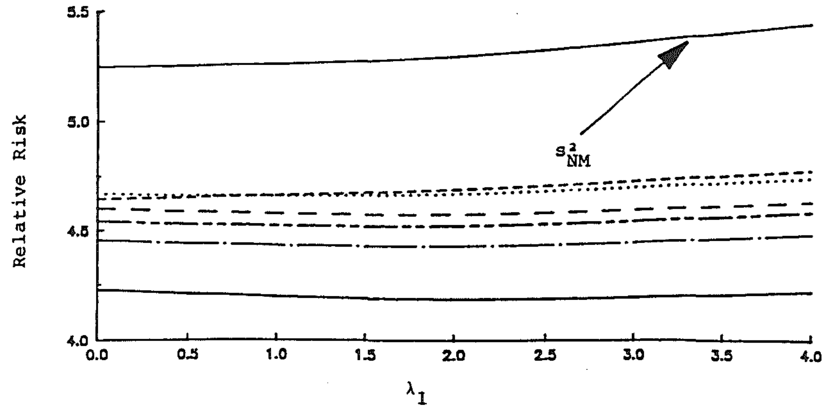

FIGURE 6.5.29: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_2 = 3$.

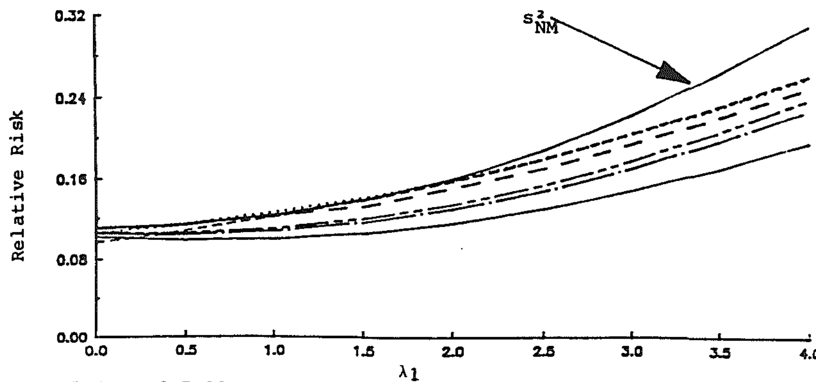

FIGURE 6.5.30: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_2 = 3$.

second sample is extremely mis-specified then pre-testing will lead us to ignore this sample (which is the optimal strategy).

Secondly, the figures illustrate the strict dominance of the pre-test estimator which uses $c=c_j^*$ for small values of ν, regardless of the values of λ_1 and λ_2. Thirdly, they indicate that there can be a λ_2 range, $\lambda_2 \in [0, \lambda_{2j}^*)$, over which the always-pool estimator dominates. λ_{2j}^* is that value of λ_2 for which $\rho(\sigma_{e_1}^2, s_{Aj}^2) = \rho(\sigma_{e_1}^2, s_{Pj}^2 | c=c_j^*)$; j=L, ML, M. If $\lambda_2 > \lambda_{2j}^*$, then $s_{P_j}^2 | c=c_j^*$ dominates. λ_{2j}^* decreases with increases in λ_1, and will be negative for large λ_1. In such cases $s_{Pj}^2 | c=c_j^*$ strictly dominates. Our results suggest that $\lambda_{2L}^* > \lambda_{2M}^* < \lambda_{2ML}^*$.

(f) We suggested in the last section when considering the correctly specified model that if $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$ and if we adopted a pre-test strategy then, using a minimax criterion, it is preferable to employ the ML component estimators when ν is small. So, as $s_{PML}^2 | c=c_{ML}^*$ strictly dominates all of the other considered estimators, the optimal strategy is to pre-test using $c=c_{ML}^*$, even if H_0 is true. Our numerical evaluations suggest that these conclusions carry over to the omitted regressors model, irrespective of the degrees of mis-specification in each sample.

However, for relatively large values of ν and, in particular, when $e \sim N(0, \sigma_2^2\Sigma)$, if there are no excluded variables then it is preferable to employ s_{PM}^2 for $\alpha=0.05$ and s_{PL}^2 for $\alpha \geq 0.05$. When we have omitted regressors this finding no longer holds. Then, it seems that the optimal strategy is to use the ML components if $\lambda_1 > 0$, regardless of the value of λ_2. If the model of the first sample is sufficiently mis-specified then $s_{PML}^2 | c=c_{ML}^*$ strictly dominates.

When $\lambda_1=0$ but $\lambda_2>0$ our results suggest use of the ML components for $\alpha \leq 0.05$ and the M components for $\alpha > 0.05$. For a sufficiently high value of λ_2, $s_{PM}^2 | c=c_M^*$ strictly dominates all of the other considered estimators. So,

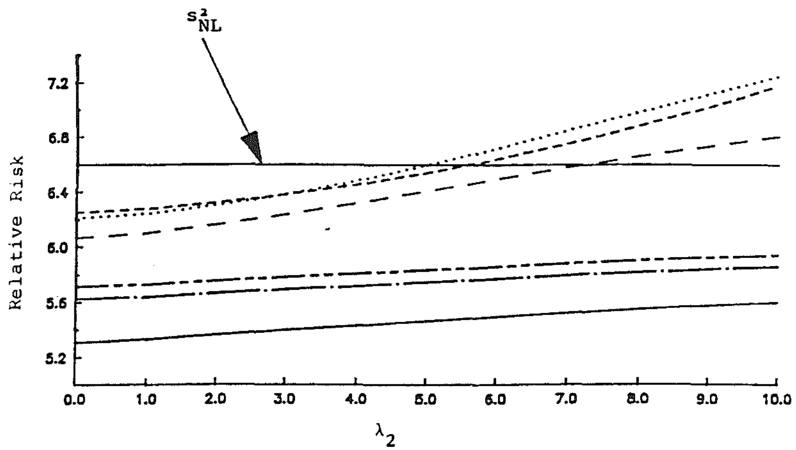

FIGURE 6.5.31: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 0$.

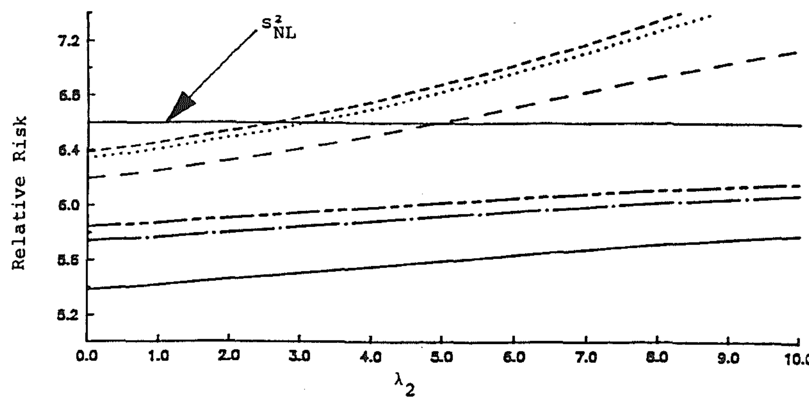

FIGURE 6.5.33: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 3$.

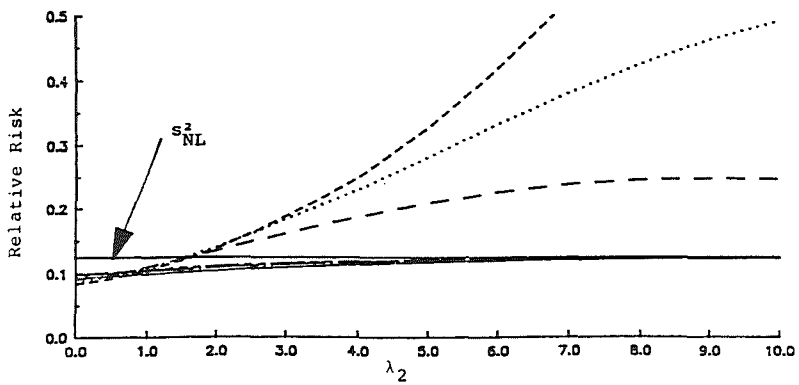

FIGURE 6.5.32: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 0$.

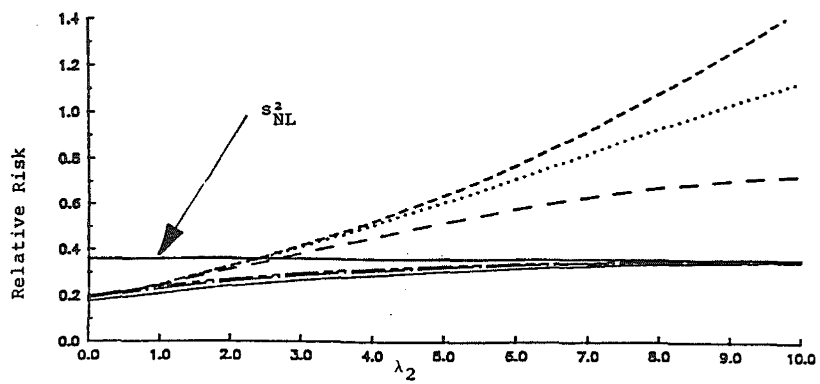

FIGURE 6.5.34: Relative risk functions for s^2_{NL}, s^2_{AL}, and s^2_{PL} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 3$.

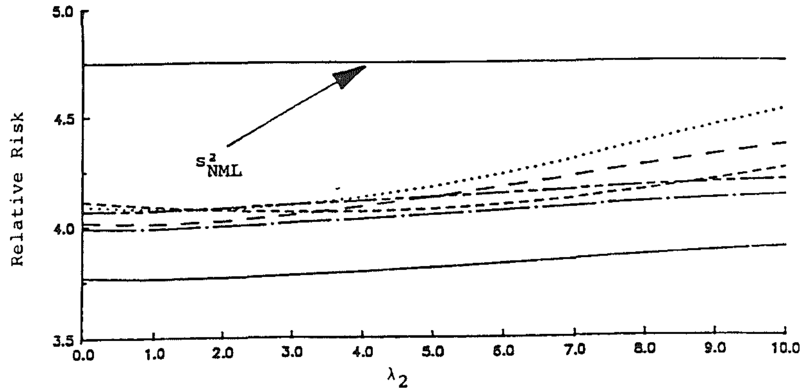

FIGURE 6.5.35: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 0$.

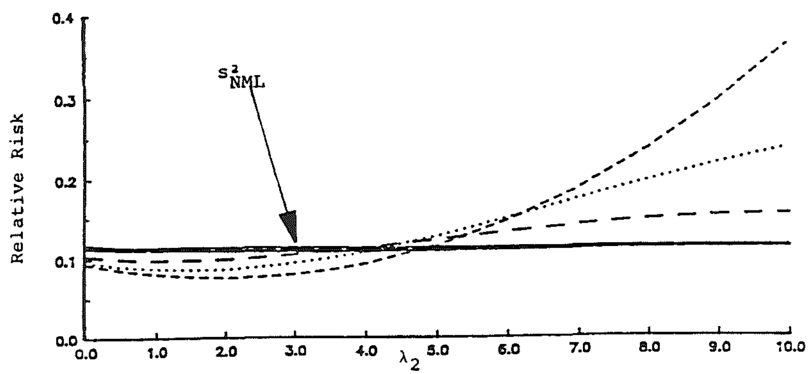

FIGURE 6.5.36: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 0$.

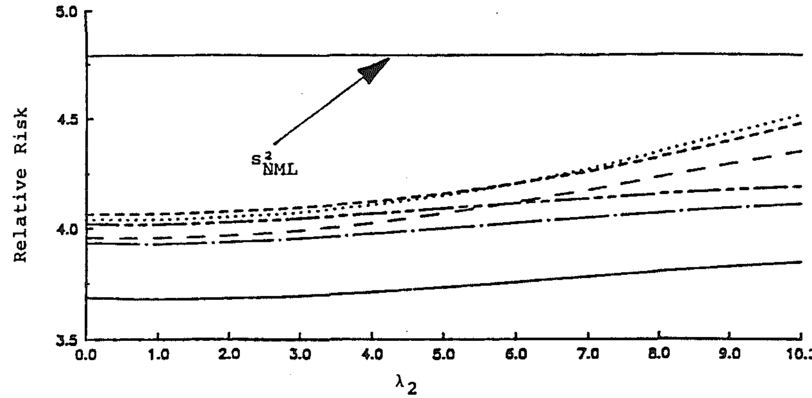

FIGURE 6.5.37: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 3$.

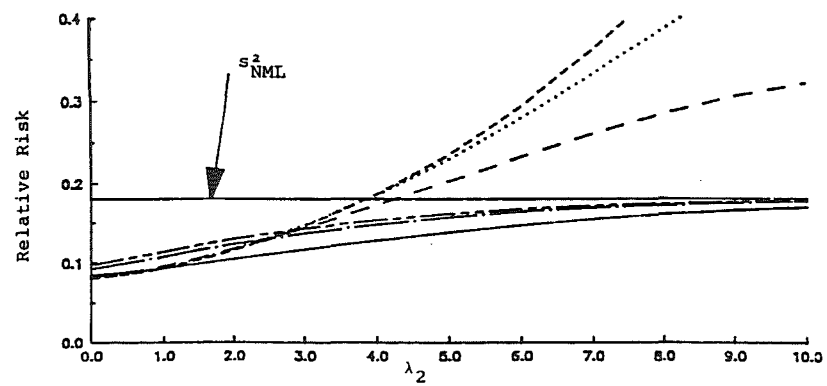

FIGURE 6.5.38: Relative risk functions for s^2_{NML}, s^2_{AML}, and s^2_{PML} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 3$.

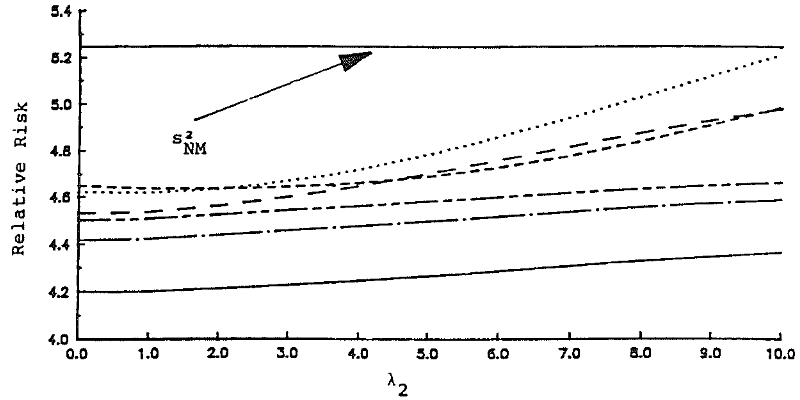

FIGURE 6.5.39: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 0$.

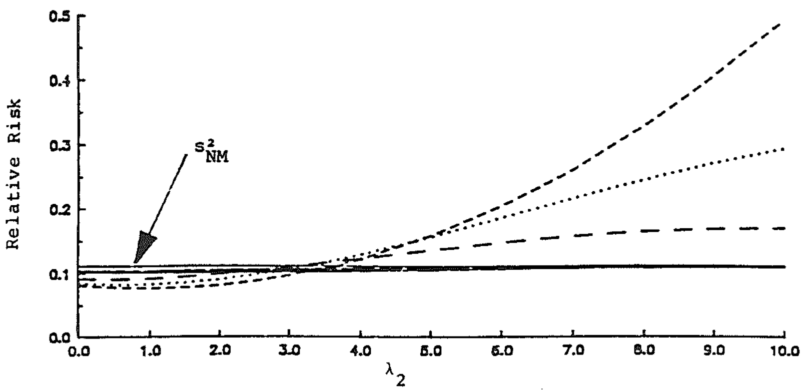

FIGURE 6.5.40: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 0$.

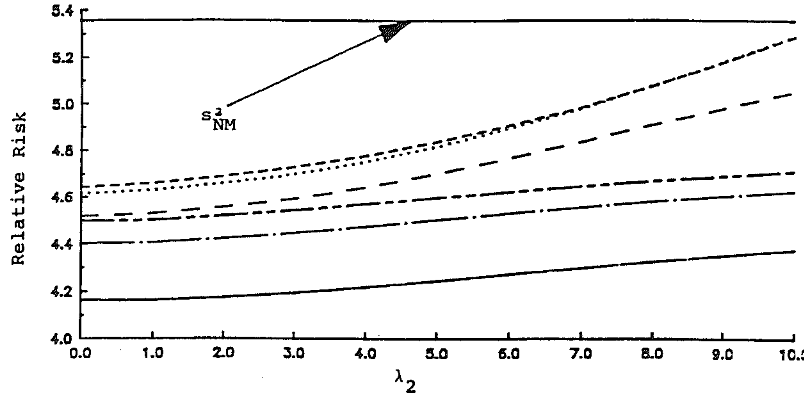

FIGURE 6.5.41: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim Mt(0, \nu\sigma_2^2/(\nu-2)\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = 5$, $\phi = 1$, $\lambda_1 = 3$.

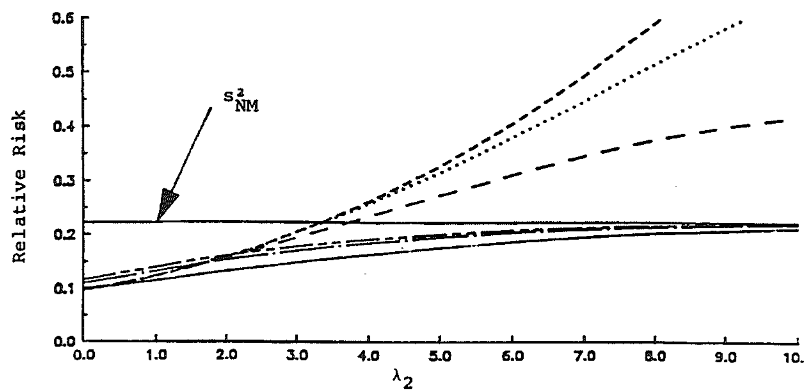

FIGURE 6.5.42: Relative risk functions for s^2_{NM}, s^2_{AM}, and s^2_{PM} when $e \sim N(0, \sigma_2^2\Sigma)$, $v_1 = 16$, $v_2 = 8$, $k = 3$, $\nu = \infty$, $\phi = 1$, $\lambda_1 = 3$.

the L components are not robust to the specification error of the design matrix or of the error distribution. When ν is small and/or the model is mis-specified through the omission of relevant variables the L components usually have the highest risk of the estimators we investigated.

In this section we have compared the risk functions of S_N^2, S_A^2, and S_P^2 when we may have mis-specified the design matrix. Our results suggest that if the specification error is severe then we should pre-test using $c=c^*$, even if the error variances are equal. The results also show that the usual L estimators are dominated by other estimators if the model is sufficiently mis-specified through the omission of relevant regressors and/or ν is small.

6.6 Concluding Remarks

In this chapter we have examined some sampling properties of estimators of the disturbance variance, after a preliminary test of homogeneity, when the joint distribution of the unobservable errors in each sample is SSD_N but it is assumed to be normal, and there is a possible specification error of the design matrix. We considered the usual least squares estimators of the error variance and we also investigated the sampling properties of the never-pool, the always-pool and the pre-test estimators whose components are the usual never-pool maximum likelihood and the minimum mean squared error estimators assuming that both the error distribution and the design matrix are correctly specified. Of course, under the investigated specification errors these estimators do not possess their desired properties.

Nevertheless, our results suggest that these estimators are preferred to the usual least squares estimators when ν is small and/or the degree of mis-specification of the design matrix is sufficiently high, whether one is interested in the bias or the risk functions of the estimators. In either of these extreme cases the pre-test estimator which uses $c=c^*$ strictly dominates and so then this estimator should be used. It is never preferable to always-pool the samples without testing the validity of the null hypothesis, nor is it optimal to ignore the prior information.

However, if the model specification is valid, that is, if the error distribution is normal and the design matrix is correctly specified, then there is no dominating estimator for all $\phi \in (0,1]$. The pre-test estimator which uses $c=c^*$ dominates for $\phi \in (0,\phi^*)$ while, for $\phi \in (\phi^*,1]$, the always-pool estimator has the smallest risk. ϕ^* is that value of ϕ for which $\rho_N(\sigma_1^2,S_A^2)=\rho_N(\sigma_1^2,S_P^2|c=c^*)$. Our results suggest that then, on the basis of a minimax criterion, it is preferable to employ the M components for $\alpha=0.01$ and the L components for $\alpha \geq 0.05$.

We should recall, though, that the results discussed in this chapter apply to a one-sided alternative hypothesis. It remains for future research to consider the two-sided alternative case.

This chapter concludes the research presented in this thesis. We provide some final remarks in the next chapter.

APPENDIX 6.1

BIAS AND RISK FUNCTIONS OF THE L, ML, AND M ESTIMATORS

Some special cases of the theorems and the corollaries of Chapter Six are given in this Appendix. We consider the so-called L, ML, and M never-pool, always-pool, and pre-test estimators of the error variance in the first sample. These estimators are discussed in Chapter Six, and for the case of Mt regression disturbances we numerically evaluate the risk functions given here.

The notation s^2_{PL}, s^2_{PML}, s^2_{PM} is used to signify that the never-pool components of these pre-test estimators are the corresponding L, ML, or M estimators of $\sigma^2_{e_1}$.[1]

<u>Theorem 6.3.1</u> (Special Cases)

The following special cases are obtained from Theorem 6.3.1 by substituting in the appropriate value of μ.

(i) Least Squares (L) Components (μ=-k).

$$\text{bias}\left(s^2_{NL}\right) = 2\theta_1\phi/v_1 \ , \tag{A6.1}$$

$$\text{bias}\left(s^2_{AL}\right) = \left(v_2E(\tau^2)(1-\phi)+2(\phi\theta_1+\theta_2)\right)/(v_1+v_2) \ , \tag{A6.2}$$

$$\text{bias}\left(s^2_{PL}\right) = \left(2\phi\theta_1(v_1+v_2)+v_1v_2\int_0^\infty \tau^2\left[Q^{d\tau}_{20}-\phi Q^{d\tau}_{02}\right]f(\tau)d\tau\right.$$

[1] More correctly, the researcher believes this to be so. As we discussed in Chapter Six, the ML and M estimators do not possess these properties for the broader distribution of errors that we consider.

$$+\ 2\int_0^{\infty}\left[v_1\theta_2 Q_{40}^{d\tau}-v_2\phi\theta_1 Q_{04}^{d\tau}\right]f(\tau)d\tau\Big)\Big/\Big(v_1(v_1+v_2)\Big)\ . \tag{A6.3}$$

(ii) Maximum Likelihood (ML) Components (μ=0):

$$\mathrm{bias}\left(s_{NML}^2\right) = \phi\left(2\theta_1-kE(\tau^2)\right)/T_1\ , \tag{A6.4}$$

$$\mathrm{bias}\left(s_{AML}^2\right) = \left(v_2E(\tau^2)(1-\phi)-2\phi E(\tau^2)k+2(\phi\theta_1+\theta_2)\right)/T\ , \tag{A6.5}$$

$$\mathrm{bias}\left(s_{PML}^2\right) = \Big(\phi T\left[2\theta_1-kE(\tau^2)\right]+\int_0^{\infty}\tau^2\left[v_2T_1Q_{20}^{d\tau}-v_1\phi T_2Q_{02}^{d\tau}\right]f(\tau)d\tau$$

$$+2\int_0^{\infty}\left[\theta_2T_1Q_{40}^{d\tau}-\theta_1\phi T_2Q_{04}^{d\tau}\right]f(\tau)d\tau\Big)/(T_1T)\ . \tag{A6.6}$$

(iii) Minimum Mean Squared Error (M) Components (μ=-k+2):

$$\mathrm{bias}\left(s_{NM}^2\right) = 2\phi\left(\theta_1-E(\tau^2)\right)/(v_1+2)\ , \tag{A6.7}$$

$$\mathrm{bias}\left(s_{AM}^2\right) = \left(v_2E(\tau^2)(1-\phi)-4\phi E(\tau^2)+2(\phi\theta_1+\theta_2)\right)/(v_1+v_2+4)\ , \tag{A6.8}$$

$$\mathrm{bias}\left(s_{PM}^2\right) = \Big(2\phi(v_1+v_2+4)\left[\theta_1-E(\tau^2)\right]+\int_0^{\infty}\Big\{\tau^2\Big[v_2(v_1+2)Q_{20}^{d\tau}$$

$$-v_1\phi(v_2+2)Q_{02}^{d\tau}\Big]+2\left[\theta_2(v_1+2)Q_{40}^{d\tau}-\theta_1\phi(v_2+2)Q_{04}^{d\tau}\right]\Big\}f(\tau)d\tau\Big)$$

$$\Big/\Big((v_1+2)(v_1+v_2+4)\Big)\ . \tag{A6.9}$$

<u>Corollary 6.3.2</u> (Special Cases)

The following special cases are obtained from Corollary 6.3.2 by substituting in the appropriate value of μ.

(i) L Components (μ=-k):

$$\mathrm{bias}_{Mt}\left(s_{NL}^2\right) = 2\sigma_2^2\phi\lambda_1/v_1\ , \tag{A6.10}$$

$$\mathrm{bias}_{Mt}\left(s_{AL}^2\right) = \sigma_2^2\left(v_2\nu(1-\phi)+2(\nu-2)(\phi\lambda_1+\lambda_2)\right)\Big/\Big((\nu-2)(v_1+v_2)\Big)\ , \tag{A6.11}$$

$$\text{bias}_{Mt}\left(s^2_{PL}\right) = \sigma^2_2\Big(2\phi\lambda_1(v_1+v_2)(\nu-2)+v_1v_2\nu\left(Q^d_{201}-\phi Q^d_{021}\right)$$

$$+2(\nu-2)\left[v_1\lambda_2Q^d_{402}-\phi v_2Q^d_{042}\right]\Big)\Big/\Big((\nu-2)v_1(v_1+v_2)\Big) \ . \tag{A6.12}$$

(ii) ML Components (μ=0):

$$\text{bias}_{Mt}\left(s^2_{NML}\right) = \phi\sigma^2_2\Big(2\lambda_1(\nu-2)-k\nu\Big)\Big/\Big((\nu-2)T_1\Big) \ , \tag{A6.13}$$

$$\text{bias}_{Mt}\left(s^2_{AML}\right) = \sigma^2_2\Big(v_2\nu(1-\phi)-2\phi\nu k+2(\nu-2)(\phi\lambda_1+\lambda_2)\Big)$$

$$\Big/\Big((\nu-2)T\Big), \tag{A6.14}$$

$$\text{bias}_{Mt}\left(s^2_{PML}\right) = \sigma^2_2\Big(\phi T\left[2\lambda_1(\nu-2)-\nu k\right]+v_2\nu T_1Q^d_{201}-v_1\phi\nu T_2Q^d_{021}$$

$$+2\lambda_2T_1(\nu-2)Q^d_{402}-2\lambda_1\phi T_2(\nu-2)Q^d_{042}\Big)\Big/\Big((\nu-2)T_1(T+2\mu)\Big) \ . \tag{A6.15}$$

(iii) M Components (μ=-k+2):

$$\text{bias}_{Mt}\left(s^2_{NM}\right) = 2\phi\sigma^2_2\Big(\lambda_1(\nu-2)-\nu\Big)\Big/\Big((\nu-2)(v_1+2)\Big) \ , \tag{A6.16}$$

$$\text{bias}_{Mt}\left(s^2_{AM}\right) = \sigma^2_2\Big(v_2\nu(1-\phi)-4\phi\nu+2(\nu-2)(\phi\lambda_1+\lambda_2)\Big)$$

$$\Big/\Big((\nu-2)(v_1+v_2+4)\Big) \ , \tag{A6.17}$$

$$\text{bias}_{Mt}\left(s^2_{PM}\right) = \sigma^2_2\Big(2\phi(v_1+v_2+4)\left[\lambda_1(\nu-2)-\nu\right]+v_2\nu(v_1+2)Q^d_{201}$$

$$-v_1\phi\nu(v_2+2)Q^d_{021}+2\lambda_2(v_1+2)(\nu-2)Q^d_{402}-2\lambda_1\phi(v_2+2)(\nu-2)Q^d_{042}\Big)$$

$$\Big/\Big((\nu-2)(v_1+2)(v_1+v_2+4)\Big) \ . \tag{A6.18}$$

Corollary 6.3.3 (Special Cases)

The following special cases are obtained from Corollary 6.3.3 by substituting in the appropriate value of μ.

(i) L Components (μ=-k):

$$\mathrm{bias}_N\left(s^2_{NL}\right) = 2\phi\sigma^2_2\lambda_1/v_1 , \tag{A6.19}$$

$$\mathrm{bias}_N\left(s^2_{AL}\right) = \sigma^2_2\left(v_2(1-\phi)+2(\phi\lambda_1+\lambda_2)\right)/(v_1+v_2) , \tag{A6.20}$$

$$\mathrm{bias}_N\left(s^2_{PL}\right) = \sigma^2_2\left(2\phi\lambda_1(v_1+v_2)+v_1v_2\left[Q^d_{20}-\phi Q^d_{02}\right]+2\left[v_1\lambda_2Q^d_{40} -\phi v_2Q^d_{04}\right]\right)/\left(v_1(v_1+v_2)\right) . \tag{A6.21}$$

(ii) ML Components (μ=0):

$$\mathrm{bias}_N\left(s^2_{NML}\right) = \phi\sigma^2_2(2\lambda_1-k)/T_1 , \tag{A6.22}$$

$$\mathrm{bias}_N\left(s^2_{AML}\right) = \sigma^2_2\left(v_2(1-\phi)-2\phi k+2(\phi\lambda_1+\lambda_2)\right)/T , \tag{A6.23}$$

$$\mathrm{bias}_N\left(s^2_{PML}\right) = \sigma^2_2\left(\phi T(2\lambda_1-k)+v_2T_1Q^d_{20}-v_1\phi T_2Q^d_{02}+2\lambda_2T_1Q^d_{40} -2\lambda_1\phi T_2Q^d_{04}\right)/(T_1T) . \tag{A6.24}$$

(iii) M Components (μ=-k+2)

$$\mathrm{bias}_N\left(s^2_{NM}\right) = 2\phi\sigma^2_2(\lambda_1-1)/(v_1+2) , \tag{A6.25}$$

$$\mathrm{bias}_N\left(s^2_{AM}\right) = \sigma^2_2\left(v_2(1-\phi)-4\phi+2(\phi\lambda_1+\lambda_2)\right)/(v_1+v_2+4) , \tag{A6.26}$$

$$\mathrm{bias}_N\left(s^2_{PM}\right) = \sigma^2_2\left(2\phi(v_1+v_2+4)(\lambda_1-1)+v_2(v_1+2)Q^d_{20}-v_1\phi(v_2+2)Q^d_{02} +2\lambda_2(v_1+2)Q^d_{40}-2\lambda_1\phi(v_2+2)Q^d_{04}\right)/\left((v_1+2)(v_1+v_2+4)\right) . \tag{A6.27}$$

Theorem 6.3.2 (Special Cases)

The following special cases are obtained from Theorem 6.3.2 by substituting in the appropriate value of μ.

(i) L Components (μ=-k):

$$\rho\left(\sigma^2_{e_1},s^2_{NL}\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)-v_1^2\left(E(\tau^2)\right)^2+8\theta_1E(\tau^2)+4\theta_1^2\right)/v_1^2 , \tag{A6.28}$$

$$\rho\left(\sigma^2_{e_1},s^2_{AL}\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1E(\tau^2)-2v_1(v_1+v_2)\left(E(\tau^2)\right)^2\right.\right.$$
$$\left.+\left(2\theta_1-E(\tau^2)(v_1+v_2)\right)^2\right]+2\phi\left[v_1v_2E(\tau^4)-v_2(v_1+v_2)\left(E(\tau^2)\right)^2\right.$$
$$\left.\left.+2v_2E(\tau^2)(\theta_1-\theta_2)+4\theta_1\theta_2\right]+v_2(v_2+2)E(\tau^4)+4(v_2+2)\theta_2E(\tau^2)+4\theta_2^2\right)$$
$$/(v_1+v_2)^2 , \tag{A6.29}$$

$$\rho\left(\sigma^2_{e_1},s^2_{PL}\right) = \left(\phi^2(v_1+v_2)^2\left[v_1(v_1+2)E(\tau^4)-v_1^2\left(E(\tau^2)\right)^2+8\theta_1E(\tau^2)+4\theta_1^2\right]\right.$$
$$+\int_0^\infty\left(\phi^2v_2\left[-(2v_1+v_2)\left(v_1(v_1+2)\tau^4Q^{d\tau}_{04}+4(v_1+2)\theta_1\tau^2Q^{d\tau}_{06}+4\theta_1^2Q^{d\tau}_{08}\right)\right.\right.$$
$$\left.+2E(\tau^2)v_1(v_1+2)\left(v_1\tau^2Q^{d\tau}_{02}+2\theta_1Q^{d\tau}_{04}\right)\right]+2v_1^2\phi\left[v_1v_2\tau^4Q^{d\tau}_{22}-\right.$$
$$v_2(v_1+v_2)E(\tau^2)\tau^2Q^{d\tau}_{20}+2v_1\theta_2\tau^2Q^{d\tau}_{42}-2(v_1+v_2)\theta_2E(\tau^2)Q^{d\tau}_{40}$$
$$\left.+2v_2\theta_1\tau^2Q^{d\tau}_{24}+4\theta_1\theta_2\tau^2Q^{d\tau}_{44}\right]+v_1^2\left[v_2(v_2+2)\tau^4Q^{d\tau}_{40}\right.$$
$$\left.\left.\left.+4(v_2+2)\tau^2\theta_2Q^{d\tau}_{60}+4\theta_2^2Q^{d\tau}_{80}\right]\right)f(\tau)d\tau\right)/\left(v_1^2(v_1+v_2)^2\right) . \tag{A6.30}$$

(ii) ML Components (μ=0)

$$\rho\left(\sigma^2_{e_1},s^2_{NML}\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)+T_1(T_1-2v_1)\left(E(\tau^2)\right)^2\right.$$

$$+4\theta_1 E(\tau^2)(2-k)+4\theta_1^2\Big)/T_1^2 \ , \tag{A6.31}$$

$$\rho\left(\sigma_{e_1}^2, s_{AML}^2\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1 E(\tau^2)-2v_1 T\left(E(\tau^2)\right)^2\right.\right.$$

$$\left.+\left(2\theta_1-E(\tau^2)T\right)^2\right]+2\phi\left[v_1 v_2 E(\tau^4)-v_2 T\left(E(\tau^2)\right)^2-2\theta_2 E(\tau^2)(v_2+2k)\right.$$

$$\left.\left.+2v_2\theta_1 E(\tau^2)+4\theta_1\theta_2\right]+v_2(v_2+2)E(\tau^4)+4(v_2+2)\theta_2 E(\tau^2)+4\theta_2^2\right)/T^2 \ , \tag{A6.32}$$

$$\rho\left(\sigma_{e_1}^2, s_{PML}^2\right) = \left(\phi^2 T^2\left[v_1(v_1+2)E(\tau^4)+T_1\left(E(\tau^2)\right)^2(T_1-2v_1)+4\theta_1 E(\tau^2)(2-k)\right.\right.$$

$$\left.+4\theta_1^2\right]+\int_0^\infty\left(\phi^2 T_2\left[-(2T_1+T_2)\left(v_1(v_1+2)\tau^4 Q_{04}^{d\tau}+4(v_1+2)\theta_1\tau^2 Q_{06}^{d\tau}+4\theta_1^2 Q_{08}^{d\tau}\right)\right.\right.$$

$$\left.+2T_1 TE(\tau^2)\left(v_1\tau^2 Q_{02}^{d\tau}+2\theta_1 Q_{04}^{d\tau}\right)\right]+2T_1^2\phi\left[v_1 v_2\tau^4 Q_{22}^{d\tau}-v_2 TE(\tau^2)\tau^2 Q_{20}^{d\tau}\right.$$

$$\left.+2v_1\theta_2\tau^2 Q_{42}^{d\tau}-2T\theta_2 E(\tau^2)Q_{40}^{d\tau}+2v_2\theta_1\tau^2 Q_{24}^{d\tau}+4\theta_1\theta_2 Q_{44}^{d\tau}\right]+T_1^2\left[v_2(v_2+2)\tau^4 Q_{40}^{d\tau}\right.$$

$$\left.\left.\left.+4(v_2+2)\tau^2 Q_{60}^{d\tau}+4\theta_2^2 Q_{80}^{d\tau}\right]\right)f(\tau)d\tau\right)/(T_1 T)^2 \ . \tag{A6.33}$$

(iii) M Components (μ=-k+2):

$$\rho\left(\sigma_{e_1}^2, s_{NM}^2\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)+(v_1+2)(2-v_1)\left(E(\tau^2)\right)^2+4\theta_1^2\right)$$

$$/(v_1+2)^2 \ , \tag{A6.34}$$

$$\rho\left(\sigma_{e_1}^2, s_{AM}^2\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+4(v_1+2)\theta_1 E(\tau^2)-2v_1(v_1+v_2+4)\left(E(\tau^2)\right)^2\right.\right.$$

$$\left.+\left(2\theta_1-E(\tau^2)(v_1+v_2+4)\right)^2\right]+2\phi\left[v_1 v_2 E(\tau^4)-v_2(v_1+v_2+4)\left(E(\tau^2)\right)^2\right.$$

$$\left.-2\theta_2 E(\tau^2)(v_2+4)+2v_2\theta_1 E(\tau^2)+4\theta_1\theta_2\right]+v_2(v_2+2)E(\tau^4)$$

$$\left.+4(v_2+2)\theta_2 E(\tau^2)+4\theta_2^2\right)/(v_1+v_2+4)^2 \ , \tag{A6.35}$$

$$\rho\left(\sigma^2_{e_1},s^2_{PM}\right) = \left(\phi^2(v_1+v_2+4)^2\left[v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(v_1+2)(2-v_1)+4\theta_1^2\right]\right.$$

$$+\int_0^\infty\left(\phi^2(v_2+2)\left[-(2v_1+v_2+6)\left(v_1(v_1+2)\tau^4Q^{d\tau}_{04}+4(v_1+2)\theta_1\tau^2Q^{d\tau}_{06}+4\theta_1^2Q^{d\tau}_{08}\right)\right.\right.$$

$$\left.+2E(\tau^2)(v_1+2)(v_1+v_2+4)\left(v_1\tau^2Q^{d\tau}_{02}+2\theta_1Q^{d\tau}_{04}\right)\right]+2(v_1+2)^2\phi\left[v_1v_2\tau^4Q^{d\tau}_{22}\right.$$

$$-v_2(v_1+v_2+4)E(\tau^2)\tau^2Q^{d\tau}_{20}+2v_1\theta_2\tau^2Q^{d\tau}_{42}-2(v_1+v_2+4)\theta_2E(\tau^2)Q^{d\tau}_{40}+2v_2\theta_1\tau^2Q^{d\tau}_{24}$$

$$\left.\left.+4\theta_1\theta_2Q^{d\tau}_{44}\right]+(v_1+2)^2\left[v_2(v_2+2)\tau^4Q^{d\tau}_{40}+4(v_2+2)\tau^2\theta_2Q^{d\tau}_{60}+4\theta_2^2Q^{d\tau}_{80}\right]\right)$$

$$\left.f(\tau)d\tau\right)\Big/\left((v_1+2)(v_1+v_2+4)\right)^2 . \tag{A6.36}$$

<u>Corollary 6.3.4</u> (Special Cases)

The following special cases are obtained from Corollary 6.3.4 by substituting in the appropriate value of μ.

(ii) L Components (μ=-k):

$$\rho_0\left(\sigma^2_{e_1},s^2_{NL}\right) = \phi^2\left((v_1+2)E(\tau^4)-v_1\left(E(\tau^2)\right)^2\right)/v_1 , \tag{A6.37}$$

$$\rho_0\left(\sigma^2_{e_1},s^2_{AL}\right) = \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2(v_2^2-v_1^2)\right]\right.$$

$$\left.+2\phi v_2\left[v_1E(\tau^4)-(v_1+v_2)\left(E(\tau^2)\right)^2\right]+v_2(v_2+2)E(\tau^4)\right)/(v_1+v_2)^2 , \tag{A6.38}$$

$$\rho_0\left(\sigma^2_{e_1},s^2_{PL}\right) = \left(\phi^2\left[(v_1+2)E(\tau^4)\left((v_1+v_2)^2-v_2(2v_1+v_2)Q_{04}\right)\right.\right.$$

$$\left.+v_1(v_1+v_2)\left(E(\tau^2)\right)^2\left(-(v_1+v_2)+2v_2Q_{02}\right)\right]+2v_1\phi\left[v_1v_2E(\tau^4)Q_{22}\right.$$

$$\left.\left.-v_2(v_1+v_2)\left(E(\tau^2)\right)^2Q_{20}\right]+v_1v_2(v_2+2)E(\tau^4)Q_{40}\right)\Big/\left(v_1(v_1+v_2)^2\right) . \tag{A6.39}$$

(ii) ML Components (μ=0):

$$\rho_0\left(\sigma^2_{e_1}, s^2_{NML}\right) = \phi^2\left(v_1(v_1+2)E(\tau^4)+T_1(T_1-2v_1)\left(E(\tau^2)\right)^2\right. \qquad \text{(A6.40)}$$

$$\begin{aligned}\rho_0\left(\sigma^2_{e_1}, s^2_{AML}\right) = {} & \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+T(T-2v_1)\left(E(\tau^2)\right)^2\right.\right. \\ & \left. +2v_2\phi\left[v_1E(\tau^4)-T\left(E(\tau^2)\right)^2\right]+v_2(v_2+2)E(\tau^4)\right)/T^2 , \end{aligned} \qquad \text{(A6.41)}$$

$$\begin{aligned}\rho_0\left(\sigma^2_{e_1}, s^2_{PML}\right) = {} & \left(\phi^2\left[v_1(v_1+2)E(\tau^4)\left(T^2-T_2(2T_1+T_2)Q_{04}\right)\right.\right. \\ & \left. +TT_1\left(E(\tau^2)\right)^2\left(T(T_1-2v_1)+2T_2v_1Q_{02}\right)\right]+2T_1^2\phi\left[v_1v_2E(\tau^4)Q_{22}\right. \\ & \left.\left. -v_2T\left(E(\tau^2)\right)^2Q_{20}\right]+v_2(v_2+2)T_1^2E(\tau^4)Q_{40}\right)/(T_1T)^2 . \end{aligned} \qquad \text{(A6.42)}$$

(iii) M Components (μ=-k+2):

$$\rho_0\left(\sigma^2_{e_1}, s^2_{NM}\right) = \phi^2\left(v_1E(\tau^4)+(2-v_1)\left(E(\tau^2)\right)^2\right)/(v_1+2) , \qquad \text{(A6.43)}$$

$$\begin{aligned}\rho_0\left(\sigma^2_{e_1}, s^2_{AM}\right) = {} & \left(\phi^2\left[v_1(v_1+2)E(\tau^4)+\left(E(\tau^2)\right)^2\left((v_2+4)^2-v_1^2\right)\right]\right. \\ & \left. +2v_2\phi\left[v_1E(\tau^4)-(v_1+v_2+4)\left(E(\tau^2)\right)^2\right]+v_2(v_2+2)E(\tau^4)\right)/(v_1+v_2+4)^2 , \end{aligned} \qquad \text{(A6.44)}$$

$$\begin{aligned}\rho_0\left(\sigma^2_{e_1}, s^2_{PM}\right) = {} & \left(\phi^2\left[v_1(v_1+2)E(\tau^4)\left((v_1+v_2+4)^2-(v_2+2)(2v_1+v_2+6)Q_{04}\right)\right.\right. \\ & \left. +(v_1+2)(v_1+v_2+4)\left(E(\tau^2)\right)^2\left((2-v_1)(v_1+v_2+4)+2v_1(v_2+2)Q_{02}\right)\right] \\ & +2(v_1+2)^2\phi\left[v_1v_2E(\tau^4)Q_{22}-v_2(v_1+v_2+4)\left(E(\tau^2)\right)^2Q_{20}\right. \\ & \left. +v_2(v_2+2)(v_1+2)^2E(\tau^4)Q_{40}\right)/\left((v_1+2)^2(v_1+v_2+4)^2\right) . \end{aligned} \qquad \text{(A6.45)}$$

Corollary 6.3.5 (Special Cases)

The following special cases are obtained from Corollary 6.3.5 by substituting in the appropriate values of μ.

(i) L Components (μ=-k):

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{NL}\right) = 2\phi^2\sigma^4_2\Big(\nu^2 v_1(v_1+\nu-2)+4\lambda_1\nu(\nu-2)(\nu-4)$$

$$+2\lambda_1^2(\nu-2)^2(\nu-4)\Big)/\Big(v_1^2(\nu-2)^2(\nu-4)\Big) \ , \qquad \text{(A6.46)}$$

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{AL}\right) = \sigma^4_2\Big(\phi^2\Big[v_2^2\nu^2(\nu-4)+2v_1\nu^2(v_1+\nu-2)-4\lambda_1\nu(\nu-2)(\nu-4)(v_2-2)$$

$$+4\lambda_1^2(\nu-2)^2(\nu-4)\Big]+2\phi\Big[v_2\nu^2\Big(2v_1-v_2(\nu-4)\Big)+2v_2\nu(\nu-2)(\nu-4)(\lambda_1-\lambda_2)$$

$$+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)\Big]+v_2(v_2+2)\nu^2(\nu-2)+4(v_2+2)\lambda_2\nu(\nu-2)(\nu-4)$$

$$+4\lambda_2^2(\nu-2)^2(\nu-4)\Big)/\Big((v_1+v_2)^2(\nu-2)^2(\nu-4)\Big) \ , \qquad \text{(A6.47)}$$

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{PL}\right) = \sigma^4_2\Big(2\phi^2(v_1+v_2)^2\Big[v_1\nu^2(v_1+\nu-2)+4\lambda_1\nu(\nu-2)(\nu-4)$$

$$+4\lambda_1^2(\nu-2)^2(\nu-4)\Big]+\phi^2 v_2\Big[-(\nu-2)(2v_1+v_2)\Big(v_1(v_1+2)\nu^2Q^d_{040}$$

$$+4(v_1+2)\nu(\nu-4)Q^d_{061}+4\lambda_1^2(\nu-2)(\nu-4)Q^d_{082}\Big)+2\nu(\nu-4)v_1(v_1+v_2)\Big(v_1\nu Q^d_{021}$$

$$+2\lambda_1(\nu-2)Q^d_{042}\Big)\Big]+2v_1^2\phi\Big[v_1v_2\nu^2(\nu-2)Q^d_{220}-v_2(v_1+v_2)\nu^2(\nu-4)Q^d_{201}$$

$$+2v_1\lambda_2\nu(\nu-2)(\nu-4)Q^d_{421}-2(v_1+v_2)\lambda_2\nu(\nu-2)(\nu-4)Q^d_{402}$$

$$+2v_2\lambda_1\nu(\nu-2)(\nu-4)Q^d_{241}+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)Q^d_{442}\Big]$$

$$+v_1^2(\nu-2)\Big[v_2(v_2+2)\nu^2Q^d_{400}+4(v_2+2)\lambda_2\nu(\nu-4)Q^d_{601}$$

$$+4\lambda_2^2(\nu-2)(\nu-4)Q^d_{802}\Big]\Big)\Big/\Big(v_1^2(v_1+v_2)^2(\nu-2)^2(\nu-4)\Big) \ . \qquad (A6.48)$$

(ii) ML Components (μ=0):

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{NML}\right) = \phi^2\sigma_2^4\Big(2\nu^2v_1(v_1+\nu-2)+\nu^2k^2(\nu-4)+4\lambda_1\nu(\nu-2)(\nu-4)(2-k)$$

$$+4\lambda_1^2(\nu-2)^2(\nu-4)\Big)\Big/\Big(T_1^2(\nu-2)^2(\nu-4)\Big) \ , \qquad (A6.49)$$

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{AML}\right) = \sigma_2^4\Big(\phi^2\Big[\nu^2(\nu-4)(v_2+2k)^2+2\nu^2v_1(v_1+\nu-2)$$

$$-4\lambda_1\nu(\nu-2)(\nu-4)(v_2+2k-2)+4\lambda_1^2(\nu-2)^2(\nu-4)\Big]+2\phi\Big[2v_1v_2\nu^2$$

$$-\nu^2(\nu-4)v_2(v_2+2k)+2\nu(\nu-2)(\nu-4)\Big(v_2\lambda_1-\lambda_2(v_2+2k)\Big)$$

$$+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)\Big]+v_2(v_2+2)\nu^2(\nu-2)$$

$$+4(v_2+2)\lambda_2\nu(\nu-2)(\nu-4)+4\lambda_2^2(\nu-2)^2(\nu-4)\Big)\Big/\Big(T^2(\nu-2)^2(\nu-4)\Big) \ , \qquad (A6.50)$$

$$\rho_{Mt}\left(\sigma^2_{e_1},s^2_{PML}\right) = \sigma_2^4\Big(\phi^2T^2\Big[\nu^2(\nu-4)k^2+2v_1\nu^2(v_1+\nu-2)+4\lambda_1\nu(\nu-2)(\nu-4)(2-k)$$

$$+4\lambda_1^2(\nu-2)^2(\nu-4)\Big]+\phi^2T_2\Big[-(\nu-2)(2T_1+T_2)\Big(v_1(v_1+2)\nu^2Q^d_{040}+4(v_1+2)\lambda_1\nu$$

$$.(\nu-4)Q^d_{061}+4\lambda_1^2(\nu-2)(\nu-4)Q^d_{082}\Big)+2\nu(\nu-4)T_1T\Big(v_1\nu Q^d_{021}+2\lambda_1(\nu-2)Q^d_{042}\Big)\Big]$$

$$+2T_1^2\phi\Big[v_1v_2\nu^2(\nu-2)Q^d_{220}-v_2T\nu^2(\nu-4)Q^d_{201}+2v_1\lambda_2\nu(\nu-2)(\nu-4)Q^d_{421}$$

$$-2T\lambda_2\nu(\nu-2)(\nu-4)Q^d_{402}+2v_2\lambda_1\nu(\nu-2)(\nu-4)Q^d_{241}+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)Q^d_{442}\Big]$$

$$+T_1^2(\nu-2)\Big[v_2(v_2+2)\nu^2Q^d_{400}+4(v_2+2)\lambda_2\nu(\nu-4)Q^d_{601}+4\lambda_2^2(\nu-2)(\nu-4)Q^d_{802}\Big]\Big)$$

$$\Big/\Big((T_1T_2)^2(\nu-2)^2(\nu-4)\Big) \ . \qquad (A6.51)$$

(iii) M Components (μ=-k+2):

$$\rho_{Mt}\left(\sigma_{e_1}^2, s_{NM}^2\right) = 2\phi^2\sigma_2^4\left(\nu^2 v_1(v_1+\nu-2)+2\nu^2(\nu-4)+2\lambda_1^2(\nu-2)^2(\nu-4)\right)/$$

$$\left((v_1+2)^2(\nu-2)^2(\nu-4)\right), \tag{A6.52}$$

$$\rho_{Mt}\left(\sigma_{e_1}^2, s_{AM}^2\right) = \sigma_2^4\left(\phi^2\left[\nu^2(\nu-4)(v_2+4)^2+2\nu^2 v_1(v_1+\nu-2)\right.\right.$$

$$\left.-4\lambda_1\nu(\nu-2)(\nu-4)(v_2+2)+4\lambda_1^2(\nu-2)^2(\nu-4)\right]+2\phi\left[2v_1 v_2\nu^2\right.$$

$$-\nu^2(\nu-4)v_2(v_2+4)+2\nu(\nu-2)(\nu-4)\left(v_2\lambda_1-\lambda_2(v_2+4)\right)$$

$$\left.+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)\right]+v_2(v_2+2)\nu^2(\nu-2)+4(v_2+2)\lambda_2\nu(\nu-2)(\nu-4)$$

$$\left.+4\lambda_2^2(\nu-2)^2(\nu-4)\right)/\left((v_1+v_2+4)^2(\nu-2)^2(\nu-4)\right), \tag{A6.53}$$

$$\rho_{Mt}\left(\sigma_{e_1}^2, s_{PM}^2\right) = \sigma_2^4\left(2\phi^2(v_1+v_2+4)^2\left[2\nu^2(\nu-4)+v_1\nu^2(v_1+\nu-2)\right.\right.$$

$$\left.+2\lambda_1^2(\nu-2)^2(\nu-4)\right]+\phi^2(v_2+2)\left[-(\nu-2)(2v_1+v_2+6)\left(v_1(v_1+2)\nu^2 Q_{040}^d\right.\right.$$

$$\left.+4(v_1+2)\lambda_1\nu(\nu-4)Q_{061}^d+4\lambda_1^2(\nu-2)(\nu-4)Q_{082}^d\right)$$

$$\left.\left.+2\nu(\nu-4)(v_1+2)(v_1+v_2+4)\left(v_1\nu Q_{021}^d+2\lambda_1(\nu-2)Q_{042}^d\right)\right]\right]$$

$$+2(v_1+2)^2\phi\left[v_1 v_2\nu^2(\nu-2)Q_{220}^d-v_2(v_1+v_2+4)\nu^2(\nu-4)Q_{201}^d\right.$$

$$+2v_1\lambda_2\nu(\nu-2)(\nu-4)Q_{421}^d-2(v_1+v_2+4)\lambda_2\nu(\nu-2)(\nu-4)Q_{402}^d$$

$$\left.+2v_2\lambda_1\nu(\nu-2)(\nu-4)Q_{241}^d+4\lambda_1\lambda_2(\nu-2)^2(\nu-4)Q_{442}^d\right]$$

$$+(v_1+2)^2(\nu-2)\left[v_2(v_2+2)\nu^2 Q_{400}^d+4(v_2+2)\lambda_2\nu(\nu-4)Q_{601}^d\right.$$

$$+4\lambda_2^2(\nu-2)(\nu-4)Q_{802}^d\Big]\Big]\Big/\Big((v_1+2)^2(v_1+v_2+4)^2(\nu-2)^2(\nu-4)\Big). \tag{A6.54}$$

<u>Corollary 6.3.6</u> (Special Cases)

The following special cases are obtained from Corollary 6.3.6 by substituting in the appropriate value of μ.

(i) L Components (μ=-k):

$$\rho_N\left(\sigma_1^2, s_{NL}^2\right) = 2\phi^2\sigma_2^4\left(v_1+4\lambda_1+2\lambda_1^2\right)/v_1^2 , \tag{A6.55}$$

$$\rho_N\left(\sigma_1^2, s_{AL}^2\right) = \sigma_2^4\left(\phi^2\left[(2\lambda_1-v_2)^2+2(v_1+4\lambda_1)\right]+2\phi(v_2+2\lambda_2)(2\lambda_1-v_2)\right.$$

$$\left.+v_2(v_2+2)+4(v_2+2)\lambda_2+4\lambda_2^2\right)/(v_1+v_2)^2 , \tag{A6.56}$$

$$\rho_N\left(\sigma_1^2, s_{PL}^2\right) = \sigma_2^4\left(2\phi^2(v_1+v_2)^2(v_1+4\lambda_1+2\lambda_1^2)+\phi^2 v_2\left[-(2v_1+v_2)\right.\right.$$

$$\cdot\left(v_1(v_1+2)Q_{04}^d+4(v_1+2)\lambda_1 Q_{06}^d+4\lambda_1^2 Q_{08}^d\right)+2v_1(v_1+v_2)(v_1 Q_{02}^d+2\lambda_1 Q_{04}^d)\Big]$$

$$+2v_1^2\phi\left[v_1 v_2 Q_{22}^d-v_2(v_1+v_2)Q_{20}^d+2v_1\lambda_2 Q_{42}^d-2(v_1+v_2)\lambda_2 Q_{40}^d\right.$$

$$\left.\left.+2v_2\lambda_1 Q_{24}^d+4\lambda_1\lambda_2 Q_{44}^d\right]+v_1^2\left[v_2(v_2+2)Q_{40}^d+4(v_2+2)\lambda_2 Q_{60}^d+4\lambda_2^2 Q_{80}^d\right]\right)$$

$$\Big/\left(v_1^2(v_1+v_2)^2\right) . \tag{A6.57}$$

(ii) ML Components (μ=0):

$$\rho_N\left(\sigma_1^2, s_{NML}^2\right) = \phi^2\sigma_2^4\left(2(v_1+4\lambda_1)+(2\lambda_1-k)^2\right)/T_1^2 , \tag{A6.58}$$

$$\rho_N\left(\sigma_1^2, s_{AML}^2\right) = \sigma_2^4\left(\phi^2\left[(2\lambda_1-v_2-2k)^2+2(v_1+4\lambda_1)\right]+2\phi(v_2+2\lambda_2)(2\lambda_1\right.$$

$$\left.-v_2-2k)+v_2(v_2+2)+4(v_2+2)\lambda_2+4\lambda_2^2\right)/T^2 , \tag{A6.59}$$

$$\rho_N\left(\sigma_1^2,s_{PML}^2\right) = \sigma_2^4\left(\phi^2T^2\left[k^2+2v_1+4\lambda_1(2-k)+4\lambda_1^2\right]+\phi^2T_2\left[-(2T_1+T_2)\right.\right.$$

$$\left.\cdot\left(v_1(v_1+2)Q_{04}^d+4(v_1+2)\lambda_1Q_{06}^d+4\lambda_1^2Q_{08}^d\right)+2T_1T\left(v_1Q_{02}^d+2\lambda_1Q_{04}^d\right)\right]$$

$$+2T_1^2\phi\left[v_1v_2Q_{22}^d-v_2TQ_{20}^d+2v_1\lambda_2Q_{42}^d-2T\lambda_2Q_{40}^d+2v_2\lambda_1Q_{24}^d+4\lambda_1\lambda_2Q_{44}^d\right]$$

$$\left.+T_1^2\left[v_2(v_2+2)Q_{40}^d+4(v_2+2)\lambda_2Q_{60}^d+4\lambda_2^2Q_{80}^d\right]\right)/(T_1T)^2 \ . \tag{A6.60}$$

(iii) M Components (μ=-k+2):

$$\rho_N\left(\sigma_1^2,s_{NM}^2\right) = 2\phi^2\sigma_2^4\left[(v_1+4\lambda_1)+2(\lambda_1-1)^2\right]/(v_1+2)^2 \ , \tag{A6.61}$$

$$\rho_N\left(\sigma_1^2,s_{AM}^2\right) = \sigma_2^4\left(\phi^2\left[(2\lambda_1-v_2-4)^2+2(v_1+4\lambda_1)\right]+2\phi(v_2+2\lambda_2)(2\lambda_1-v_2-4)\right.$$

$$\left.+v_2(v_2+2)+4(v_2+2)\lambda_2+4\lambda_2^2\right)/(v_1+v_2+4)^2 \ , \tag{A6.62}$$

$$\rho_N\left(\sigma_1^2,s_{PM}^2\right) = \sigma_2^4\left(2\phi^2(v_1+v_2+4)^2(v_1+2+2\lambda_1^2)+\phi^2(v_2+2)\left[-(2v_1+v_2+6)\left(v_1\right.\right.\right.$$

$$\left.\left.\cdot(v_1+2)Q_{04}^d+4(v_1+2)\lambda_1Q_{06}^d+4\lambda_1^2Q_{08}^d\right)+2(v_1+2)(v_1+v_2+4)\left(v_1Q_{02}^d+2\lambda_1Q_{04}^d\right)\right]$$

$$+2(v_1+2)^2\phi\left[v_1v_2Q_{22}^d-v_2(v_1+v_2+4)Q_{20}^d+2v_1\lambda_2Q_{42}^d-2(v_1+v_2+4)\lambda_2Q_{40}^d\right.$$

$$\left.+2v_2\lambda_1Q_{24}^d+4\lambda_1\lambda_2Q_{44}^d\right]+(v_1+2)^2\left[v_2(v_2+2)Q_{40}^d+4(v_2+2)\lambda_2Q_{60}^d\right.$$

$$\left.\left.+4\lambda_2^2Q_{80}^d\right]\right)/\left((v_1+2)^2(v_1+v_2+4)^2\right) \ . \tag{A6.63}$$

APPENDIX 6.2

TABLES OF RELATIVE RISKS OF s^2_{Nj}, s^2_{Aj}, AND s^2_{Pj}, j = L, ML, AND M.

In this Appendix we give a small sample of the numerical evaluations of the relative risks of s^2_{Nj}, s^2_{Aj}, and s^2_{Pj} (α=0.01, 0.05, 0.30, and 0.75), j=L, ML, and M. We also consider that value of α associated with c=1 for $s^2_{PL}(c^*_L)$; with $\left(v_1T_2/(v_2T_1)\right)$ for $s^2_{PML}(c^*_{ML})$ and, with $\left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right)$ for $s^2_{PM}(c^*_M)$. These critical values result in a minimum of the risk functions of s^2_{PL}, s^2_{PML}, and s^2_{PM} .

The relative risks of s^2_{NL}, s^2_{AL}, and of s^2_{PL} are given in Table A6.2.1. Tables A6.2.2 and A6.2.3 present the relative risks of s^2_{NML}, s^2_{AML}, and of s^2_{PML}, and of s^2_{NM}, s^2_{AM}, and of s^2_{PM}. In each case we consider risk as a function of ϕ for given values of λ_1 and λ_2. We recall that ϕ is the ratio of the error variances $(\sigma^2_{e_1}/\sigma^2_{e_2})$ and is equal to unity when the error variances are identical. λ_1 and λ_2 are measures of the specification error in the first and the second samples, respectively. If λ_1=λ_2=0 then there are no omitted regressors in either sample. Note that ϕ=1 implies equal error variances even if the models are mis-specified.

In these tables we consider λ_1=0, 3, 10; λ_2=0, 3, 10; ϕ= [0,0.2(0.05);0.2,0.7(0.1);0.7,1.0(0.05)] and ν=5, 10, 100, ∞. For each of the possible values of λ_1, λ_2, ϕ and ν, the tables give the relative risks of the estimators for v_1=16, v_2=8, and k=3.

Tables A6.2.4, A6.2.5, and A6.2.6, present the relative risks of s^2_{Nj}, s^2_{Aj}, and s^2_{Pj}, j = L, ML, and M, respectively, for given values of ϕ and λ_2, as a function of λ_1. v_1=16, v_2=8, ν=5, 10, ∞; λ_2= 0, 3; ϕ=1.0, 0.5, 0.1; and λ_1=[0,4(0.5);4,10(1.0)] are considered. Conversely, Tables A6.2.7, A6.2.8, and A6.2.9 give the relative risks of s^2_{Nj}, s^2_{Aj}, s^2_{Pj}, j = L, ML, and M, respectively, for given values of ϕ and λ_1 as a function of λ_2. We consider v_1=16, v_2=8; ν=5, 10, ∞; λ_1=0,3 ; ϕ=1.0, 0.5, 0.1; and λ_2=[0,4(0.5);4,10(1.0)].

Due to space constraints we have omitted legends from the tables. The relative risks are presented in the following estimator order.

L Component Tables	ML Component Tables	M Component Tables
s^2_{NL},	s^2_{NML}	s^2_{NM}
s^2_{AL},	s^2_{AML}	s^2_{AM}
s^2_{PL}: α = 0.01	s^2_{PML}: α = 0.01	s^2_{PM}: α = 0.01
s^2_{PL}: α = 0.05	s^2_{PML}: α = 0.05	s^2_{PM}: α = 0.05
s^2_{PL}: α = 0.30	s^2_{PML}: α = 0.30	s^2_{PM}: α = 0.30
s^2_{PL}: α = 0.75	s^2_{PML}: α = 0.75	s^2_{PM}: α = 0.75
s^2_{PL}: c = 1	s^2_{PML}: $c=v_1T_2/(v_2T_1)$	s^2_{PM}: $c=v_1(v_2+2)/(v_2(v_1+2))$

The following values of α correspond to a nominal critical value of c^*_L, c^*_{ML}, and of c^*_M when v_1=16, v_2=8, T_1=19 and T_2=11:

	$\underline{\alpha}$
$c^*_L = 1$	0.473
$c^*_{ML} = (v_1T_2)/(v_2T_1) = 1.1579$	0.380
$c^*_M = \left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right) = 1.1111$	0.406 .

TABLE A6.2.1: Relative risks of s^2_{NL}, s^2_{AL}, and s^2_{PL}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	0.05	0.10	0.15	0.20	0.30	0.40	ϕ 0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0164	0.0661	0.1483	0.2639	0.5939	1.0556	1.6494	2.3750	3.2328	3.7108	4.2222	4.7664	5.3439	5.9539	6.5972
	1.2581	1.3750	1.5081	1.6575	2.0047	2.4167	2.8936	3.4353	4.0417	4.3692	4.7131	5.0731	5.3250	5.8414	6.2500
	0.0172	0.0847	0.2328	0.4656	1.0947	1.7944	2.4906	3.1800	3.8811	4.2419	4.6117	4.9922	5.3847	5.7900	6.2086
$\nu=5, \lambda_1=0, \lambda_2=0$	0.0167	0.0689	0.1658	0.3147	0.7667	1.3725	2.0650	2.8031	3.5706	3.9647	4.3667	4.7769	5.1964	5.6258	6.0661
	0.0164	0.0658	0.1483	0.2633	0.5886	1.0350	1.5911	2.2453	2.9853	3.3839	3.8006	4.2336	4.6822	5.1461	5.6242
	0.0164	0.0658	0.1483	0.2633	0.5897	1.0386	1.6003	2.2639	3.0169	3.4239	3.8492	4.2917	4.7508	5.2253	5.7144
	0.0164	0.0658	0.1483	0.2628	0.5856	1.0244	1.5667	2.1992	2.9114	3.2944	3.6942	4.1111	4.5411	4.9872	5.4481
	0.0020	0.0078	0.0177	0.0313	0.0703	0.1250	0.1953	0.2813	0.3828	0.4395	0.5000	0.5645	0.6328	0.7052	0.7813
	0.2848	0.2830	0.2838	0.2870	0.3016	0.3264	0.3617	0.4073	0.4636	0.4955	0.5302	0.5673	0.6070	0.6494	0.6944
	0.0020	0.0113	0.0325	0.0659	0.1503	0.2344	0.3095	0.3792	0.4491	0.4855	0.5233	0.5628	0.6042	0.6480	0.6938
$\nu=10, \lambda_1=0, \lambda_2=0$	0.0020	0.0083	0.0206	0.0397	0.0977	0.1719	0.2522	0.3339	0.4166	0.4586	0.5016	0.5455	0.5908	0.6377	0.6863
	0.0020	0.0078	0.0175	0.0311	0.0695	0.1219	0.1869	0.2628	0.3483	0.3942	0.4422	0.4922	0.5439	0.5975	0.6528
	0.0020	0.0078	0.0175	0.0313	0.0697	0.1225	0.1883	0.2655	0.3527	0.3995	0.4486	0.4997	0.5527	0.6073	0.6639
	0.0020	0.0078	0.0175	0.0311	0.0691	0.1205	0.1833	0.2564	0.3384	0.3825	0.4286	0.4766	0.5264	0.5783	0.6320
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0248	0.0386	0.0556	0.0757	0.0869	0.0989	0.1116	0.1252	0.1395	0.1545
	0.1370	0.1273	0.1185	0.1107	0.0977	0.0884	0.0829	0.0810	0.0828	0.0851	0.0882	0.0924	0.0975	0.1034	0.1103
	0.0004	0.0029	0.0094	0.0194	0.0421	0.0594	0.0699	0.0766	0.0827	0.0860	0.0900	0.0944	0.0998	0.1058	0.1127
$\nu=100, \lambda_1=0, \lambda_2=0$	0.0004	0.0018	0.0046	0.0094	0.0233	0.0396	0.0545	0.0673	0.0787	0.0842	0.0900	0.0959	0.1024	0.1092	0.1166
	0.0004	0.0016	0.0034	0.0061	0.0136	0.0237	0.0360	0.0502	0.0658	0.0741	0.0829	0.0919	0.1013	0.1111	0.1213
	0.0004	0.0016	0.0034	0.0061	0.0137	0.0239	0.0364	0.0509	0.0670	0.0756	0.0845	0.0938	0.1034	0.1134	0.1238
	0.0004	0.0016	0.0034	0.0061	0.0135	0.0233	0.0350	0.0483	0.0632	0.0711	0.0796	0.0883	0.0975	0.1070	0.1171
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0313	0.0450	0.0613	0.0703	0.0800	0.0903	0.1013	0.1128	0.1250
	0.1282	0.1183	0.1093	0.1011	0.0872	0.0767	0.0694	0.0656	0.0650	0.0660	0.0678	0.0704	0.0739	0.0782	0.0833
	0.0004	0.0025	0.0082	0.0171	0.0367	0.0508	0.0584	0.0623	0.0655	0.0674	0.0698	0.0728	0.0764	0.0807	0.0857
$\nu=\infty, \lambda_1=0, \lambda_2=0$	0.0003	0.0014	0.0039	0.0079	0.0197	0.0332	0.0451	0.0547	0.0629	0.0667	0.0707	0.0750	0.0796	0.0846	0.0902
	0.0003	0.0012	0.0028	0.0050	0.0110	0.0191	0.0290	0.0401	0.0525	0.0591	0.0660	0.0732	0.0806	0.0884	0.0965
	0.0003	0.0012	0.0028	0.0050	0.0111	0.0193	0.0293	0.0408	0.0536	0.0604	0.0675	0.0747	0.0824	0.0903	0.0985
	0.0003	0.0012	0.0028	0.0049	0.0109	0.0187	0.0280	0.0386	0.0503	0.0566	0.0632	0.0701	0.0774	0.0850	0.0931
	0.0172	0.0689	0.1550	0.2758	0.6206	1.1031	1.7236	2.4819	3.3781	3.8781	4.4122	4.9811	5.5842	6.2219	6.8942
	1.2717	1.4014	1.5464	1.7072	2.0747	2.5044	2.9958	3.5494	4.1647	4.4956	4.8419	5.2036	5.5808	5.9736	6.3819
	0.0183	0.0931	0.2578	0.5119	1.1775	1.8981	2.6047	3.3011	4.0075	4.3700	4.7414	5.1231	5.5161	5.9214	6.3394
$\nu=5, \lambda_1=3, \lambda_2=0$	0.0175	0.0728	0.1769	0.3375	0.8197	1.4536	2.1667	2.9175	3.6931	4.0900	4.4936	4.9050	5.3247	5.7542	6.1936
	0.0172	0.0689	0.1550	0.2750	0.6139	1.0769	1.6514	2.3228	3.0781	3.4833	3.9056	4.3433	4.7958	5.2625	5.7425
	0.0172	0.0689	0.1550	0.2750	0.6153	1.0814	1.6625	2.3450	3.1156	3.5300	3.9619	4.4103	4.8742	5.3525	5.8444
	0.0172	0.0689	0.1547	0.2744	0.6100	1.0642	1.6219	2.2689	2.9928	3.3806	3.7842	4.2031	4.6367	5.0844	5.5461
	0.0027	0.0105	0.0234	0.0416	0.0936	0.1663	0.2598	0.3741	0.5092	0.5845	0.6650	0.7508	0.8417	0.9378	1.0391
	0.2950	0.3028	0.3128	0.3250	0.3556	0.3947	0.4423	0.4986	0.5634	0.5991	0.6367	0.6766	0.7186	0.7628	0.8091
	0.0030	0.0183	0.0531	0.1030	0.2139	0.3125	0.3964	0.4731	0.5492	0.5884	0.6288	0.6706	0.7141	0.7594	0.8064
$\nu=10, \lambda_1=3, \lambda_2=0$	0.0027	0.0117	0.0302	0.0591	0.1406	0.2359	0.3316	0.4236	0.5134	0.5584	0.6039	0.6500	0.6970	0.7453	0.7948
	0.0027	0.0103	0.0233	0.0413	0.0914	0.1581	0.2381	0.3288	0.4272	0.4788	0.5317	0.5859	0.6414	0.6980	0.7558
	0.0027	0.0103	0.0233	0.0414	0.0917	0.1594	0.2413	0.3344	0.4361	0.4894	0.5444	0.6005	0.6578	0.7164	0.7759
	0.0027	0.0103	0.0233	0.0411	0.0902	0.1545	0.2306	0.3158	0.4081	0.4567	0.5067	0.5580	0.6108	0.6648	0.7203
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0626	0.0977	0.1407	0.1915	0.2198	0.2501	0.2824	0.3165	0.3528	0.3909
	0.1454	0.1437	0.1425	0.1420	0.1429	0.1461	0.1516	0.1596	0.1699	0.1760	0.1826	0.1899	0.1977	0.2063	0.2153
	0.0012	0.0094	0.0275	0.0512	0.0948	0.1235	0.1417	0.1556	0.1685	0.1752	0.1823	0.1898	0.1978	0.2064	0.2155
$\nu=100, \lambda_1=3, \lambda_2=0$	0.0010	0.0049	0.0134	0.0267	0.0608	0.0942	0.1214	0.1431	0.1616	0.1702	0.1789	0.1875	0.1965	0.2056	0.2152
	0.0009	0.0039	0.0087	0.0154	0.0335	0.0565	0.0825	0.1096	0.1368	0.1504	0.1637	0.1768	0.1899	0.2028	0.2157
	0.0009	0.0040	0.0087	0.0154	0.0338	0.0575	0.0844	0.1131	0.1421	0.1565	0.1707	0.1846	0.1985	0.2121	0.2256
	0.0009	0.0039	0.0087	0.0153	0.0327	0.0541	0.0778	0.1020	0.1265	0.1388	0.1510	0.1632	0.1753	0.1877	0.2002
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0575	0.0898	0.1294	0.1761	0.2021	0.2300	0.2596	0.2911	0.3243	0.3594
	0.1364	0.1344	0.1329	0.1319	0.1316	0.1333	0.1372	0.1431	0.1510	0.1558	0.1611	0.1669	0.1733	0.1801	0.1875
	0.0012	0.0089	0.0261	0.0484	0.0884	0.1137	0.1289	0.1399	0.1500	0.1554	0.1610	0.1670	0.1735	0.1804	0.1878
$\nu=\infty, \lambda_1=3, \lambda_2=0$	0.0009	0.0046	0.0125	0.0250	0.0567	0.0870	0.1109	0.1293	0.1445	0.1515	0.1585	0.1655	0.1727	0.1802	0.1879
	0.0009	0.0036	0.0080	0.0142	0.0307	0.0517	0.0749	0.0991	0.1229	0.1346	0.1461	0.1573	0.1684	0.1793	0.1901
	0.0009	0.0036	0.0080	0.0142	0.0310	0.0525	0.0769	0.1024	0.1279	0.1404	0.1527	0.1648	0.1765	0.1881	0.1994
	0.0009	0.0036	0.0080	0.0140	0.0299	0.0493	0.0704	0.0918	0.1131	0.1236	0.1340	0.1444	0.1548	0.1652	0.1757
	0.0217	0.0869	0.1953	0.3472	0.7814	1.3889	2.1703	3.1250	4.2536	4.8828	5.5556	6.2717	7.0314	7.8342	8.6806
	1.3044	1.4675	1.6469	1.8425	2.2825	2.7869	3.3564	3.9908	4.6897	5.0636	5.4536	5.8600	6.2825	6.7211	7.1758
	0.0250	0.1353	0.3625	0.6833	1.4528	2.2447	3.0142	3.7764	4.5550	4.9564	5.3681	5.7911	6.2269	6.6764	7.1397
$\nu=5, \lambda_1=10, \lambda_2=0$	0.0222	0.0964	0.2378	0.4500	1.0472	1.7844	2.5844	3.4133	4.2647	4.6997	5.1425	5.5939	6.0550	6.5278	7.0094
	0.0217	0.0867	0.1947	0.3444	0.7628	1.3231	2.0036	2.7833	3.6458	4.1036	4.5772	5.0658	5.5686	6.0844	6.6136
	0.0217	0.0867	0.1947	0.3450	0.7661	1.3328	2.0247	2.8217	3.7336	4.1756	4.6622	5.1639	5.6803	6.2100	6.7525
	0.0217	0.0867	0.1942	0.3431	0.7542	1.2983	1.9528	2.6983	3.5206	3.9572	4.4094	4.8769	5.3589	5.8550	6.3650
	0.0069	0.0273	0.0616	0.1094	0.2461	0.4375	0.6836	0.9844	1.3398	1.5381	1.7500	1.9756	2.2148	2.4678	2.7344
	0.3200	0.3542	0.3917	0.4328	0.5255	0.6319	0.7523	0.8866	1.0347	1.1141	1.1967	1.2830	1.3727	1.4658	1.5625
	0.0083	0.0414	0.1045	0.1855	0.3505	0.4966	0.6322	0.7689	0.9139	0.9903	1.0698	1.1525	1.2383	1.3273	1.4198
$\nu=10, \lambda_1=10, \lambda_2=0$	0.0072	0.0347	0.0880	0.1616	0.3338	0.5073	0.6738	0.8373	1.0036	1.0892	1.1770	1.2670	1.3598	1.4555	1.5541
	0.0069	0.0273	0.0609	0.1067	0.2295	0.3830	0.5559	0.7408	0.9336	1.0323	1.1327	1.2345	1.3380	1.4431	1.5502
	0.0069	0.0273	0.0609	0.1072	0.2320	0.3897	0.5688	0.7609	0.9613	1.0634	1.1672	1.2720	1.3781	1.4858	1.5948
	0.0069	0.0272	0.0605	0.1053	0.2233	0.3678	0.5292	0.7023	0.8848	0.9792	1.0758	1.1744	1.2755	1.3788	1.4845
	0.0051	0.0204	0.0458	0.0814	0.1833	0.3258	0.5090	0.7329	0.9976	1.1453	1.3030	1.4709	1.6491	1.8375	2.0359
	0.1660	0.1867	0.2096	0.2348	0.2920	0.3583	0.4336	0.5180	0.6115	0.6617	0.7141	0.7687	0.8257	0.8849	0.9465
	0.0078	0.0488	0.1132	0.1753	0.2726	0.3521	0.4316	0.5173	0.6113	0.6615	0.7140	0.7687	0.8257	0.8849	0.9464
$\nu=100, \lambda_1=10, \lambda_2=0$	0.0055	0.0276	0.0697	0.1236	0.2335	0.3309	0.4213	0.5124	0.6090	0.6599	0.7129	0.7680	0.8252	0.8845	0.9462
	0.0051	0.0203	0.0451	0.0786	0.1659	0.2696	0.3804	0.4934	0.6076	0.6652	0.7236	0.7827	0.8429	0.9043	0.9671
	0.0051	0.0203	0.0453	0.0791	0.1684	0.2760	0.3920	0.5105	0.6292	0.6887	0.7483	0.8084	0.8692	0.9310	0.9938
	0.0051	0.0202	0.0447	0.0773	0.1596	0.2555	0.3579	0.4640	0.5737	0.6303	0.6880	0.7472	0.8079	0.8702	0.9343
	0.0050	0.0200	0.0450	0.0800	0.1800	0.3200	0.5000	0.7200	0.9800	1.1250	1.2800	1.4450	1.6200	1.8050	2.0000
	0.1567	0.1767	0.1989	0.2233	0.2789	0.3433	0.4167	0.4989	0.5900	0.6389	0.6900	0.7433	0.7989	0.8567	0.9167
	0.0077	0.0483	0.1111	0.1709	0.2629	0.3387	0.4152	0.4985	0.5898	0.6388	0.6899	0.7433	0.7989	0.8566	0.9167
$\nu=\infty, \lambda_1=10, \lambda_2=0$	0.0054	0.0272	0.0687	0.1214	0.2276	0.3205	0.4069	0.4947	0.5882	0.6378	0.6892	0.7427	0.7987	0.8565	0.9165
	0.0050	0.0199	0.0443	0.0772	0.1625	0.2636	0.3710	0.4803	0.5903	0.6459	0.7021	0.7592	0.8174	0.8767	0.9376
	0.0050	0.0199	0.0445	0.0777	0.1650	0.2700	0.3827	0.4972	0.6115	0.6688	0.7262	0.7841	0.8427	0.9023	0.9629
	0.0050	0.0199	0.0439	0.0758	0.1563	0.2496	0.3488	0.4514	0.5576	0.6122	0.6682	0.7255	0.7843	0.8448	0.9069

TABLE A6.2.1 (continued)

							ϕ								
Estimator	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0164	0.0661	0.1483	0.2639	0.5939	1.0556	1.6494	2.3750	3.2328	3.7108	4.2222	4.7664	5.3439	5.9539	6.5972
	1.6539	1.7569	1.8761	2.0117	2.3311	2.7153	3.1644	3.6783	4.2569	4.5706	4.9006	5.2467	5.6089	5.9872	6.3819
	0.0172	0.0819	0.2242	0.4550	1.1092	1.8667	2.6231	3.3567	4.0808	4.4456	4.8156	5.1928	5.5792	5.9756	6.3839
$\nu=5, \lambda_1=0, \lambda_2=3$	0.0167	0.0683	0.1633	0.3097	0.7614	1.3842	2.1100	2.8881	3.6928	4.1025	4.5172	4.9372	5.3636	5.7969	6.2383
	0.0164	0.0658	0.1483	0.2633	0.5894	1.0381	1.5989	2.2608	3.0117	3.4169	3.8403	4.2806	4.7372	5.2086	5.6947
	0.0164	0.0658	0.1483	0.2636	0.5903	1.0411	1.6072	2.2778	3.0414	3.4547	3.8869	4.3372	4.8042	5.2869	5.7847
	0.0164	0.0658	0.1483	0.2631	0.5869	1.0286	1.5764	2.2175	2.9403	3.3292	3.7350	4.1569	4.5942	5.0464	5.5128
	0.0020	0.0078	0.0177	0.0313	0.0703	0.1250	0.1953	0.2813	0.3828	0.4395	0.5000	0.5645	0.6328	0.7052	0.7813
	0.5973	0.5850	0.5753	0.5683	0.5619	0.5659	0.5805	0.6053	0.6406	0.6622	0.6864	0.7131	0.7425	0.7744	0.8091
	0.0020	0.0095	0.0272	0.0581	0.1556	0.2753	0.3914	0.4939	0.5833	0.6244	0.6641	0.7030	0.7417	0.7808	0.8205
$\nu=10, \lambda_1=0, \lambda_2=3$	0.0020	0.0081	0.0192	0.0366	0.0933	0.1763	0.2759	0.3825	0.4894	0.5420	0.5938	0.6448	0.6955	0.7458	0.7959
	0.0020	0.0078	0.0175	0.0313	0.0700	0.1234	0.1909	0.2708	0.3620	0.4114	0.4630	0.5169	0.5727	0.6303	0.6898
	0.0020	0.0078	0.0175	0.0313	0.0700	0.1238	0.1917	0.2725	0.3652	0.4155	0.4681	0.5233	0.5803	0.6395	0.7006
	0.0020	0.0078	0.0175	0.0313	0.0697	0.1227	0.1886	0.2661	0.3538	0.4009	0.4503	0.5016	0.5547	0.6095	0.6661
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0248	0.0386	0.0556	0.0757	0.0869	0.0989	0.1116	0.1252	0.1395	0.1545
	0.4036	0.3854	0.3681	0.3517	0.3217	0.2955	0.2729	0.2540	0.2388	0.2326	0.2273	0.2229	0.2194	0.2169	0.2153
	0.0004	0.0019	0.0057	0.0134	0.0427	0.0835	0.1239	0.1567	0.1800	0.1886	0.1953	0.2009	0.2052	0.2090	0.2122
$\nu=100, \lambda_1=0, \lambda_2=3$	0.0004	0.0016	0.0037	0.0073	0.0198	0.0404	0.0671	0.0960	0.1243	0.1375	0.1499	0.1614	0.1721	0.1821	0.1914
	0.0004	0.0016	0.0034	0.0061	0.0138	0.0246	0.0381	0.0544	0.0729	0.0831	0.0937	0.1047	0.1163	0.1282	0.1405
	0.0004	0.0016	0.0034	0.0061	0.0138	0.0246	0.0382	0.0546	0.0734	0.0837	0.0945	0.1059	0.1178	0.1299	0.1426
	0.0004	0.0016	0.0034	0.0061	0.0138	0.0245	0.0377	0.0535	0.0713	0.0810	0.0911	0.1015	0.1123	0.1235	0.1350
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0313	0.0450	0.0613	0.0703	0.0800	0.0903	0.1013	0.1128	0.1250
	0.3907	0.3725	0.3551	0.3386	0.3081	0.2808	0.2569	0.2364	0.2192	0.2118	0.2053	0.1996	0.1947	0.1907	0.1875
	0.0003	0.0015	0.0047	0.0113	0.0369	0.0734	0.1099	0.1393	0.1596	0.1667	0.1721	0.1761	0.1791	0.1813	0.1829
$\nu=\infty, \lambda_1=0, \lambda_2=3$	0.0003	0.0013	0.0030	0.0059	0.0163	0.0337	0.0566	0.0817	0.1060	0.1173	0.1278	0.1375	0.1463	0.1544	0.1618
	0.0003	0.0013	0.0028	0.0050	0.0112	0.0199	0.0309	0.0440	0.0591	0.0673	0.0760	0.0850	0.0944	0.1041	0.1140
	0.0003	0.0013	0.0028	0.0050	0.0112	0.0199	0.0309	0.0442	0.0595	0.0679	0.0768	0.0859	0.0955	0.1056	0.1158
	0.0003	0.0012	0.0028	0.0050	0.0112	0.0198	0.0306	0.0433	0.0579	0.0656	0.0739	0.0824	0.0912	0.1003	0.1096
	0.0172	0.0689	0.1550	0.2758	0.6206	1.1031	1.7236	2.4819	3.3781	3.8781	4.4122	4.9811	5.5842	6.2219	6.8942
	1.6736	1.7958	1.9333	2.0864	2.4386	2.8531	3.3292	3.8675	4.4675	4.7906	5.1294	5.4836	5.8531	6.2383	6.6389
	0.0181	0.0881	0.2456	0.5006	1.2089	2.0033	2.7778	3.5203	4.2517	4.6208	4.9958	5.3786	5.7706	6.1731	6.5875
$\nu=5, \lambda_1=3, \lambda_2=3$	0.0172	0.0719	0.1731	0.3303	0.8153	1.4764	2.2331	3.0314	3.8478	4.2611	4.6783	5.1006	5.5289	5.9642	6.4075
	0.0172	0.0689	0.1550	0.2753	0.6153	1.0819	1.6636	2.3467	3.1172	3.5317	3.9631	4.4108	4.8739	5.3508	5.8411
	0.0172	0.0689	0.1550	0.2753	0.6164	1.0858	1.6733	2.3667	3.1519	3.5756	4.0172	4.4761	4.9508	5.4403	5.9436
	0.0172	0.0689	0.1550	0.2747	0.6122	1.0711	1.6372	2.2961	3.0347	3.4306	3.8428	4.2700	4.7117	5.1672	5.6364
	0.0027	0.0105	0.0234	0.0416	0.0936	0.1663	0.2598	0.3741	0.5092	0.5845	0.6650	0.7508	0.8417	0.9378	1.0391
	0.6138	0.6175	0.6233	0.6313	0.6534	0.6844	0.7236	0.7716	0.8280	0.8594	0.8930	0.9288	0.9666	1.0066	1.0486
	0.0027	0.0141	0.0427	0.0931	0.2411	0.4036	0.5466	0.6650	0.7659	0.8127	0.8580	0.9027	0.9475	0.9927	1.0388
$\nu=10, \lambda_1=3, \lambda_2=3$	0.0027	0.0109	0.0267	0.0525	0.1372	0.2564	0.3909	0.5256	0.6533	0.7139	0.7728	0.8302	0.8864	0.9417	0.9969
	0.0027	0.0103	0.0234	0.0414	0.0928	0.1630	0.2503	0.3520	0.4656	0.5261	0.5886	0.6528	0.7186	0.7858	0.8542
	0.0027	0.0103	0.0234	0.0416	0.0930	0.1636	0.2519	0.3556	0.4720	0.5342	0.5988	0.6653	0.7336	0.8034	0.8745
	0.0027	0.0103	0.0233	0.0414	0.0922	0.1611	0.2455	0.3425	0.4495	0.5061	0.5644	0.6242	0.6855	0.7478	0.8114
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0626	0.0977	0.1407	0.1915	0.2198	0.2501	0.2824	0.3165	0.3528	0.3909
	0.4182	0.4142	0.4109	0.4082	0.4044	0.4031	0.4042	0.4076	0.4135	0.4173	0.4217	0.4267	0.4323	0.4385	0.4453
	0.0010	0.0055	0.0180	0.0424	0.1198	0.2067	0.2793	0.3325	0.3700	0.3846	0.3975	0.4090	0.4195	0.4294	0.4390
$\nu=100, \lambda_1=3, \lambda_2=3$	0.0009	0.0041	0.0102	0.0206	0.0578	0.1134	0.1770	0.2385	0.2930	0.3171	0.3390	0.3591	0.3776	0.3946	0.4106
	0.0009	0.0040	0.0087	0.0156	0.0350	0.0614	0.0944	0.1327	0.1749	0.1972	0.2200	0.2432	0.2668	0.2904	0.3141
	0.0009	0.0040	0.0087	0.0156	0.0350	0.0617	0.0951	0.1341	0.1777	0.2009	0.2247	0.2491	0.2738	0.2988	0.3240
	0.0009	0.0040	0.0087	0.0156	0.0348	0.0607	0.0924	0.1284	0.1674	0.1877	0.2082	0.2291	0.2500	0.2709	0.2919
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0575	0.0898	0.1294	0.1761	0.2021	0.2300	0.2596	0.2911	0.3243	0.3594
	0.4051	0.4010	0.3975	0.3944	0.3899	0.3875	0.3872	0.3889	0.3927	0.3954	0.3986	0.4023	0.4066	0.4114	0.4167
	0.0009	0.0051	0.0168	0.0397	0.1133	0.1962	0.2653	0.3155	0.3502	0.3636	0.3749	0.3850	0.3941	0.4025	0.4105
$\nu=\infty, \lambda_1=3, \lambda_2=3$	0.0009	0.0038	0.0094	0.0191	0.0538	0.1061	0.1661	0.2242	0.2752	0.2976	0.3179	0.3363	0.3531	0.3684	0.3827
	0.0009	0.0036	0.0081	0.0143	0.0321	0.0565	0.0869	0.1221	0.1610	0.1814	0.2024	0.2237	0.2452	0.2668	0.2884
	0.0009	0.0036	0.0081	0.0144	0.0322	0.0568	0.0875	0.1234	0.1635	0.1848	0.2068	0.2291	0.2519	0.2748	0.2979
	0.0009	0.0036	0.0081	0.0143	0.0320	0.0559	0.0850	0.1181	0.1540	0.1725	0.1913	0.2103	0.2293	0.2483	0.2674
	0.0217	0.0869	0.1953	0.3472	0.7814	1.3889	2.1703	3.1250	4.2536	4.8828	5.5556	6.2717	7.0314	7.8342	8.6806
	1.7211	1.8911	2.0775	2.2800	2.7339	3.2522	3.8356	4.4839	5.1967	5.5775	5.9744	6.3878	6.8172	7.2628	7.7244
	0.0233	0.1217	0.3481	0.6992	1.5892	2.5044	3.3661	4.1919	5.0147	5.4331	5.8594	6.2958	6.7433	7.2033	7.6767
$\nu=5, \lambda_1=10, \lambda_2=3$	0.0219	0.0925	0.2275	0.4394	1.0778	1.8986	2.7917	3.7050	4.6256	5.0900	5.5586	6.0333	6.5147	7.0047	7.5044
	0.0217	0.0867	0.1950	0.3461	0.7719	1.3508	2.0633	2.8883	3.8050	4.2922	4.7967	5.3161	5.8500	6.3969	6.9561
	0.0217	0.0867	0.1950	0.3464	0.7736	1.3575	2.0803	2.9217	3.8614	4.3622	4.8814	5.4167	5.9669	6.5308	7.1072
	0.0217	0.0867	0.1950	0.3453	0.7664	1.3317	2.0192	2.8069	3.6772	4.1389	4.6167	5.1094	5.6164	6.1372	6.6711
	0.0069	0.0273	0.0616	0.1094	0.2461	0.4375	0.6836	0.9844	1.3398	1.5381	1.7500	1.9756	2.2148	2.4678	2.7344
	0.6533	0.6980	0.7459	0.7975	0.9109	1.0381	1.1794	1.3345	1.5034	1.5931	1.6864	1.7830	1.8831	1.9867	2.0938
	0.0075	0.0442	0.1348	0.2725	0.5863	0.8595	1.0844	1.2839	1.4759	1.5728	1.6711	1.7716	1.8744	1.9802	2.0888
$\nu=10, \lambda_1=10, \lambda_2=3$	0.0069	0.0300	0.0769	0.1533	0.3805	0.6502	0.9144	1.1597	1.3898	1.5016	1.6127	1.7236	1.8350	1.9478	2.0622
	0.0069	0.0273	0.0614	0.1089	0.2420	0.4205	0.6355	0.8770	1.1364	1.2706	1.4067	1.5444	1.6831	1.8228	1.9630
	0.0069	0.0273	0.0614	0.1091	0.2428	0.4236	0.6431	0.8923	1.1622	1.3022	1.4448	1.5891	1.7345	1.8808	2.0278
	0.0069	0.0273	0.0614	0.1084	0.2394	0.4116	0.6150	0.8400	1.0794	1.2030	1.3286	1.4559	1.5848	1.7153	1.8473
	0.0051	0.0204	0.0458	0.0814	0.1833	0.3258	0.5090	0.7329	0.9976	1.1453	1.3030	1.4709	1.6491	1.8375	2.0359
	0.4535	0.4865	0.5217	0.5592	0.6411	0.7320	0.8320	0.9411	1.0592	1.1217	1.1865	1.2534	1.3228	1.3943	1.4681
	0.0056	0.0337	0.1044	0.2112	0.4461	0.6381	0.7892	0.9217	1.0504	1.1157	1.1823	1.2506	1.3208	1.3930	1.4672
$\nu=100, \lambda_1=10, \lambda_2=3$	0.0051	0.0224	0.0580	0.1167	0.2907	0.4918	0.6813	0.8510	1.0068	1.0818	1.1562	1.2305	1.3054	1.3811	1.4581
	0.0051	0.0203	0.0457	0.0811	0.1801	0.3128	0.4717	0.6488	0.8368	0.9330	1.0302	1.1277	1.2251	1.3226	1.4198
	0.0051	0.0203	0.0457	0.0812	0.1809	0.3152	0.4778	0.6612	0.8577	0.9588	1.0610	1.1638	1.2667	1.3694	1.4719
	0.0051	0.0203	0.0457	0.0808	0.1782	0.3057	0.4551	0.6187	0.7906	0.8785	0.9672	1.0566	1.1467	1.2373	1.3285
	0.0050	0.0200	0.0450	0.0800	0.1800	0.3200	0.5000	0.7200	0.9800	1.1250	1.2800	1.4450	1.6200	1.8050	2.0000
	0.4400	0.4725	0.5072	0.5442	0.6247	0.7142	0.8125	0.9197	1.0358	1.0972	1.1608	1.2267	1.2947	1.3650	1.4375
	0.0055	0.0331	0.1026	0.2076	0.4379	0.6254	0.7726	0.9021	1.0280	1.0920	1.1572	1.2242	1.2931	1.3639	1.4368
$\nu=\infty, \lambda_1=10, \lambda_2=3$	0.0051	0.0220	0.0570	0.1147	0.2858	0.4831	0.6687	0.8347	0.9869	1.0603	1.1330	1.2057	1.2790	1.3531	1.4286
	0.0050	0.0200	0.0449	0.0797	0.1770	0.3073	0.4634	0.6373	0.8220	0.9165	1.0118	1.1075	1.2030	1.2986	1.3940
	0.0050	0.0200	0.0450	0.0797	0.1777	0.3097	0.4695	0.6495	0.8426	0.9418	1.0422	1.1429	1.2440	1.3448	1.4452
	0.0050	0.0200	0.0450	0.0794	0.1750	0.3004	0.4471	0.6078	0.7766	0.8627	0.9497	1.0375	1.1258	1.2146	1.3041

TABLE A6.2.1 (continued)

Estimator	ϕ 0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0164	0.0661	0.1483	0.2639	0.5939	1.0556	1.6494	2.3750	3.2328	3.7108	4.2222	4.7664	5.3439	5.9539	6.5972
	3.0636	3.1342	3.2211	3.3242	3.5786	3.8981	4.2825	4.7314	5.2453	5.5267	5.8242	6.1378	6.4675	6.8136	7.1758
	0.0169	0.0781	0.2092	0.4256	1.0817	1.9172	2.8167	3.7219	4.6136	5.0539	5.4922	5.9292	6.3664	6.8053	7.2472
$\nu=5, \lambda_1=0, \lambda_2=10$	0.0167	0.0678	0.1600	0.3006	0.7389	1.3669	2.1339	2.9906	3.9019	4.3711	4.8469	5.3289	5.8161	6.3086	6.8058
	0.0164	0.0658	0.1483	0.2636	0.5906	1.0425	1.6108	2.2867	3.0594	3.4786	3.9183	4.3772	4.8544	5.3489	5.8594
	0.0164	0.0658	0.1483	0.2636	0.5914	1.0447	1.6175	2.3003	3.0839	3.5100	3.9578	4.4256	4.9125	5.4175	5.9394
	0.0164	0.0658	0.1483	0.2633	0.5886	1.0353	1.5931	2.2511	2.9986	3.4028	3.8258	4.2667	4.7244	5.1983	5.6881
	0.0020	0.0078	0.0177	0.0313	0.0703	0.1250	0.1953	0.2813	0.3828	0.4395	0.5000	0.5645	0.6328	0.7052	0.7813
	1.8127	1.7761	1.7420	1.7106	1.6556	1.6111	1.5770	1.5533	1.5400	1.5372	1.5370	1.5395	1.5445	1.5522	1.5625
	0.0020	0.0083	0.0209	0.0431	0.1256	0.2623	0.4400	0.6380	0.8380	0.9345	1.0275	1.1161	1.2002	1.2795	1.3542
$\nu=10, \lambda_1=0, \lambda_2=10$	0.0020	0.0078	0.0181	0.0331	0.0813	0.1575	0.2634	0.3959	0.5486	0.6305	0.7147	0.8006	0.8875	0.9748	1.0619
	0.0020	0.0078	0.0175	0.0313	0.0702	0.1247	0.1941	0.2781	0.3761	0.4302	0.4873	0.5475	0.6109	0.6769	0.7456
	0.0020	0.0078	0.0175	0.0313	0.0703	0.1247	0.1944	0.2788	0.3773	0.4319	0.4895	0.5505	0.6145	0.6814	0.7513
	0.0020	0.0078	0.0175	0.0313	0.0702	0.1244	0.1933	0.2763	0.3725	0.4252	0.4809	0.5394	0.6006	0.6644	0.7305
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0248	0.0386	0.0556	0.0757	0.0869	0.0989	0.1116	0.1252	0.1395	0.1545
	1.5118	1.4737	1.4366	1.4004	1.3307	1.2648	1.2025	1.1439	1.0890	1.0629	1.0378	1.0135	0.9903	0.9679	0.9465
	0.0004	0.0016	0.0035	0.0069	0.0205	0.0517	0.1071	0.1848	0.2767	0.3246	0.3721	0.4184	0.4623	0.5033	0.5411
$\nu=100, \lambda_1=0, \lambda_2=10$	0.0004	0.0016	0.0034	0.0062	0.0145	0.0278	0.0486	0.0789	0.1197	0.1439	0.1703	0.1987	0.2284	0.2595	0.2913
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0247	0.0386	0.0556	0.0756	0.0868	0.0987	0.1113	0.1247	0.1388	0.1536
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0247	0.0386	0.0556	0.0757	0.0868	0.0987	0.1114	0.1248	0.1390	0.1539
	0.0004	0.0016	0.0034	0.0061	0.0140	0.0247	0.0386	0.0556	0.0755	0.0866	0.0984	0.1109	0.1241	0.1381	0.1525
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0313	0.0450	0.0613	0.0703	0.0800	0.0903	0.1013	0.1128	0.1250
	1.4893	1.4517	1.4149	1.3789	1.3094	1.2433	1.1806	1.1211	1.0650	1.0382	1.0122	0.9871	0.9628	0.9393	0.9167
	0.0003	0.0013	0.0028	0.0054	0.0160	0.0413	0.0885	0.1578	0.2419	0.2865	0.3311	0.3748	0.4166	0.4558	0.4917
$\nu=\infty, \lambda_1=0, \lambda_2=10$	0.0003	0.0013	0.0028	0.0050	0.0116	0.0221	0.0386	0.0631	0.0973	0.1177	0.1406	0.1652	0.1913	0.2188	0.2471
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0313	0.0450	0.0613	0.0702	0.0799	0.0901	0.1010	0.1125	0.1246
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0313	0.0450	0.0612	0.0703	0.0800	0.0901	0.1011	0.1127	0.1246
	0.0003	0.0013	0.0028	0.0050	0.0113	0.0200	0.0312	0.0449	0.0612	0.0701	0.0797	0.0900	0.1007	0.1119	0.1239
	0.0172	0.0689	0.1550	0.2758	0.6206	1.1031	1.7236	2.4819	3.3781	3.8781	4.4122	4.9811	5.5842	6.2219	6.8942
	3.0981	3.2022	3.3219	3.4572	3.7739	4.1525	4.5931	5.0956	5.6600	5.9656	6.2864	6.6228	6.9744	7.3417	7.7244
	0.0178	0.0825	0.2233	0.4586	1.1794	2.0975	3.0733	4.0339	4.9581	5.4072	5.8500	6.2886	6.7256	7.1628	7.6025
$\nu=5, \lambda_1=3, \lambda_2=10$	0.0172	0.0711	0.1681	0.3167	0.7844	1.4583	2.2808	3.1936	4.1544	4.6444	5.1383	5.6350	6.1339	6.6347	7.1381
	0.0172	0.0689	0.1550	0.2753	0.6169	1.0886	1.6811	2.3842	3.1861	3.6203	4.0750	4.5489	5.0406	5.5489	6.0731
	0.0172	0.0689	0.1550	0.2756	0.6178	1.0911	1.6883	2.3994	3.2133	3.6556	4.1192	4.6028	5.1056	5.6261	6.1631
	0.0172	0.0689	0.1550	0.2750	0.6147	1.0806	1.6611	2.3444	3.1181	3.5353	3.9708	4.4242	4.8939	5.3792	5.8794
	0.0027	0.0105	0.0234	0.0416	0.0936	0.1663	0.2598	0.3741	0.5092	0.5845	0.6650	0.7508	0.8417	0.9378	1.0391
	1.8436	1.8377	1.8338	1.8319	1.8347	1.8461	1.8661	1.8945	1.9316	1.9531	1.9770	2.0030	2.0311	2.0614	2.0938
	0.0027	0.0113	0.0294	0.0628	0.1941	0.4127	0.6869	0.9752	1.2469	1.3714	1.4872	1.5945	1.6936	1.7850	1.8698
$\nu=10, \lambda_1=3, \lambda_2=10$	0.0027	0.0105	0.0242	0.0448	0.1130	0.2253	0.3834	0.5795	0.8006	0.9161	1.0328	1.1495	1.2650	1.3784	1.4892
	0.0027	0.0105	0.0234	0.0416	0.0934	0.1656	0.2577	0.3688	0.4978	0.5686	0.6433	0.7217	0.8036	0.8888	0.9772
	0.0027	0.0105	0.0234	0.0416	0.0934	0.1658	0.2581	0.3698	0.5000	0.5714	0.6472	0.7267	0.8102	0.8970	0.9872
	0.0027	0.0105	0.0234	0.0416	0.0933	0.1652	0.2563	0.3655	0.4913	0.5598	0.6317	0.7070	0.7853	0.8663	0.9498
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0626	0.0977	0.1407	0.1915	0.2198	0.2501	0.2824	0.3165	0.3528	0.3909
	1.5409	1.5317	1.5231	1.5151	1.5008	1.4891	1.4796	1.4724	1.4678	1.4664	1.4655	1.4653	1.4656	1.4666	1.4681
	0.0009	0.0040	0.0095	0.0195	0.0676	0.1736	0.3361	0.5293	0.7237	0.8146	0.8991	0.9766	1.0468	1.1096	1.1657
$\nu=100, \lambda_1=3, \lambda_2=10$	0.0009	0.0040	0.0089	0.0159	0.0387	0.0791	0.1447	0.2383	0.3563	0.4219	0.4903	0.5603	0.6308	0.7007	0.7694
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0625	0.0977	0.1405	0.1909	0.2188	0.2485	0.2801	0.3133	0.3481	0.3845
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0625	0.0977	0.1406	0.1911	0.2191	0.2490	0.2807	0.3141	0.3493	0.3862
	0.0009	0.0040	0.0087	0.0156	0.0352	0.0625	0.0976	0.1401	0.1899	0.2175	0.2467	0.2774	0.3097	0.3433	0.3782
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0575	0.0898	0.1294	0.1761	0.2021	0.2300	0.2596	0.2911	0.3243	0.3594
	1.5183	1.5094	1.5010	1.4931	1.4788	1.4667	1.4566	1.4486	1.4427	1.4405	1.4389	1.4378	1.4372	1.4371	1.4375
	0.0009	0.0036	0.0085	0.0174	0.0608	0.1596	0.3149	0.5022	0.6924	0.7816	0.8649	0.9414	1.0106	1.0725	1.1276
$\nu=\infty, \lambda_1=3, \lambda_2=10$	0.0009	0.0036	0.0081	0.0147	0.0352	0.0718	0.1320	0.2194	0.3312	0.3939	0.4596	0.5269	0.5951	0.6629	0.7294
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0575	0.0898	0.1292	0.1756	0.2014	0.2289	0.2580	0.2888	0.3210	0.3546
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0575	0.0898	0.1292	0.1758	0.2016	0.2291	0.2585	0.2895	0.3220	0.3560
	0.0009	0.0036	0.0081	0.0144	0.0323	0.0574	0.0898	0.1290	0.1749	0.2003	0.2274	0.2558	0.2858	0.3169	0.3494
	0.0217	0.0869	0.1953	0.3472	0.7814	1.3889	2.1703	3.1250	4.2536	4.8828	5.5556	6.2717	7.0314	7.8342	8.6806
	3.1794	3.3658	3.5683	3.7869	4.2731	4.8242	5.4397	6.1203	6.8658	7.2628	7.6758	8.1053	8.5508	9.0128	9.4908
	0.0225	0.1053	0.2922	0.6142	1.6206	2.8717	4.1192	5.2686	6.3278	6.8364	7.3381	7.8372	8.3378	8.8431	9.3553
$\nu=5, \lambda_1=10, \lambda_2=10$	0.0217	0.0894	0.2128	0.4047	1.0203	1.9192	3.0083	4.1869	5.3850	5.9803	6.5708	7.1572	7.7400	8.3206	8.9011
	0.0217	0.0867	0.1953	0.3467	0.7767	1.3697	2.1136	2.9936	3.9933	4.5325	5.0956	5.6803	6.2850	6.9081	7.5475
	0.0217	0.0867	0.1953	0.3469	0.7778	1.3733	2.1233	3.0144	4.0314	4.5817	5.1575	5.7567	6.3772	7.0178	7.6764
	0.0217	0.0867	0.1950	0.3464	0.7739	1.3589	2.0861	2.9378	3.8969	4.4114	4.9467	5.5011	6.0733	6.6619	7.2661
	0.0069	0.0273	0.0616	0.1094	0.2461	0.4375	0.6836	0.9844	1.3398	1.5381	1.7500	1.9756	2.2148	2.4678	2.7344
	1.9172	1.9861	2.0584	2.1342	2.2963	2.4722	2.6620	2.8658	3.0833	3.1973	3.3148	3.4358	3.5602	3.6881	3.8194
	0.0069	0.0303	0.0834	0.1877	0.5911	1.1742	1.7733	2.2959	2.7306	2.9217	3.0998	3.2681	3.4294	3.5859	3.7395
$\nu=10, \lambda_1=10, \lambda_2=10$	0.0069	0.0277	0.0644	0.1211	0.3181	0.6472	1.0881	1.5877	2.0939	2.3388	2.5753	2.8027	3.0209	3.2306	3.4327
	0.0069	0.0273	0.0616	0.1094	0.2456	0.4355	0.6767	0.9666	1.3005	1.4823	1.6731	1.8723	2.0791	2.2923	2.5116
	0.0069	0.0273	0.0616	0.1094	0.2458	0.4359	0.6783	0.9702	1.3083	1.4931	1.6877	1.8913	2.1033	2.3227	2.5491
	0.0069	0.0273	0.0616	0.1092	0.2453	0.4338	0.6717	0.9545	1.2763	1.4498	1.6308	1.8183	2.0116	2.2097	2.4122
	0.0051	0.0204	0.0458	0.0814	0.1833	0.3258	0.5090	0.7329	0.9976	1.1453	1.3030	1.4709	1.6491	1.8375	2.0359
	1.6102	1.6719	1.7359	1.8023	1.9417	2.0902	2.2477	2.4143	2.5900	2.6813	2.7748	2.8706	2.9686	3.0689	3.1715
	0.0051	0.0212	0.0555	0.1259	0.4347	0.9228	1.4426	1.8989	2.2739	2.4361	2.5853	2.7245	2.8561	2.9823	3.1049
$\nu=100, \lambda_1=10, \lambda_2=10$	0.0051	0.0204	0.0466	0.0861	0.2251	0.4743	0.8301	1.2486	1.6803	1.8900	2.0924	2.2860	2.4709	2.6473	2.8158
	0.0051	0.0204	0.0458	0.0814	0.1832	0.3255	0.5076	0.7286	0.9860	1.1277	1.2772	1.4340	1.5873	1.7674	1.9427
	0.0051	0.0204	0.0458	0.0814	0.1832	0.3255	0.5080	0.7298	0.9893	1.1323	1.2838	1.4434	1.6104	1.7843	1.9644
	0.0051	0.0204	0.0458	0.0814	0.1832	0.3250	0.5058	0.7233	0.9740	1.1103	1.2531	1.4017	1.5555	1.7138	1.8759
	0.0050	0.0200	0.0450	0.0800	0.1800	0.3200	0.5000	0.7200	0.9800	1.1250	1.2800	1.4450	1.6200	1.8050	2.0000
	1.5872	1.6489	1.7128	1.7789	1.9178	2.0656	2.2222	2.3878	2.5622	2.6528	2.7456	2.8406	2.9378	3.0372	3.1389
	0.0050	0.0207	0.0539	0.1222	0.4249	0.9075	1.4229	1.8761	2.2487	2.4097	2.5578	2.6961	2.8266	2.9517	3.0732
$\nu=\infty, \lambda_1=10, \lambda_2=10$	0.0050	0.0200	0.0457	0.0843	0.2198	0.4642	0.8152	1.2295	1.6578	1.8658	2.0667	2.2591	2.4427	2.6178	2.7850
	0.0050	0.0200	0.0450	0.0800	0.1779	0.3197	0.4990	0.7163	0.9699	1.1095	1.2569	1.4117	1.5734	1.7412	1.9146
	0.0050	0.0200	0.0450	0.0800	0.1800	0.3198	0.4993	0.7174	0.9728	1.1139	1.2632	1.4205	1.5853	1.7568	1.9349
	0.0050	0.0200	0.0450	0.0800	0.1799	0.3194	0.4973	0.7115	0.9587	1.0933	1.2348	1.3809	1.5329	1.6894	1.8498

TABLE A6.2.2: Relative risks of s^2_{NML}, s^2_{AML}, and s^2_{PML}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	0.05	0.10	0.15	0.20	0.30	0.40	ϕ 0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0119	0.0475	0.1069	0.1900	0.4272	0.7597	1.3119	1.7092	2.3264	2.6706	3.0386	3.4303	3.8456	4.2847	4.7475
	0.7914	0.8544	0.9300	1.0178	1.2308	1.4933	1.8056	2.1675	2.5789	2.8033	3.0400	3.2892	3.5508	3.8247	4.1111
	0.0122	0.0581	0.1542	0.3017	0.6947	1.1331	1.5761	2.0247	2.4906	2.7339	2.9853	3.2461	3.5167	3.7978	4.0897
$\nu=5, \lambda_1=0, \lambda_2=0$	0.0119	0.0492	0.1156	0.2153	0.5092	0.8969	1.3406	1.8186	2.3231	2.5856	2.8550	3.1325	3.4181	3.7125	4.0164
	0.0119	0.0475	0.1067	0.1894	0.4231	0.7428	1.1400	1.6053	2.1297	2.4117	2.7056	3.0108	3.3269	3.6533	3.9894
	0.0119	0.0475	0.1067	0.1894	0.4242	0.7461	1.1483	1.6217	2.1578	2.4467	2.7483	3.0619	3.3867	3.7222	4.0681
	0.0119	0.0475	0.1067	0.1886	0.4183	0.7275	1.1050	1.5419	2.0303	2.2925	2.5658	2.8503	3.1456	3.4514	3.7678
	0.0014	0.0059	0.0133	0.0238	0.0534	0.0948	0.1483	0.2134	0.2906	0.3336	0.3795	0.4284	0.4803	0.5352	0.5930
	0.1745	0.1667	0.1617	0.1595	0.1636	0.1789	0.2055	0.2433	0.2923	0.3211	0.3527	0.3870	0.4242	0.4641	0.5069
	0.0016	0.0077	0.0208	0.0403	0.0888	0.1378	0.1855	0.2350	0.2900	0.3205	0.3531	0.3883	0.4258	0.4661	0.5091
$\nu=10, \lambda_1=0, \lambda_2=0$	0.0016	0.0063	0.0147	0.0273	0.0641	0.1106	0.1630	0.2197	0.2814	0.3147	0.3497	0.3867	0.4259	0.4675	0.5114
	0.0014	0.0059	0.0133	0.0236	0.0528	0.0930	0.1430	0.2022	0.2700	0.3070	0.3459	0.3870	0.4300	0.4750	0.5220
	0.0014	0.0059	0.0133	0.0236	0.0530	0.0933	0.1439	0.2039	0.2728	0.3103	0.3500	0.3916	0.4352	0.4808	0.5284
	0.0014	0.0059	0.0133	0.0236	0.0523	0.0911	0.1392	0.1958	0.2606	0.2961	0.3338	0.3736	0.4155	0.4597	0.5061
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0488	0.0664	0.0762	0.0867	0.0980	0.1099	0.1223	0.1356
	0.0825	0.0718	0.0626	0.0547	0.0430	0.0367	0.0357	0.0402	0.0501	0.0571	0.0654	0.0751	0.0861	0.0985	0.1122
	0.0004	0.0020	0.0056	0.0107	0.0213	0.0295	0.0359	0.0436	0.0545	0.0614	0.0697	0.0790	0.0899	0.1019	0.1154
$\nu=100, \lambda_1=0, \lambda_2=0$	0.0003	0.0015	0.0034	0.0066	0.0149	0.0246	0.0348	0.0460	0.0595	0.0673	0.0761	0.0859	0.0968	0.1090	0.1223
	0.0003	0.0014	0.0030	0.0054	0.0121	0.0213	0.0330	0.0472	0.0639	0.0734	0.0836	0.0946	0.1064	0.1191	0.1328
	0.0003	0.0014	0.0030	0.0054	0.0121	0.0213	0.0331	0.0474	0.0642	0.0736	0.0838	0.0946	0.1063	0.1189	0.1322
	0.0003	0.0014	0.0030	0.0054	0.0120	0.0210	0.0325	0.0465	0.0635	0.0732	0.0838	0.0953	0.1078	0.1212	0.1358
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0557	0.0639	0.0727	0.0821	0.0920	0.1025	0.1136
	0.0771	0.0665	0.0573	0.0492	0.0370	0.0299	0.0278	0.0308	0.0388	0.0447	0.0519	0.0604	0.0701	0.0811	0.0933
	0.0003	0.0017	0.0049	0.0093	0.0181	0.0243	0.0289	0.0347	0.0434	0.0493	0.0563	0.0644	0.0739	0.0845	0.0964
$\nu=\infty, \lambda_1=0, \lambda_2=0$	0.0003	0.0012	0.0029	0.0055	0.0125	0.0204	0.0286	0.0377	0.0489	0.0555	0.0630	0.0715	0.0811	0.0917	0.1035
	0.0003	0.0011	0.0026	0.0045	0.0101	0.0179	0.0277	0.0397	0.0540	0.0621	0.0709	0.0804	0.0907	0.1018	0.1137
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0179	0.0278	0.0398	0.0541	0.0622	0.0709	0.0802	0.0904	0.1013	0.1129
	0.0003	0.0011	0.0025	0.0045	0.0100	0.0176	0.0273	0.0393	0.0539	0.0624	0.0716	0.0817	0.0927	0.1045	0.1174
	0.0119	0.0481	0.1078	0.1917	0.4314	0.7667	1.1981	1.7250	2.3481	2.6956	3.0669	3.4622	3.8814	4.3247	4.7919
	0.7997	0.8700	0.9514	1.0442	1.2636	1.5281	1.8378	2.1925	2.5922	2.8092	3.0372	3.2764	3.5272	3.7892	4.0622
	0.0125	0.0611	0.1644	0.3208	0.7267	1.1681	1.6072	2.0469	2.5006	2.7361	2.9789	3.2300	3.4897	3.7589	4.0378
$\nu=5, \lambda_1=3, \lambda_2=0$	0.0119	0.0500	0.1186	0.2219	0.5239	0.9169	1.3600	1.8314	2.3244	2.5794	2.8406	3.1083	3.3833	3.6661	3.9572
	0.0119	0.0478	0.1078	0.1911	0.4264	0.7469	1.1433	1.6047	2.1219	2.3983	2.6861	2.9836	3.2911	3.6075	3.9325
	0.0119	0.0478	0.1078	0.1911	0.4275	0.7508	1.1528	1.6236	2.1536	2.4381	2.7342	3.0411	3.3581	3.6844	4.0200
	0.0119	0.0478	0.1075	0.1903	0.4208	0.7289	1.1031	1.5325	2.0100	2.2650	2.5300	2.8050	3.0897	3.3842	3.6878
	0.0016	0.0066	0.0147	0.0261	0.0586	0.1042	0.1628	0.2344	0.3191	0.3663	0.4167	0.4705	0.5275	0.5877	0.6511
	0.1808	0.1784	0.1781	0.1798	0.1892	0.2066	0.2320	0.2656	0.3072	0.3311	0.3569	0.3847	0.4145	0.4464	0.4803
	0.0019	0.0106	0.0295	0.0559	0.1134	0.1645	0.2095	0.2538	0.3011	0.3266	0.3538	0.3825	0.4130	0.4453	0.4795
$\nu=10, \lambda_1=3, \lambda_2=0$	0.0017	0.0072	0.0177	0.0336	0.0772	0.1277	0.1794	0.2317	0.2859	0.3142	0.3438	0.3745	0.4066	0.4403	0.4756
	0.0016	0.0066	0.0147	0.0259	0.0572	0.0991	0.1494	0.2066	0.2695	0.3030	0.3375	0.3731	0.4102	0.4481	0.4875
	0.0016	0.0066	0.0147	0.0259	0.0575	0.1000	0.1516	0.2105	0.2756	0.3102	0.3459	0.3828	0.4209	0.4602	0.5006
	0.0016	0.0066	0.0145	0.0256	0.0558	0.0948	0.1409	0.1930	0.2503	0.2811	0.3131	0.3466	0.3814	0.4177	0.4555
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0503	0.0725	0.0987	0.1133	0.1288	0.1455	0.1631	0.1817	0.2014
	0.0877	0.0815	0.0762	0.0715	0.0646	0.0604	0.0592	0.0611	0.0658	0.0693	0.0735	0.0785	0.0842	0.0906	0.0978
	0.0006	0.0048	0.0135	0.0245	0.0422	0.0515	0.0564	0.0606	0.0662	0.0699	0.0741	0.0791	0.0848	0.0912	0.0983
$\nu=100, \lambda_1=3, \lambda_2=0$	0.0005	0.0025	0.0066	0.0125	0.0270	0.0399	0.0496	0.0577	0.0656	0.0700	0.0748	0.0801	0.0860	0.0925	0.0996
	0.0005	0.0020	0.0045	0.0079	0.0171	0.0285	0.0411	0.0542	0.0677	0.0743	0.0812	0.0882	0.0954	0.1028	0.1105
	0.0005	0.0020	0.0045	0.0079	0.0173	0.0292	0.0425	0.0565	0.0708	0.0780	0.0853	0.0926	0.1001	0.1077	0.1155
	0.0005	0.0020	0.0045	0.0077	0.0161	0.0259	0.0364	0.0474	0.0589	0.0651	0.0714	0.0782	0.0854	0.0930	0.1011
	0.0005	0.0018	0.0041	0.0072	0.0162	0.0288	0.0450	0.0648	0.0882	0.1013	0.1152	0.1301	0.1458	0.1625	0.1801
	0.0821	0.0760	0.0706	0.0658	0.0582	0.0533	0.0511	0.0516	0.0547	0.0572	0.0604	0.0643	0.0689	0.0741	0.0800
	0.0006	0.0045	0.0128	0.0228	0.0386	0.0459	0.0490	0.0515	0.0552	0.0579	0.0612	0.0650	0.0695	0.0747	0.0805
$\nu=\infty, \lambda_1=3, \lambda_2=0$	0.0005	0.0022	0.0060	0.0115	0.0245	0.0355	0.0434	0.0493	0.0552	0.0585	0.0621	0.0663	0.0709	0.0762	0.0820
	0.0005	0.0018	0.0040	0.0071	0.0152	0.0252	0.0360	0.0471	0.0580	0.0635	0.0691	0.0747	0.0804	0.0863	0.0925
	0.0004	0.0018	0.0040	0.0071	0.0154	0.0258	0.0373	0.0492	0.0611	0.0670	0.0728	0.0788	0.0847	0.0908	0.0971
	0.0004	0.0002	0.0040	0.0069	0.0142	0.0227	0.0315	0.0405	0.0499	0.0548	0.0600	0.0655	0.0713	0.0776	0.0843
	0.0142	0.0567	0.1275	0.2269	0.5103	0.9072	1.4178	2.0417	2.7789	3.1900	3.6294	4.0972	4.5936	5.1181	5.6711
	0.8200	0.9092	1.0089	1.1186	1.3686	1.6592	1.9908	2.3631	2.7758	2.9978	3.2297	3.4717	3.7242	3.9867	4.2592
	0.0158	0.0831	0.2183	0.4086	0.8631	1.3297	1.7833	2.2333	2.6944	2.9331	3.1781	3.4303	3.6908	3.9597	4.2375
$\nu=5, \lambda_1=10, \lambda_2=0$	0.0144	0.0614	0.1481	0.2756	0.6294	1.0639	1.5344	2.0231	2.5264	2.7844	3.0475	3.3161	3.5911	3.8731	4.1622
	0.0142	0.0567	0.1272	0.2250	0.4981	0.8619	1.3014	1.8014	2.3489	2.6378	2.9350	3.2403	3.5531	3.8725	4.1989
	0.0142	0.0567	0.1272	0.2256	0.5003	0.8697	1.3192	1.8333	2.3997	2.6989	3.0075	3.3244	3.6492	3.9811	4.3194
	0.0142	0.0567	0.1267	0.2231	0.4864	0.8294	1.2350	1.6908	2.1881	2.4506	2.7214	3.0008	3.2886	3.5844	3.8883
	0.0039	0.0156	0.0352	0.0625	0.1406	0.2500	0.3906	0.5625	0.7656	0.8789	1.0000	1.1289	1.2656	1.4102	1.5625
	0.1963	0.2089	0.2234	0.2395	0.2769	0.3211	0.3720	0.4298	0.4945	0.5294	0.5659	0.6042	0.6442	0.6858	0.7292
	0.0053	0.0322	0.0788	0.1302	0.2206	0.2933	0.3581	0.4227	0.4906	0.5266	0.5639	0.6027	0.6430	0.6848	0.7284
$\nu=10, \lambda_1=10, \lambda_2=0$	0.0041	0.0191	0.0472	0.0848	0.1703	0.2530	0.3297	0.4036	0.4781	0.5164	0.5556	0.5959	0.6375	0.6805	0.7248
	0.0039	0.0156	0.0347	0.0606	0.1289	0.2117	0.3017	0.3945	0.4878	0.5347	0.5814	0.6284	0.6756	0.7233	0.7714
	0.0039	0.0156	0.0348	0.0609	0.1308	0.2170	0.3119	0.4105	0.5097	0.5592	0.6086	0.6580	0.7073	0.7567	0.8066
	0.0039	0.0155	0.0342	0.0588	0.1209	0.1931	0.2702	0.3503	0.4330	0.4755	0.5186	0.5627	0.6078	0.6541	0.7013
	0.0028	0.0113	0.0254	0.0452	0.1017	0.1809	0.2826	0.4070	0.5539	0.6359	0.7236	0.8168	0.9158	1.0203	1.1306
	0.1004	0.1071	0.1148	0.1233	0.1429	0.1658	0.1921	0.2219	0.2551	0.2730	0.2918	0.3113	0.3318	0.3531	0.3753
	0.0044	0.0266	0.0599	0.0902	0.1325	0.1626	0.1912	0.2217	0.2550	0.2730	0.2918	0.3113	0.3318	0.3531	0.3753
$\nu=100, \lambda_1=10, \lambda_2=0$	0.0030	0.0148	0.0362	0.0625	0.1123	0.1522	0.1864	0.2195	0.2542	0.2724	0.2913	0.3111	0.3316	0.3530	0.3753
	0.0028	0.0112	0.0249	0.0431	0.0887	0.1395	0.1895	0.2363	0.2797	0.3005	0.3211	0.3415	0.3620	0.3829	0.4040
	0.0028	0.0112	0.0250	0.0435	0.0908	0.1448	0.1991	0.2499	0.2968	0.3188	0.3404	0.3614	0.3823	0.4032	0.4242
	0.0028	0.0111	0.0243	0.0411	0.0808	0.1226	0.1635	0.2036	0.2436	0.2640	0.2847	0.3058	0.3275	0.3498	0.3727
	0.0028	0.0111	0.0250	0.0444	0.1000	0.1777	0.2777	0.3990	0.5443	0.6248	0.7109	0.8026	0.8998	1.0025	1.1108
	0.0946	0.1012	0.1086	0.1168	0.1357	0.1579	0.1833	0.2121	0.2441	0.2614	0.2795	0.2984	0.3181	0.3386	0.3600
	0.0043	0.0262	0.0588	0.0878	0.1273	0.1556	0.1827	0.2120	0.2441	0.2613	0.2794	0.2984	0.3181	0.3386	0.3600
$\nu=\infty, \lambda_1=10, \lambda_2=0$	0.0030	0.0146	0.0357	0.0614	0.1092	0.1467	0.1789	0.2103	0.2435	0.2610	0.2792	0.2982	0.3181	0.3386	0.3600
	0.0028	0.0110	0.0245	0.0423	0.0869	0.1361	0.1840	0.2284	0.2694	0.2890	0.3084	0.3277	0.3472	0.3668	0.3870
	0.0028	0.0111	0.0246	0.0427	0.0889	0.1414	0.1935	0.2419	0.2861	0.3069	0.3272	0.3469	0.3665	0.3862	0.4060
	0.0028	0.0109	0.0239	0.0403	0.0788	0.1191	0.1583	0.1965	0.2347	0.2541	0.2739	0.2941	0.3149	0.3362	0.3582

TABLE A6.2.2 (continued)

Estimator	0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
							ϕ								
	0.0119	0.0475	0.1069	0.1900	0.4272	0.7597	1.1869	1.7092	2.3264	2.6706	3.0386	3.4303	3.8456	4.2847	4.7475
	1.0381	1.0856	1.1456	1.2178	1.3997	1.6311	1.9122	2.2431	2.6233	2.8322	3.0533	3.2869	3.5331	3.7914	4.0622
	0.0122	0.0567	0.1506	0.2983	0.7083	1.1772	1.6461	2.1058	2.5678	2.8039	3.0458	3.2947	3.5519	3.8181	4.0942
$\nu=5,\lambda_1=0,\lambda_2=3$	0.0119	0.0489	0.1147	0.2136	0.5092	0.9078	1.3683	1.8625	2.3775	2.6417	2.9111	3.1858	3.4669	3.7544	4.0494
	0.0119	0.0475	0.1067	0.1894	0.4236	0.7447	1.1447	1.6139	2.1433	2.4278	2.7244	3.0319	3.3500	3.6778	4.0147
	0.0119	0.0475	0.1067	0.1894	0.4244	0.7478	1.1522	1.6297	2.1706	2.4622	2.7667	3.0831	3.4106	3.7483	4.0958
	0.0119	0.0475	0.1067	0.1889	0.4194	0.7306	1.1114	1.5517	2.0433	2.3067	2.5806	2.8647	3.1592	3.4633	3.7775
	0.0014	0.0059	0.0133	0.0238	0.0534	0.0948	0.1483	0.2134	0.2906	0.3336	0.3795	0.4284	0.4803	0.5352	0.5930
	0.3695	0.3500	0.3334	0.3195	0.3003	0.2922	0.2955	0.3098	0.3356	0.3527	0.3727	0.3953	0.4208	0.4491	0.4803
	0.0016	0.0069	0.0184	0.0378	0.0953	0.1631	0.2277	0.2853	0.3384	0.3647	0.3911	0.4184	0.4472	0.4773	0.5094
$\nu=10,\lambda_1=0,\lambda_2=3$	0.0016	0.0061	0.0141	0.0263	0.0638	0.1159	0.1777	0.2434	0.3109	0.3452	0.3797	0.4148	0.4506	0.4872	0.5250
	0.0014	0.0059	0.0133	0.0238	0.0531	0.0938	0.1448	0.2058	0.2753	0.3131	0.3528	0.3944	0.4375	0.4825	0.5289
	0.0014	0.0059	0.0133	0.0238	0.0531	0.0939	0.1456	0.2072	0.2778	0.3164	0.3569	0.3992	0.4434	0.4894	0.5370
	0.0014	0.0059	0.0133	0.0236	0.0528	0.0925	0.1417	0.1997	0.2652	0.3008	0.3380	0.3769	0.4175	0.4598	0.5039
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0488	0.0664	0.0762	0.0867	0.0980	0.1099	0.1223	0.1356
	0.2491	0.2289	0.2101	0.1926	0.1619	0.1365	0.1165	0.1019	0.0928	0.0903	0.0890	0.0892	0.0907	0.0936	0.0978
	0.0003	0.0016	0.0041	0.0087	0.0246	0.0445	0.0627	0.0766	0.0867	0.0909	0.0949	0.0988	0.1031	0.1078	0.1132
$\nu=100,\lambda_1=0,\lambda_2=3$	0.0003	0.0014	0.0031	0.0058	0.0145	0.0271	0.0426	0.0590	0.0755	0.0836	0.0916	0.0996	0.1078	0.1161	0.1247
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0216	0.0336	0.0482	0.0651	0.0744	0.0844	0.0950	0.1060	0.1177	0.1299
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0337	0.0483	0.0654	0.0749	0.0849	0.0955	0.1067	0.1185	0.1308
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0214	0.0332	0.0474	0.0638	0.0729	0.0826	0.0928	0.1035	0.1150	0.1270
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0557	0.0639	0.0727	0.0821	0.0920	0.1025	0.1136
	0.2411	0.2212	0.2026	0.1852	0.1544	0.1285	0.1078	0.0921	0.0815	0.0781	0.0759	0.0750	0.0754	0.0771	0.0800
	0.0003	0.0013	0.0034	0.0074	0.0210	0.0384	0.0543	0.0661	0.0742	0.0774	0.0803	0.0832	0.0864	0.0899	0.0940
$\nu=\infty,\lambda_1=0,\lambda_2=3$	0.0003	0.0011	0.0026	0.0049	0.0121	0.0228	0.0359	0.0498	0.0637	0.0705	0.0773	0.0840	0.0907	0.0977	0.1050
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0181	0.0282	0.0405	0.0548	0.0627	0.0712	0.0802	0.0897	0.0996	0.1101
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0181	0.0282	0.0406	0.0550	0.0630	0.0722	0.0805	0.0900	0.1001	0.1107
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0180	0.0280	0.0400	0.0539	0.0616	0.0699	0.0786	0.0879	0.0977	0.1081
	0.0119	0.0481	0.1078	0.1917	0.4314	0.7667	1.1981	1.7250	2.3481	2.6956	3.0669	3.4622	3.8814	4.3247	4.7919
	1.0503	1.1092	1.1792	1.2603	1.4567	1.6981	1.9844	2.3161	2.6928	2.8981	3.1144	3.3422	3.5814	3.8317	4.0933
	0.0125	0.0589	0.1592	0.3172	0.7469	1.2219	1.6850	2.1339	2.5839	2.8144	3.0503	3.2931	3.5439	3.8031	4.0717
$\nu=5,\lambda_1=3,\lambda_2=3$	0.0119	0.0494	0.1172	0.2194	0.5247	0.9308	1.3914	1.8769	2.3775	2.6331	2.8931	3.1578	3.4283	3.7050	3.9889
	0.0119	0.0478	0.1078	0.1911	0.4269	0.7492	1.1486	1.6142	2.1358	2.4147	2.7042	3.0036	3.3119	3.6289	3.9539
	0.0119	0.0478	0.1078	0.1914	0.4281	0.7528	1.1572	1.6322	2.1669	2.4539	2.7522	3.0614	3.3803	3.7081	4.0442
	0.0119	0.0478	0.1075	0.1906	0.4222	0.7328	1.1100	1.5431	2.0228	2.2783	2.5433	2.8178	3.1011	3.3933	3.6942
	0.0016	0.0066	0.0147	0.0261	0.0586	0.1042	0.1628	0.2344	0.3191	0.3663	0.4167	0.4705	0.5275	0.5877	0.6511
	0.3798	0.3698	0.3619	0.3558	0.3498	0.3520	0.3620	0.3803	0.4066	0.4227	0.4409	0.4611	0.4833	0.5073	0.5336
	0.0017	0.0086	0.0250	0.0528	0.1313	0.2127	0.2811	0.3367	0.3850	0.4083	0.4317	0.4556	0.4806	0.5067	0.5342
$\nu=10,\lambda_1=3,\lambda_2=3$	0.0016	0.0067	0.0163	0.0313	0.0780	0.1409	0.2095	0.2770	0.3411	0.3720	0.4027	0.4331	0.4636	0.4945	0.5261
	0.0016	0.0066	0.0147	0.0259	0.0580	0.1017	0.1563	0.2183	0.2877	0.3244	0.3623	0.4012	0.4411	0.4819	0.5233
	0.0016	0.0066	0.0147	0.0259	0.0581	0.1022	0.1570	0.2211	0.2927	0.3308	0.3702	0.4108	0.4525	0.4950	0.5384
	0.0016	0.0066	0.0147	0.0259	0.0573	0.0991	0.1494	0.2063	0.2683	0.3009	0.3345	0.3691	0.4045	0.4408	0.4781
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0503	0.0725	0.0987	0.1133	0.1288	0.1455	0.1631	0.1817	0.2014
	0.2582	0.2466	0.2356	0.2255	0.2074	0.1923	0.1801	0.1709	0.1645	0.1625	0.1612	0.1607	0.1609	0.1617	0.1634
	0.0005	0.0028	0.0093	0.0216	0.0585	0.0966	0.1244	0.1413	0.1505	0.1534	0.1558	0.1579	0.1599	0.1621	0.1647
$\nu=100,\lambda_1=3,\lambda_2=3$	0.0005	0.0021	0.0052	0.0103	0.0279	0.0526	0.0790	0.1027	0.1220	0.1302	0.1372	0.1437	0.1496	0.1551	0.1605
	0.0005	0.0020	0.0045	0.0080	0.0180	0.0315	0.0481	0.0672	0.0880	0.0988	0.1097	0.1208	0.1319	0.1431	0.1542
	0.0005	0.0020	0.0045	0.0080	0.0180	0.0317	0.0486	0.0682	0.0900	0.1013	0.1130	0.1248	0.1367	0.1488	0.1608
	0.0005	0.0020	0.0045	0.0080	0.0177	0.0305	0.0457	0.0623	0.0796	0.0884	0.0973	0.1061	0.1150	0.1239	0.1329
	0.0005	0.0018	0.0041	0.0072	0.0162	0.0288	0.0450	0.0648	0.0882	0.1013	0.1152	0.1301	0.1458	0.1625	0.1801
	0.2501	0.2387	0.2279	0.2178	0.1996	0.1840	0.1711	0.1609	0.1533	0.1506	0.1484	0.1470	0.1462	0.1461	0.1467
	0.0005	0.0026	0.0085	0.0199	0.0548	0.0908	0.1168	0.1319	0.1395	0.1416	0.1431	0.1442	0.1453	0.1465	0.1479
$\nu=\infty,\lambda_1=3,\lambda_2=3$	0.0005	0.0019	0.0047	0.0093	0.0254	0.0483	0.0727	0.0944	0.1117	0.1188	0.1250	0.1303	0.1352	0.1395	0.1438
	0.0004	0.0018	0.0041	0.0072	0.0161	0.0282	0.0430	0.0601	0.0786	0.0881	0.0979	0.1077	0.1175	0.1273	0.1369
	0.0004	0.0018	0.0040	0.0072	0.0161	0.0283	0.0435	0.0610	0.0804	0.0905	0.1009	0.1114	0.1220	0.1326	0.1432
	0.0005	0.0018	0.0040	0.0072	0.0159	0.0273	0.0408	0.0555	0.0708	0.0784	0.0862	0.0938	0.1016	0.1091	0.1168
	0.0142	0.0567	0.1275	0.2269	0.5103	0.9072	1.4178	2.0417	2.7789	3.1900	3.6294	4.0972	4.5936	5.1181	5.6711
	1.0800	1.1669	1.2644	1.3719	1.6175	1.9036	2.2308	2.5986	3.0069	3.2267	3.4564	3.6961	3.9464	4.2067	4.4769
	0.0150	0.0764	0.2122	0.4189	0.9347	1.4603	1.9536	2.4269	2.9003	3.1419	3.3892	3.6425	3.9033	4.1719	4.4492
$\nu=5,\lambda_1=10,\lambda_2=3$	0.0142	0.0597	0.1439	0.2722	0.6472	1.1203	1.6317	2.1542	2.6828	2.9503	3.2211	3.4961	3.7758	4.0614	4.3533
	0.0142	0.0567	0.1275	0.2258	0.5028	0.8769	1.3333	1.8561	2.4306	2.7331	3.0447	3.3639	3.6900	4.0225	4.3611
	0.0142	0.0567	0.1275	0.2261	0.5044	0.8828	1.3481	1.8850	2.4789	2.7931	3.1172	3.4497	3.7897	4.1367	4.4897
	0.0142	0.0567	0.1272	0.2247	0.4944	0.8494	1.2722	1.7472	2.2642	2.5358	2.8158	3.1036	3.3992	3.7022	4.0125
	0.0039	0.0156	0.0352	0.0625	0.1406	0.2500	0.3906	0.5625	0.7656	0.8789	1.0000	1.1289	1.2656	1.4102	1.5625
	0.4045	0.4189	0.4350	0.4528	0.4936	0.5411	0.5955	0.6566	0.7245	0.7611	0.7992	0.8392	0.8808	0.9241	0.9692
	0.0044	0.0250	0.0747	0.1486	0.3105	0.4425	0.5441	0.6297	0.7103	0.7506	0.7916	0.8334	0.8766	0.9209	0.9667
$\nu=10,\lambda_1=10,\lambda_2=3$	0.0039	0.0170	0.0427	0.0834	0.2000	0.3320	0.4561	0.5666	0.6672	0.7155	0.7630	0.8102	0.8577	0.9056	0.9544
	0.0039	0.0156	0.0350	0.0622	0.1375	0.2370	0.3542	0.4823	0.6155	0.6823	0.7491	0.8153	0.8809	0.9458	1.0100
	0.0039	0.0156	0.0352	0.0622	0.1381	0.2395	0.3606	0.4950	0.6363	0.7080	0.7798	0.8513	0.9220	0.9922	1.0613
	0.0039	0.0156	0.0350	0.0616	0.1338	0.2248	0.3274	0.4356	0.5461	0.6017	0.6573	0.7131	0.7691	0.8253	0.8819
	0.0028	0.0113	0.0254	0.0452	0.1017	0.1809	0.2826	0.4070	0.5539	0.6359	0.7236	0.8168	0.9158	1.0203	1.1306
	0.2803	0.2908	0.3023	0.3146	0.3417	0.3723	0.4063	0.4437	0.4845	0.5062	0.5287	0.5522	0.5764	0.6015	0.6276
	0.0031	0.0187	0.0572	0.1141	0.2331	0.3217	0.3841	0.4341	0.4804	0.5035	0.5270	0.5510	0.5757	0.6010	0.6272
$\nu=100,\lambda_1=10,\lambda_2=3$	0.0028	0.0124	0.0314	0.0622	0.1496	0.2450	0.3290	0.3992	0.4596	0.4877	0.5151	0.5421	0.5690	0.5961	0.6236
	0.0028	0.0113	0.0254	0.0450	0.0994	0.1710	0.2543	0.3437	0.4342	0.4788	0.5224	0.5649	0.6061	0.6461	0.6847
	0.0028	0.0113	0.0254	0.0450	0.1000	0.1729	0.2595	0.3539	0.4514	0.4998	0.5476	0.5942	0.6397	0.6839	0.7266
	0.0028	0.0112	0.0253	0.0446	0.0964	0.1612	0.2324	0.3055	0.3778	0.4132	0.4480	0.4824	0.5165	0.5501	0.5835
	0.0028	0.0111	0.0250	0.0444	0.1000	0.1777	0.2777	0.3990	0.5443	0.6248	0.7109	0.8026	0.8998	1.0025	1.1108
	0.2720	0.2825	0.2939	0.3061	0.3330	0.3632	0.3967	0.4334	0.4735	0.4947	0.5168	0.5397	0.5634	0.5880	0.6133
	0.0031	0.0184	0.0563	0.1122	0.2290	0.3155	0.3761	0.4248	0.4698	0.4924	0.5153	0.5387	0.5628	0.5876	0.6132
$\nu=\infty,\lambda_1=10,\lambda_2=3$	0.0028	0.0121	0.0309	0.0611	0.1472	0.2408	0.3231	0.3915	0.4504	0.4777	0.5043	0.5306	0.5568	0.5831	0.6099
	0.0028	0.0111	0.0249	0.0442	0.0977	0.1680	0.2499	0.3376	0.4265	0.4701	0.5128	0.5544	0.5947	0.6337	0.6715
	0.0028	0.0111	0.0250	0.0442	0.0982	0.1700	0.2550	0.3478	0.4434	0.4909	0.5376	0.5833	0.6280	0.6711	0.7128
	0.0028	0.0111	0.0248	0.0437	0.0947	0.1583	0.2283	0.3000	0.3707	0.4053	0.4394	0.4731	0.5063	0.5391	0.5716

TABLE A6.2.2 (continued)

Estimator	0.05	0.10	0.15	0.20	0.30	0.40	ϕ 0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0119	0.0475	0.1069	0.1900	0.4272	0.7597	1.3119	1.7092	2.3264	2.6706	3.0386	3.4303	3.8456	4.2847	4.7475
	1.9247	1.9358	1.9594	1.9956	2.1047	2.2636	2.4722	2.7303	3.0381	3.2106	3.3956	3.5928	3.8025	4.0247	4.2592
	0.0122	0.0547	0.1425	0.2833	0.7014	1.2247	1.7817	2.3389	2.8875	3.1589	3.4297	3.7008	3.9733	4.2486	4.5272
$\nu=5, \lambda_1=0, \lambda_2=10$	0.0119	0.0486	0.1131	0.2092	0.5006	0.9078	1.3981	1.9417	2.5178	2.8144	3.1158	3.4214	3.7311	4.0447	4.3631
	0.0119	0.0475	0.1067	0.1897	0.4247	0.7483	1.1542	1.6339	2.1794	2.4742	2.7822	3.1028	3.4353	3.7783	4.1319
	0.0119	0.0475	0.1067	0.1897	0.4253	0.7506	1.1600	1.6467	2.2025	2.5036	2.8192	3.1478	3.4892	3.8422	4.2061
	0.0119	0.0475	0.1067	0.1892	0.4214	0.7369	1.1264	1.5806	2.0906	2.3644	2.6497	2.9461	3.2528	3.5694	3.8961
	0.0014	0.0059	0.0133	0.0238	0.0534	0.0948	0.1483	0.2134	0.2906	0.3336	0.3795	0.4284	0.4803	0.5352	0.5930
	1.1356	1.0889	1.0450	1.0039	0.9303	0.8678	0.8166	0.7766	0.7478	0.7377	0.7303	0.7258	0.7242	0.7253	0.7292
	0.0016	0.0063	0.0153	0.0303	0.0836	0.1672	0.2719	0.3847	0.4955	0.5480	0.5980	0.6453	0.6898	0.7317	0.7714
$\nu=10, \lambda_1=0, \lambda_2=10$	0.0014	0.0059	0.0136	0.0247	0.0588	0.1106	0.1798	0.2639	0.3589	0.4092	0.4608	0.5131	0.5659	0.6189	0.6717
	0.0014	0.0059	0.0133	0.0238	0.0533	0.0945	0.1472	0.2109	0.2852	0.3259	0.3692	0.4148	0.4580	0.5127	0.5647
	0.0014	0.0059	0.0133	0.0238	0.0533	0.0947	0.1475	0.2114	0.2863	0.3275	0.3713	0.4173	0.4659	0.5166	0.5695
	0.0014	0.0059	0.0133	0.0238	0.0531	0.0941	0.1459	0.2081	0.2797	0.3189	0.3602	0.4033	0.4484	0.4955	0.5441
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0488	0.0664	0.0762	0.0867	0.0980	0.1099	0.1223	0.1356
	0.9488	0.9064	0.8654	0.8257	0.7504	0.6805	0.6162	0.5572	0.5035	0.4788	0.4553	0.4334	0.4126	0.3933	0.3753
	0.0003	0.0014	0.0031	0.0057	0.0155	0.0349	0.0662	0.1071	0.1526	0.1752	0.1970	0.2173	0.2359	0.2527	0.2674
$\nu=100, \lambda_1=0, \lambda_2=10$	0.0003	0.0014	0.0030	0.0054	0.0125	0.0231	0.0382	0.0585	0.0841	0.0986	0.1140	0.1302	0.1469	0.1640	0.1813
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0487	0.0663	0.0761	0.0866	0.0978	0.1095	0.1219	0.1349
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0488	0.0664	0.0762	0.0866	0.0978	0.1096	0.1220	0.1352
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0217	0.0338	0.0487	0.0662	0.0759	0.0862	0.0971	0.1087	0.1209	0.1337
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0557	0.0639	0.0727	0.0821	0.0920	0.1025	0.1136
	0.9349	0.8932	0.8528	0.8137	0.7392	0.6699	0.6056	0.5463	0.4921	0.4669	0.4430	0.4204	0.3990	0.3789	0.3600
	0.0003	0.0011	0.0026	0.0047	0.0126	0.0284	0.0549	0.0910	0.1320	0.1526	0.1725	0.1914	0.2086	0.2241	0.2375
$\nu=\infty, \lambda_1=0, \lambda_2=10$	0.0003	0.0011	0.0026	0.0045	0.0104	0.0191	0.0315	0.0483	0.0697	0.0820	0.0953	0.1092	0.1237	0.1386	0.1536
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0557	0.0638	0.0726	0.0820	0.0918	0.1023	0.1133
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0556	0.0639	0.0727	0.0819	0.0919	0.1024	0.1134
	0.0003	0.0011	0.0026	0.0045	0.0102	0.0182	0.0284	0.0409	0.0555	0.0637	0.0724	0.0817	0.0913	0.1017	0.1125
	0.0119	0.0481	0.1078	0.1917	0.4314	0.7667	1.1981	1.7250	2.3481	2.6956	3.0669	3.4622	3.8814	4.3247	4.7919
	1.9464	1.9781	2.0211	2.0753	2.2178	2.4053	2.6378	2.9156	3.2381	3.4164	3.6061	3.8069	4.0189	4.2425	4.4769
	0.0122	0.0558	0.1478	0.2972	0.7436	1.2983	1.8764	2.4372	2.9722	3.2319	3.4886	3.7436	3.9992	4.2564	4.5172
$\nu=5, \lambda_1=3, \lambda_2=10$	0.0119	0.0492	0.1147	0.2131	0.5139	0.9350	1.4394	1.9917	2.5681	2.8608	3.1556	3.4519	3.7500	4.0494	4.3514
	0.0119	0.0478	0.1078	0.1914	0.4283	0.7542	1.1619	1.6422	2.1858	2.4786	2.7836	3.1003	3.4272	3.7642	4.1100
	0.0119	0.0478	0.1078	0.1914	0.4289	0.7567	1.1686	1.6564	2.2114	2.5111	2.8244	3.1503	3.4875	3.8353	4.1928
	0.0119	0.0478	0.1075	0.1908	0.4247	0.7417	1.1311	1.5828	2.0867	2.3558	2.6356	2.9247	3.2231	3.5303	3.8456
	0.0016	0.0066	0.0147	0.0261	0.0586	0.1042	0.1628	0.2344	0.3191	0.3663	0.4167	0.4705	0.5275	0.5877	0.6511
	1.1553	1.1273	1.1014	1.0777	1.0358	1.0022	0.9766	0.9589	0.9494	0.9477	0.9480	0.9503	0.9547	0.9608	0.9692
	0.0017	0.0070	0.0181	0.0380	0.1133	0.2334	0.3778	0.5223	0.6513	0.7078	0.7589	0.8048	0.8461	0.8834	0.9173
$\nu=10, \lambda_1=3, \lambda_2=10$	0.0016	0.0066	0.0150	0.0277	0.0683	0.1327	0.2198	0.3245	0.4386	0.4967	0.5547	0.6117	0.6673	0.7213	0.7734
	0.0016	0.0066	0.0147	0.0261	0.0584	0.1036	0.1611	0.2302	0.3098	0.3534	0.3992	0.4472	0.4972	0.5489	0.6022
	0.0016	0.0066	0.0147	0.0261	0.0586	0.1038	0.1614	0.2311	0.3117	0.3559	0.4027	0.4516	0.5028	0.5559	0.6108
	0.0016	0.0066	0.0147	0.0259	0.0583	0.1030	0.1589	0.2252	0.3003	0.3408	0.3830	0.4267	0.4717	0.5180	0.5653
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0503	0.0725	0.0987	0.1133	0.1288	0.1455	0.1631	0.1817	0.2014
	0.9672	0.9426	0.9189	0.8959	0.8520	0.8111	0.7731	0.7381	0.7061	0.6911	0.6769	0.6635	0.6508	0.6388	0.6276
	0.0005	0.0021	0.0049	0.0101	0.0353	0.0897	0.1701	0.2616	0.3480	0.3860	0.4198	0.4491	0.4740	0.4946	0.5115
$\nu=100, \lambda_1=3, \lambda_2=10$	0.0005	0.0020	0.0046	0.0082	0.0199	0.0402	0.0723	0.1166	0.1707	0.1998	0.2295	0.2592	0.2883	0.3165	0.3433
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0503	0.0723	0.0982	0.1125	0.1277	0.1438	0.1607	0.1784	0.1968
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0503	0.0724	0.0983	0.1128	0.1281	0.1443	0.1614	0.1793	0.1980
	0.0005	0.0020	0.0046	0.0080	0.0181	0.0322	0.0501	0.0717	0.0968	0.1106	0.1249	0.1399	0.1555	0.1715	0.1879
	0.0005	0.0018	0.0041	0.0072	0.0162	0.0288	0.0450	0.0648	0.0882	0.1013	0.1152	0.1301	0.1458	0.1625	0.1801
	0.9532	0.9293	0.9061	0.8836	0.8404	0.8000	0.7622	0.7271	0.6947	0.6794	0.6649	0.6510	0.6378	0.6252	0.6133
	0.0005	0.0018	0.0043	0.0088	0.0312	0.0815	0.1581	0.2464	0.3308	0.3681	0.4013	0.4302	0.4547	0.4749	0.4913
$\nu=\infty, \lambda_1=3, \lambda_2=10$	0.0005	0.0018	0.0040	0.0073	0.0176	0.0356	0.0646	0.1055	0.1561	0.1837	0.2119	0.2402	0.2681	0.2951	0.3208
	0.0004	0.0018	0.0041	0.0072	0.0162	0.0288	0.0450	0.0647	0.0879	0.1007	0.1144	0.1289	0.1442	0.1601	0.1767
	0.0005	0.0018	0.0041	0.0072	0.0162	0.0288	0.0450	0.0647	0.0880	0.1009	0.1146	0.1293	0.1447	0.1609	0.1777
	0.0005	0.0018	0.0041	0.0072	0.0162	0.0288	0.0449	0.0643	0.0869	0.0993	0.1123	0.1258	0.1399	0.1544	0.1693
	0.0142	0.0567	0.1275	0.2269	0.5103	0.9072	1.4178	2.0417	2.7789	3.1900	3.6294	4.0972	4.5936	5.1181	5.6711
	1.9978	2.0797	2.1717	2.2742	2.5092	2.7853	3.1019	3.4592	3.8575	4.0717	4.2964	4.5311	4.7758	5.0311	5.2964
	0.0147	0.0675	0.1839	0.3792	0.9717	1.6847	2.3772	3.0039	3.5775	3.8531	4.1253	4.3975	4.6717	4.9494	5.2325
$\nu=5, \lambda_1=10, \lambda_2=10$	0.0142	0.0581	0.1369	0.2567	0.6294	1.1556	1.7758	2.4333	3.0928	3.4183	3.7408	4.0608	4.3792	4.6972	5.0158
	0.0142	0.0567	0.1275	0.2264	0.5064	0.8908	1.3700	1.9311	2.5611	2.8981	3.2475	3.6078	3.9778	4.3567	4.7427
	0.0142	0.0567	0.1275	0.2264	0.5075	0.8942	1.3789	1.9503	2.5961	2.9431	3.3039	3.6772	4.0617	4.4558	4.8589
	0.0142	0.0567	0.1272	0.2258	0.5017	0.8739	1.3278	1.8486	2.4225	2.7261	3.0389	3.3603	3.6892	4.0256	4.3683
	0.0039	0.0156	0.0352	0.0625	0.1406	0.2500	0.3906	0.5625	0.7656	0.8789	1.0000	1.1289	1.2656	1.4102	1.5625
	1.2017	1.2200	1.2400	1.2617	1.3103	1.3656	1.4277	1.4966	1.5723	1.6127	1.6548	1.6986	1.7442	1.7914	1.8403
	0.0039	0.0173	0.0475	0.1063	0.3284	0.6384	0.9428	1.1941	1.3905	1.4728	1.5475	1.6164	1.6813	1.7433	1.8034
$\nu=10, \lambda_1=10, \lambda_2=10$	0.0039	0.0158	0.0366	0.0686	0.1767	0.3514	0.5777	0.8248	1.0661	1.1795	1.2869	1.3883	1.4838	1.5739	1.6594
	0.0039	0.0156	0.0352	0.0625	0.1403	0.2483	0.3850	0.5480	0.7338	0.8341	0.9384	1.0464	1.1575	1.2711	1.3864
	0.0039	0.0156	0.0352	0.0625	0.1403	0.2488	0.3864	0.5513	0.7406	0.8434	0.9509	1.0628	1.1784	1.2972	1.4184
	0.0039	0.0156	0.0352	0.0623	0.1397	0.2456	0.3772	0.5298	0.6984	0.7873	0.8783	0.9708	1.0645	1.1591	1.2538
	0.0028	0.0113	0.0254	0.0452	0.1017	0.1809	0.2826	0.4070	0.5539	0.6359	0.7236	0.8168	0.9158	1.0203	1.1306
	1.0110	1.0305	1.0508	1.0721	1.1169	1.1653	1.2171	1.2723	1.3309	1.3614	1.3929	1.4251	1.4584	1.4923	1.5272
	0.0028	0.0118	0.0309	0.0702	0.2401	0.5018	0.7696	0.9924	1.1636	1.2334	1.2955	1.3512	1.4023	1.4500	1.4953
$\nu=100, \lambda_1=10, \lambda_2=10$	0.0028	0.0113	0.0258	0.0477	0.1231	0.2548	0.4375	0.6458	0.8529	0.9504	1.0425	1.1286	1.2090	1.2838	1.3536
	0.0028	0.0113	0.0254	0.0452	0.1017	0.1807	0.2815	0.4035	0.5448	0.6219	0.7029	0.7875	0.8749	0.9650	1.0572
	0.0028	0.0113	0.0254	0.0452	0.1017	0.1808	0.2819	0.4045	0.5475	0.6259	0.7087	0.7954	0.8858	0.9793	1.0755
	0.0028	0.0113	0.0254	0.0452	0.1016	0.1798	0.2786	0.3951	0.5260	0.5955	0.6671	0.7401	0.8142	0.8890	0.9641
	0.0028	0.0111	0.0250	0.0444	0.1000	0.1777	0.2777	0.3990	0.5443	0.6248	0.7109	0.8026	0.8998	1.0025	1.1108
	0.9969	1.0168	1.0375	1.0590	1.1046	1.1534	1.2056	1.2610	1.3197	1.3503	1.3817	1.4139	1.4470	1.4809	1.5156
	0.0028	0.0115	0.0300	0.0681	0.2349	0.4941	0.7604	0.9825	1.1533	1.2230	1.2848	1.3406	1.3915	1.4390	1.4840
$\nu=\infty, \lambda_1=10, \lambda_2=10$	0.0028	0.0111	0.0254	0.0467	0.1203	0.2497	0.4304	0.6370	0.8431	0.9402	1.0320	1.1179	1.1980	1.2726	1.3421
	0.0028	0.0111	0.0250	0.0444	0.0999	0.1775	0.2768	0.3969	0.5362	0.6125	0.6925	0.7761	0.8628	0.9521	1.0435
	0.0028	0.0111	0.0250	0.0444	0.1000	0.1776	0.2771	0.3978	0.5387	0.6162	0.6979	0.7836	0.8730	0.9654	1.0608
	0.0028	0.0111	0.0250	0.0444	0.0999	0.1768	0.2742	0.3892	0.5185	0.5874	0.6583	0.7307	0.8042	0.8785	0.9529

TABLE A6.2.3: Relative risks of s^2_{NM}, s^2_{AM}, and s^2_{PM}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	ϕ 0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0131	0.0525	0.1181	0.2100	0.4722	0.8394	1.3117	1.8889	2.5711	2.9514	3.3581	3.7908	4.2500	4.7353	5.2469
	0.9136	0.9903	1.0803	1.1836	1.4303	1.7303	2.0833	2.4897	2.9494	3.1994	3.4625	3.7392	4.0289	4.3319	4.6486
	0.0136	0.0653	0.1747	0.3444	0.7978	1.3028	1.8094	2.3183	2.8428	3.1153	3.3961	3.6864	3.9594	4.2989	4.6222
$\nu=5, \lambda_1=0, \lambda_2=0$	0.0131	0.0544	0.1289	0.2414	0.5758	1.0192	1.5258	2.0689	2.6392	2.9344	3.2369	3.5472	3.8661	4.1939	4.5314
	0.0131	0.0525	0.1181	0.2094	0.4678	0.8214	1.2608	1.7761	2.3572	2.6694	2.9950	3.3333	3.6833	4.0450	4.4172
	0.0131	0.0525	0.1181	0.2094	0.4686	0.8247	1.2697	1.7936	2.3869	2.7067	3.0406	3.3878	3.7472	4.1186	4.5011
	0.0131	0.0525	0.1178	0.2086	0.4633	0.8069	1.2281	1.7158	2.2619	2.5550	2.8606	3.1781	3.5075	3.8486	4.2008
	0.0016	0.0064	0.0144	0.0255	0.0573	0.1019	0.1592	0.2292	0.3119	0.3581	0.4073	0.4600	0.5156	0.5745	0.6366
	0.2033	0.1967	0.1930	0.1917	0.1977	0.2142	0.2419	0.2802	0.3294	0.3581	0.3895	0.4236	0.4603	0.4998	0.5420
	0.0017	0.0086	0.0236	0.0466	0.1034	0.1605	0.2138	0.2670	0.3242	0.3552	0.3883	0.4234	0.4609	0.5009	0.5436
$\nu=10, \lambda_1=0, \lambda_2=0$	0.0016	0.0067	0.0161	0.0303	0.0717	0.1247	0.1830	0.2445	0.3098	0.3444	0.3805	0.4183	0.4580	0.4997	0.5436
	0.0016	0.0064	0.0144	0.0253	0.0567	0.0995	0.1528	0.2156	0.2869	0.3256	0.3663	0.4088	0.4533	0.4995	0.5478
	0.0016	0.0064	0.0144	0.0255	0.0569	0.1000	0.1539	0.2177	0.2903	0.3297	0.3711	0.4144	0.4597	0.5069	0.5558
	0.0016	0.0064	0.0142	0.0253	0.0561	0.0977	0.1489	0.2088	0.2766	0.3136	0.3525	0.3934	0.4364	0.4816	0.5286
	0.0003	0.0014	0.0031	0.0054	0.0122	0.0216	0.0337	0.0486	0.0661	0.0759	0.0864	0.0976	0.1093	0.1218	0.1349
	0.0966	0.0861	0.0767	0.0685	0.0558	0.0480	0.0450	0.0468	0.0534	0.0585	0.0649	0.0724	0.0811	0.0911	0.1022
	0.0004	0.0022	0.0065	0.0126	0.0258	0.0354	0.0421	0.0483	0.0567	0.0622	0.0685	0.0760	0.0845	0.0943	0.1053
$\nu=100, \lambda_1=0, \lambda_2=0$	0.0003	0.0015	0.0036	0.0070	0.0163	0.0270	0.0375	0.0480	0.0597	0.0663	0.0736	0.0816	0.0907	0.1007	0.1116
	0.0003	0.0014	0.0030	0.0054	0.0120	0.0210	0.0324	0.0458	0.0614	0.0701	0.0793	0.0892	0.0998	0.1110	0.1230
	0.0003	0.0014	0.0030	0.0054	0.0121	0.0211	0.0326	0.0461	0.0620	0.0707	0.0800	0.0899	0.1004	0.1116	0.1235
	0.0003	0.0014	0.0030	0.0053	0.0119	0.0206	0.0314	0.0446	0.0599	0.0686	0.0779	0.0880	0.0989	0.1106	0.1231
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0278	0.0400	0.0544	0.0625	0.0711	0.0803	0.0900	0.1003	0.1111
	0.0904	0.0798	0.0704	0.0620	0.0488	0.0400	0.0357	0.0359	0.0406	0.0446	0.0498	0.0561	0.0635	0.0720	0.0816
	0.0003	0.0019	0.0056	0.0109	0.0220	0.0294	0.0339	0.0382	0.0443	0.0485	0.0536	0.0598	0.0670	0.0752	0.0846
$\nu=\infty, \lambda_1=0, \lambda_2=0$	0.0003	0.0012	0.0030	0.0058	0.0136	0.0223	0.0306	0.0388	0.0479	0.0532	0.0591	0.0658	0.0734	0.0818	0.0912
	0.0003	0.0011	0.0025	0.0044	0.0099	0.0173	0.0266	0.0377	0.0507	0.0579	0.0656	0.0739	0.0828	0.0924	0.1027
	0.0003	0.0011	0.0025	0.0044	0.0099	0.0174	0.0268	0.0380	0.0511	0.0583	0.0661	0.0744	0.0832	0.0927	0.1028
	0.0003	0.0011	0.0025	0.0044	0.0097	0.0169	0.0259	0.0367	0.0496	0.0569	0.0648	0.0733	0.0826	0.0927	0.1036
	0.0133	0.0536	0.1206	0.2144	0.4822	0.8572	1.3394	1.9289	2.6256	3.0139	3.4292	3.8711	4.3400	4.8356	5.3581
	0.9233	1.0086	1.1064	1.2161	1.4728	1.7783	2.1331	2.5369	2.9900	3.2347	3.4919	3.7614	4.0431	4.3372	4.6433
	0.0142	0.0697	0.1886	0.3703	0.8425	1.3547	1.8611	2.3650	2.8814	3.1483	3.4228	3.7061	3.9986	4.3014	4.6144
$\nu=5, \lambda_1=3, \lambda_2=0$	0.0136	0.0561	0.1342	0.2522	0.6003	1.0544	1.5653	2.1069	2.6703	2.9606	3.2569	3.5603	3.8708	4.1897	4.5172
	0.0133	0.0536	0.1203	0.2136	0.4767	0.8353	1.2789	1.7956	2.3747	2.6844	3.0067	3.3403	3.6844	4.0389	4.4033
	0.0133	0.0536	0.1206	0.2139	0.4778	0.8394	1.2892	1.8158	2.4089	2.7272	3.0586	3.4019	3.7567	4.1219	4.4972
	0.0133	0.0536	0.1203	0.2128	0.4714	0.8183	1.2406	1.7264	2.2667	2.5553	2.8556	3.1667	3.4886	3.8211	4.1639
	0.0019	0.0075	0.0169	0.0298	0.0673	0.1197	0.1869	0.2692	0.3664	0.4206	0.4786	0.5402	0.6056	0.6748	0.7477
	0.2106	0.2106	0.2127	0.2166	0.2305	0.2522	0.2820	0.3197	0.3653	0.3911	0.4189	0.4486	0.4803	0.5141	0.5497
	0.0022	0.0125	0.0353	0.0675	0.1380	0.2000	0.2536	0.3044	0.3572	0.3852	0.4145	0.4455	0.4780	0.5123	0.5486
$\nu=10, \lambda_1=3, \lambda_2=0$	0.0019	0.0083	0.0208	0.0398	0.0923	0.1533	0.2148	0.2756	0.3369	0.3684	0.4008	0.4344	0.4691	0.5052	0.5117
	0.0019	0.0075	0.0167	0.0297	0.0656	0.1134	0.1709	0.2358	0.3066	0.3439	0.3823	0.4219	0.4625	0.5041	0.5469
	0.0019	0.0075	0.0167	0.0297	0.0659	0.1145	0.1734	0.2403	0.3138	0.3523	0.3922	0.4333	0.4753	0.5184	0.5627
	0.0019	0.0075	0.0167	0.0294	0.0642	0.1092	0.1622	0.2216	0.2863	0.3205	0.3561	0.3928	0.4311	0.4706	0.5114
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0615	0.0886	0.1206	0.1384	0.1575	0.1778	0.1993	0.2221	0.2460
	0.1027	0.0975	0.0930	0.0891	0.0834	0.0803	0.0799	0.0820	0.0868	0.0902	0.0942	0.0989	0.1042	0.1103	0.1169
	0.0008	0.0058	0.0170	0.0309	0.0546	0.0680	0.0753	0.0807	0.0867	0.0904	0.0945	0.0993	0.1047	0.1108	0.1173
$\nu=100, \lambda_1=3, \lambda_2=0$	0.0006	0.0030	0.0081	0.0158	0.0348	0.0521	0.0652	0.0754	0.0847	0.0893	0.0943	0.0996	0.1054	0.1116	0.1184
	0.0006	0.0025	0.0055	0.0097	0.0209	0.0350	0.0504	0.0663	0.0822	0.0900	0.0978	0.1056	0.1134	0.1213	0.1294
	0.0006	0.0025	0.0055	0.0097	0.0211	0.0357	0.0520	0.0690	0.0861	0.0945	0.1029	0.1112	0.1194	0.1278	0.1361
	0.0006	0.0025	0.0054	0.0095	0.0199	0.0323	0.0454	0.0586	0.0722	0.0790	0.0861	0.0934	0.1009	0.1087	0.1168
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0356	0.0556	0.0800	0.1089	0.1250	0.1422	0.1606	0.1800	0.2006	0.2222
	0.0962	0.0910	0.0863	0.0822	0.0759	0.0718	0.0702	0.0708	0.0738	0.0762	0.0792	0.0827	0.0869	0.0916	0.0969
	0.0008	0.0055	0.0160	0.0290	0.0503	0.0614	0.0665	0.0700	0.0740	0.0766	0.0797	0.0833	0.0874	0.0921	0.0974
$\nu=\infty, \lambda_1=3, \lambda_2=0$	0.0006	0.0028	0.0075	0.0147	0.0319	0.0470	0.0579	0.0658	0.0727	0.0762	0.0799	0.0839	0.0883	0.0932	0.0986
	0.0006	0.0022	0.0050	0.0087	0.0188	0.0312	0.0447	0.0583	0.0715	0.0779	0.0844	0.0906	0.0969	0.1033	0.1096
	0.0006	0.0022	0.0050	0.0088	0.0190	0.0319	0.0462	0.0608	0.0752	0.0822	0.0891	0.0959	0.1025	0.1092	0.1159
	0.0006	0.0022	0.0049	0.0085	0.0178	0.0286	0.0398	0.0509	0.0621	0.0677	0.0734	0.0793	0.0854	0.0917	0.0983
	0.0161	0.0647	0.1458	0.2592	0.5833	1.0369	1.6203	2.3333	3.1758	3.6458	4.1481	4.6828	5.2500	5.8494	6.4814
	0.9469	1.0550	1.1747	1.3061	1.6036	1.9478	2.3383	2.7756	3.2592	3.5183	3.7892	4.0717	4.3656	4.6714	4.9886
	0.0183	0.0967	0.2558	0.4794	1.0142	1.5633	2.0961	2.6242	3.1639	3.4425	3.7286	4.0231	4.3267	4.6397	4.9631
$\nu=5, \lambda_1=10, \lambda_2=0$	0.0164	0.0706	0.1714	0.3208	0.7369	1.2478	1.8011	2.3747	2.9644	3.2664	3.5739	3.8878	4.2086	4.5372	4.8739
	0.0161	0.0647	0.1453	0.2572	0.5689	0.9850	1.4878	2.0603	2.6889	3.0208	3.3628	3.7144	4.0750	4.4439	4.8206
	0.0161	0.0647	0.1456	0.2578	0.5717	0.9936	1.5069	2.0950	2.7436	3.0867	3.4408	3.8047	4.1781	4.5597	4.9494
	0.0161	0.0647	0.1447	0.2550	0.5578	0.9536	1.4236	1.9536	2.5331	2.8392	3.1556	3.4817	3.8178	4.1631	4.5178
	0.0047	0.0188	0.0420	0.0748	0.1684	0.2994	0.4678	0.6736	0.9169	1.0525	1.1975	1.3519	1.5156	1.6888	1.8711
	0.2284	0.2466	0.2666	0.2888	0.3392	0.3980	0.4650	0.5405	0.6241	0.6689	0.7159	0.7578	0.8163	0.8694	0.9247
	0.0064	0.0391	0.0956	0.1589	0.2722	0.3647	0.4483	0.5317	0.6194	0.6656	0.7134	0.7631	0.8148	0.8683	0.9239
$\nu=10, \lambda_1=10, \lambda_2=0$	0.0050	0.0231	0.0575	0.1042	0.2109	0.3155	0.4134	0.5083	0.6041	0.6531	0.7033	0.7550	0.8081	0.8630	0.9195
	0.0047	0.0186	0.0416	0.0727	0.1552	0.2558	0.3664	0.4816	0.5984	0.6575	0.7167	0.7763	0.8364	0.8970	0.9586
	0.0047	0.0186	0.0417	0.0731	0.1572	0.2617	0.3777	0.4992	0.6227	0.6847	0.7467	0.8091	0.8714	0.9342	0.9973
	0.0047	0.0186	0.0411	0.0709	0.1472	0.2373	0.3350	0.4370	0.5430	0.5972	0.6527	0.7092	0.7669	0.8259	0.8864
	0.0034	0.0137	0.0308	0.0548	0.1233	0.2191	0.3424	0.4930	0.6711	0.7704	0.8765	0.9895	1.1093	1.2360	1.3695
	0.1175	0.1277	0.1390	0.1514	0.1798	0.2129	0.2506	0.2929	0.3399	0.3651	0.3914	0.4189	0.4475	0.4774	0.5083
	0.0052	0.0322	0.0733	0.1116	0.1672	0.2091	0.2494	0.2925	0.3397	0.3650	0.3913	0.4189	0.4475	0.4774	0.5083
$\nu=100, \lambda_1=10, \lambda_2=0$	0.0036	0.0180	0.0447	0.0779	0.1423	0.1959	0.2431	0.2897	0.3384	0.3641	0.3908	0.4185	0.4473	0.4772	0.5082
	0.0034	0.0136	0.0302	0.0525	0.1088	0.1729	0.2378	0.3003	0.3602	0.3894	0.4186	0.4478	0.4772	0.5071	0.5376
	0.0034	0.0136	0.0304	0.0529	0.1110	0.1787	0.2481	0.3153	0.3790	0.4098	0.4400	0.4700	0.4999	0.5299	0.5602
	0.0034	0.0135	0.0297	0.0506	0.1010	0.1561	0.2116	0.2670	0.3230	0.3515	0.3806	0.4102	0.4406	0.4719	0.5041
	0.0034	0.0135	0.0303	0.0538	0.1211	0.2153	0.3364	0.4844	0.6594	0.7569	0.8612	0.9723	1.0900	1.2145	1.3457
	0.1108	0.1206	0.1316	0.1437	0.1712	0.2033	0.2398	0.2808	0.3263	0.3508	0.3763	0.4030	0.4308	0.4597	0.4898
	0.0052	0.0319	0.0720	0.1087	0.1609	0.2004	0.2390	0.2806	0.3262	0.3507	0.3763	0.4030	0.4308	0.4597	0.4898
$\nu=\infty, \lambda_1=10, \lambda_2=0$	0.0036	0.0178	0.0441	0.0765	0.1385	0.1892	0.2340	0.2785	0.3254	0.3502	0.3759	0.4027	0.4308	0.4597	0.4898
	0.0034	0.0134	0.0297	0.0515	0.1065	0.1688	0.2313	0.2911	0.3482	0.3760	0.4038	0.4316	0.4598	0.4883	0.5176
	0.0034	0.0134	0.0298	0.0519	0.1088	0.1745	0.2416	0.3060	0.3667	0.3959	0.4246	0.4530	0.4815	0.5101	0.5390
	0.0034	0.0133	0.0291	0.0496	0.0987	0.1520	0.2054	0.2586	0.3123	0.3396	0.3676	0.3962	0.4255	0.4556	0.4866

TABLE A6.2.3 (continued)

Estimator	φ 0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0131	0.0525	0.1181	0.2100	0.4722	0.8394	1.3117	1.8889	2.5711	2.9514	3.3581	3.7908	4.2500	4.7353	5.2469
	1.1994	1.2608	1.3356	1.4233	1.6394	1.9086	2.2314	2.6072	3.0364	3.2708	3.5186	3.7800	4.0544	4.3422	4.6433
	0.0136	0.0633	0.1700	0.3392	0.8119	1.3544	1.8958	2.4242	2.9511	3.2192	3.4925	3.7731	4.0619	4.3600	4.6686
$\nu=5,\lambda_1=0,\lambda_2=3$	0.0131	0.0542	0.1278	0.2389	0.5750	1.0308	1.5586	2.1242	2.7114	3.0119	3.3175	3.6281	3.9450	4.2686	4.5994
	0.0131	0.0525	0.1181	0.2094	0.4683	0.8236	1.2667	1.7869	2.3747	2.6908	3.0203	3.3622	3.7161	4.0808	4.4561
	0.0131	0.0525	0.1181	0.2094	0.4692	0.8267	1.2744	1.8033	2.4033	2.7269	3.0647	3.4161	3.7800	4.1553	4.5417
	0.0131	0.0525	0.1178	0.2089	0.4647	0.8106	1.2356	1.7286	2.2803	2.5758	2.8836	3.2028	3.5333	3.8750	4.2272
	0.0016	0.0064	0.0144	0.0255	0.0573	0.1019	0.1592	0.2292	0.3119	0.3581	0.4073	0.4600	0.5156	0.5745	0.6366
	0.4291	0.4111	0.3956	0.3831	0.3659	0.3597	0.3642	0.3797	0.4059	0.4231	0.4430	0.4656	0.4909	0.5191	0.5497
	0.0016	0.0075	0.0206	0.0427	0.1100	0.1900	0.2664	0.3339	0.3944	0.4233	0.4520	0.4811	0.5111	0.5422	0.5750
$\nu=10,\lambda_1=0,\lambda_2=3$	0.0016	0.0066	0.0153	0.0288	0.0706	0.1300	0.2003	0.2752	0.3509	0.3888	0.4266	0.4645	0.5030	0.5417	0.5814
	0.0016	0.0064	0.0144	0.0255	0.0570	0.1006	0.1555	0.2205	0.2948	0.3352	0.3773	0.4214	0.4672	0.5145	0.5636
	0.0016	0.0064	0.0144	0.0255	0.0570	0.1008	0.1561	0.2220	0.2977	0.3388	0.3819	0.4269	0.4738	0.5223	0.5727
	0.0016	0.0064	0.0144	0.0253	0.0567	0.0994	0.1523	0.2145	0.2848	0.3227	0.3623	0.4036	0.4466	0.4911	0.5373
	0.0003	0.0014	0.0031	0.0054	0.0122	0.0216	0.0337	0.0486	0.0661	0.0759	0.0864	0.0976	0.1093	0.1218	0.1349
	0.2894	0.2695	0.2507	0.2331	0.2017	0.1751	0.1534	0.1364	0.1243	0.1201	0.1170	0.1152	0.1145	0.1152	0.1169
	0.0003	0.0016	0.0044	0.0097	0.0285	0.0530	0.0760	0.0935	0.1055	0.1100	0.1138	0.1172	0.1207	0.1242	0.1281
$\nu=100,\lambda_1=0,\lambda_2=3$	0.0003	0.0014	0.0032	0.0060	0.0153	0.0294	0.0469	0.0655	0.0836	0.0923	0.1006	0.1087	0.1166	0.1244	0.1322
	0.0003	0.0014	0.0030	0.0054	0.0121	0.0214	0.0334	0.0477	0.0643	0.0735	0.0831	0.0933	0.1039	0.1151	0.1267
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0216	0.0334	0.0479	0.0647	0.0739	0.0837	0.0940	0.1049	0.1162	0.1281
	0.0003	0.0014	0.0030	0.0054	0.0121	0.0213	0.0330	0.0469	0.0628	0.0715	0.0807	0.0904	0.1006	0.1112	0.1223
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0278	0.0400	0.0544	0.0625	0.0711	0.0803	0.0900	0.1003	0.1111
	0.2802	0.2604	0.2418	0.2243	0.1927	0.1655	0.1429	0.1247	0.1110	0.1059	0.1018	0.0989	0.0971	0.0965	0.0969
	0.0003	0.0013	0.0036	0.0081	0.0244	0.0460	0.0662	0.0813	0.0911	0.0945	0.0972	0.0996	0.1018	0.1041	0.1067
$\nu=\infty,\lambda_1=0,\lambda_2=3$	0.0003	0.0011	0.0026	0.0049	0.0126	0.0245	0.0393	0.0551	0.0703	0.0776	0.0845	0.0911	0.0975	0.1039	0.1102
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0177	0.0276	0.0394	0.0532	0.0608	0.0688	0.0773	0.0863	0.0956	0.1053
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0177	0.0276	0.0395	0.0535	0.0611	0.0693	0.0779	0.0869	0.0964	0.1063
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0176	0.0272	0.0388	0.0520	0.0593	0.0670	0.0751	0.0837	0.0926	0.1020
	0.0133	0.0536	0.1206	0.2144	0.4822	0.8572	1.3394	1.9289	2.6256	3.0139	3.4292	3.8711	4.3400	4.8356	5.3581
	1.2136	1.2883	1.3753	1.4742	1.7094	1.9936	2.3269	2.7094	3.1408	3.3750	3.6217	3.8803	4.1514	4.4347	4.7303
	0.0139	0.0667	0.1819	0.3650	0.8658	1.4219	1.9633	2.4853	3.0053	3.2697	3.5400	3.8172	4.1025	4.3969	4.7011
$\nu=5,\lambda_1=3,\lambda_2=3$	0.0133	0.0556	0.1319	0.2486	0.6003	1.0711	1.6064	2.1703	2.7494	3.0442	3.3428	3.6464	3.9558	4.2714	4.5944
	0.0133	0.0536	0.1206	0.2139	0.4778	0.8386	1.2864	1.8094	2.3964	2.7103	3.0367	3.3744	3.7225	4.0803	4.4475
	0.0133	0.0536	0.1206	0.2139	0.4786	0.8422	1.2956	1.8283	2.4292	2.7519	3.0878	3.4358	3.7947	4.1644	4.5433
	0.0133	0.0536	0.1203	0.2131	0.4731	0.8233	1.2503	1.7422	2.2889	2.5803	2.8828	3.1956	3.5189	3.8519	4.1944
	0.0019	0.0075	0.0169	0.0298	0.0673	0.1197	0.1869	0.2692	0.3664	0.4206	0.4786	0.5402	0.6056	0.6748	0.7477
	0.4409	0.4341	0.4292	0.4263	0.4264	0.4344	0.4503	0.4742	0.5061	0.5250	0.5459	0.5688	0.5936	0.6205	0.6492
	0.0020	0.0100	0.0294	0.0628	0.1584	0.2594	0.3453	0.4150	0.4745	0.5027	0.5305	0.5584	0.5870	0.6166	0.6473
$\nu=10,\lambda_1=3,\lambda_2=3$	0.0019	0.0078	0.0189	0.0366	0.0925	0.1688	0.2530	0.3358	0.4139	0.4513	0.4878	0.5238	0.5594	0.5952	0.6313
	0.0019	0.0075	0.0169	0.0298	0.0667	0.1169	0.1791	0.2511	0.3309	0.3733	0.4169	0.4616	0.5073	0.5539	0.6014
	0.0019	0.0075	0.0169	0.0298	0.0669	0.1175	0.1805	0.2542	0.3366	0.3803	0.4256	0.4723	0.5202	0.5689	0.6184
	0.0019	0.0075	0.0167	0.0297	0.0659	0.1144	0.1730	0.2392	0.3116	0.3495	0.3886	0.4286	0.4695	0.5114	0.5542
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0615	0.0886	0.1206	0.1384	0.1575	0.1778	0.1993	0.2221	0.2460
	0.3000	0.2900	0.2807	0.2721	0.2569	0.2442	0.2342	0.2268	0.2220	0.2206	0.2198	0.2198	0.2203	0.2216	0.2234
	0.0006	0.0034	0.0115	0.0267	0.0734	0.1231	0.1612	0.1859	0.2006	0.2057	0.2098	0.2133	0.2166	0.2197	0.2229
$\nu=100,\lambda_1=3,\lambda_2=3$	0.0006	0.0026	0.0064	0.0128	0.0350	0.0668	0.1017	0.1338	0.1605	0.1717	0.1817	0.1906	0.1987	0.2061	0.2129
	0.0006	0.0025	0.0055	0.0098	0.0220	0.0385	0.0589	0.0824	0.1080	0.1213	0.1348	0.1485	0.1622	0.1759	0.1895
	0.0006	0.0025	0.0055	0.0098	0.0220	0.0387	0.0595	0.0836	0.1103	0.1242	0.1386	0.1532	0.1678	0.1825	0.1973
	0.0006	0.0025	0.0055	0.0098	0.0218	0.0376	0.0564	0.0774	0.0993	0.1105	0.1217	0.1329	0.1440	0.1551	0.1662
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0356	0.0556	0.0800	0.1089	0.1250	0.1422	0.1606	0.1800	0.2006	0.2222
	0.2906	0.2808	0.2715	0.2629	0.2473	0.2341	0.2232	0.2147	0.2085	0.2063	0.2042	0.2037	0.2032	0.2034	0.2041
	0.0006	0.0032	0.0105	0.0247	0.0690	0.1161	0.1520	0.1747	0.1877	0.1918	0.1949	0.1974	0.1995	0.2015	0.2036
$\nu=\infty,\lambda_1=3,\lambda_2=3$	0.0006	0.0023	0.0058	0.0116	0.0321	0.0618	0.0943	0.1240	0.1484	0.1585	0.1674	0.1751	0.1820	0.1881	0.1938
	0.0006	0.0022	0.0050	0.0089	0.0198	0.0348	0.0533	0.0744	0.0975	0.1094	0.1216	0.1338	0.1460	0.1582	0.1702
	0.0006	0.0022	0.0050	0.0089	0.0199	0.0350	0.0537	0.0755	0.0995	0.1121	0.1251	0.1381	0.1513	0.1645	0.1776
	0.0006	0.0022	0.0050	0.0089	0.0196	0.0339	0.0510	0.0697	0.0894	0.0993	0.1093	0.1191	0.1289	0.1387	0.1483
	0.0161	0.0647	0.1458	0.2592	0.5833	1.0369	1.6203	2.3333	3.1758	3.6458	4.1481	4.6828	5.2500	5.8494	6.4814
	1.2478	1.3561	1.4758	1.6072	1.9047	2.2489	2.6394	3.0767	3.5600	3.8192	4.0900	4.3725	4.6667	4.9725	5.2897
	0.0172	0.0883	0.2478	0.4914	1.1028	1.7269	2.3128	2.8739	3.4339	3.7194	4.0106	4.3092	4.6158	4.9314	5.2567
$\nu=5,\lambda_1=10,\lambda_2=3$	0.0164	0.0683	0.1658	0.3161	0.7586	1.3194	1.9269	2.5469	3.1728	3.4892	3.8089	4.1333	4.4628	4.7986	5.1414
	0.0161	0.0647	0.1456	0.2583	0.5753	1.0039	1.5283	2.1303	2.7939	3.1442	3.5050	3.8756	4.2544	4.6414	5.0353
	0.0161	0.0647	0.1456	0.2586	0.5769	1.0100	1.5439	2.1611	2.8458	3.2086	3.5828	3.9678	4.3617	4.7639	5.1736
	0.0161	0.0647	0.1453	0.2569	0.5672	0.9783	1.4703	2.0261	2.6328	2.9528	3.2825	3.6214	3.9694	4.3264	4.6922
	0.0047	0.0188	0.0420	0.0748	0.1684	0.2994	0.4678	0.6736	0.9169	1.0525	1.1975	1.3519	1.5156	1.6888	1.8711
	0.4695	0.4914	0.5153	0.5413	0.5994	0.6658	0.7405	0.8236	0.9148	0.9636	1.0145	1.0673	1.1223	1.1794	1.2386
	0.0052	0.0300	0.0902	0.1802	0.3800	0.5469	0.6781	0.7906	0.8973	0.9506	1.0048	1.0602	1.1170	1.1753	1.2355
$\nu=10,\lambda_1=10,\lambda_2=3$	0.0047	0.0203	0.0516	0.1014	0.2455	0.4116	0.5695	0.7123	0.8434	0.9066	0.9688	1.0308	1.0930	1.1558	1.2195
	0.0047	0.0188	0.0420	0.0744	0.1648	0.2850	0.4273	0.5844	0.7491	0.8325	0.9163	0.9998	1.0831	1.1658	1.2480
	0.0047	0.0188	0.0420	0.0745	0.1656	0.2878	0.4344	0.5981	0.7720	0.8606	0.9500	1.0394	1.1284	1.2169	1.3047
	0.0047	0.0188	0.0419	0.0739	0.1614	0.2734	0.4014	0.5386	0.6803	0.7522	0.8244	0.8970	0.9700	1.0434	1.1173
	0.0034	0.0137	0.0308	0.0548	0.1233	0.2191	0.3424	0.4930	0.6711	0.7704	0.8765	0.9895	1.1093	1.2360	1.3695
	0.3255	0.3416	0.3589	0.3772	0.4175	0.4625	0.5121	0.5662	0.6251	0.6562	0.6885	0.7219	0.7565	0.7923	0.8291
	0.0037	0.0226	0.0693	0.1389	0.2869	0.4009	0.4846	0.5541	0.6197	0.6526	0.6861	0.7203	0.7554	0.7915	0.8287
$\nu=100,\lambda_1=10,\lambda_2=3$	0.0034	0.0150	0.0383	0.0762	0.1852	0.3067	0.4164	0.5103	0.5933	0.6323	0.6707	0.7087	0.7467	0.7849	0.8237
	0.0034	0.0136	0.0307	0.0545	0.1207	0.2081	0.3110	0.4227	0.5377	0.5950	0.6517	0.7075	0.7624	0.8160	0.8686
	0.0034	0.0136	0.0308	0.0546	0.1212	0.2103	0.3166	0.4339	0.5563	0.6180	0.6792	0.7397	0.7993	0.8576	0.9145
	0.0034	0.0136	0.0306	0.0540	0.1179	0.1988	0.2899	0.3855	0.4819	0.5298	0.5775	0.6247	0.6718	0.7187	0.7654
	0.0034	0.0135	0.0303	0.0538	0.1211	0.2153	0.3364	0.4844	0.6594	0.7569	0.8612	0.9723	1.0900	1.2145	1.3457
	0.3159	0.3318	0.3489	0.3671	0.4069	0.4512	0.5000	0.5533	0.6110	0.6416	0.6733	0.7061	0.7400	0.7751	0.8112
	0.0037	0.0222	0.0682	0.1366	0.2818	0.3931	0.4745	0.5423	0.6063	0.6385	0.6712	0.7047	0.7392	0.7745	0.8110
$\nu=\infty,\lambda_1=10,\lambda_2=3$	0.0034	0.0147	0.0377	0.0749	0.1822	0.3014	0.4088	0.5006	0.5815	0.6196	0.6570	0.6940	0.7311	0.7685	0.8065
	0.0034	0.0135	0.0302	0.0535	0.1186	0.2046	0.3056	0.4153	0.5282	0.5844	0.6400	0.6955	0.7483	0.8009	0.8524
	0.0034	0.0135	0.0302	0.0536	0.1191	0.2067	0.3112	0.4263	0.5466	0.6070	0.6671	0.7264	0.7847	0.8419	0.8977
	0.0034	0.0134	0.0301	0.0532	0.1159	0.1954	0.2848	0.3786	0.4732	0.5201	0.5668	0.6131	0.6592	0.7050	0.7508

TABLE A6.2.3 (continued)

Estimator	ϕ 0.05	0.10	0.15	0.20	0.30	0.40	0.50	0.60	0.70	0.75	0.80	0.85	0.90	0.95	1.00
	0.0131	0.0525	0.1181	0.2100	0.4722	0.8394	1.3117	1.8889	2.5711	2.9514	3.3581	3.7908	4.2500	4.7353	5.2469
	2.2233	2.2489	2.2878	2.3400	2.4847	2.6825	2.9336	3.2381	3.5958	3.7947	4.0067	4.2322	4.4711	4.7233	4.9886
	0.0133	0.0608	0.1600	0.3208	0.8003	1.4036	2.0478	2.6928	3.3269	3.6406	3.9531	4.2650	4.5783	4.8936	5.2122
$\nu=5, \lambda_1=0, \lambda_2=10$	0.0131	0.0536	0.1256	0.2333	0.5631	1.0272	1.5883	2.2111	2.8717	3.2117	3.5567	3.9061	4.2600	4.6181	4.9806
	0.0131	0.0525	0.1181	0.2094	0.4694	0.8275	1.2769	1.8089	2.4144	2.7422	3.0847	3.4417	3.8117	4.1942	4.5886
	0.0131	0.0525	0.1181	0.2097	0.4700	0.8297	1.2831	1.8222	2.4383	2.7728	3.1231	3.4886	3.8681	4.2608	4.6661
	0.0131	0.0525	0.1178	0.2092	0.4664	0.8172	1.2514	1.7594	2.3319	2.6397	2.9606	3.2944	3.6400	3.9969	4.3650
	0.0016	0.0064	0.0144	0.0255	0.0573	0.1019	0.1592	0.2292	0.3119	0.3581	0.4073	0.4600	0.5156	0.5745	0.6366
	1.3130	1.2681	1.2261	1.1867	1.1159	1.0561	1.0072	0.9689	0.9417	0.9320	0.9252	0.9209	0.9195	0.9208	0.9247
	0.0016	0.0067	0.0166	0.0334	0.0938	0.1905	0.3127	0.4459	0.5777	0.6405	0.7002	0.7567	0.8100	0.8600	0.9072
$\nu=10, \lambda_1=0, \lambda_2=10$	0.0016	0.0064	0.0147	0.0267	0.0641	0.1217	0.1995	0.2950	0.4033	0.4608	0.5197	0.5795	0.6398	0.7005	0.7608
	0.0016	0.0064	0.0144	0.0255	0.0572	0.1016	0.1581	0.2264	0.3061	0.3498	0.3963	0.4452	0.4964	0.5500	0.6058
	0.0016	0.0064	0.0144	0.0255	0.0572	0.1016	0.1583	0.2270	0.3072	0.3514	0.3984	0.4480	0.4998	0.5542	0.6109
	0.0016	0.0064	0.0144	0.0255	0.0572	0.1011	0.1569	0.2238	0.3009	0.3433	0.3877	0.4344	0.4830	0.5336	0.5861
	0.0003	0.0014	0.0031	0.0054	0.0122	0.0216	0.0337	0.0486	0.0661	0.0759	0.0864	0.0976	0.1093	0.1218	0.1349
	1.0962	1.0545	1.0138	0.9744	0.8992	0.8289	0.7634	0.7027	0.6469	0.6208	0.5959	0.5722	0.5497	0.5284	0.5083
	0.0003	0.0014	0.0031	0.0058	0.0163	0.0383	0.0750	0.1243	0.1803	0.2087	0.2362	0.2621	0.2862	0.3081	0.3276
$\nu=100, \lambda_1=0, \lambda_2=10$	0.0003	0.0014	0.0030	0.0054	0.0125	0.0234	0.0395	0.0615	0.0901	0.1064	0.1240	0.1425	0.1619	0.1817	0.2017
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0216	0.0337	0.0485	0.0660	0.0758	0.0862	0.0973	0.1089	0.1213	0.1343
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0216	0.0337	0.0485	0.0661	0.0758	0.0862	0.0974	0.1091	0.1214	0.1345
	0.0003	0.0014	0.0030	0.0054	0.0122	0.0216	0.0337	0.0485	0.0659	0.0756	0.0858	0.0967	0.1082	0.1203	0.1330
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0278	0.0400	0.0544	0.0625	0.0711	0.0803	0.0900	0.1003	0.1111
	1.0802	1.0390	0.9989	0.9600	0.8855	0.8155	0.7500	0.6890	0.6324	0.6059	0.5804	0.5561	0.5329	0.5108	0.4898
	0.0003	0.0011	0.0025	0.0047	0.0130	0.0308	0.0619	0.1055	0.1562	0.1823	0.2076	0.2319	0.2544	0.2749	0.2930
$\nu=\infty, \lambda_1=0, \lambda_2=10$	0.0003	0.0011	0.0025	0.0044	0.0102	0.0190	0.0319	0.0499	0.0737	0.0875	0.1026	0.1186	0.1353	0.1526	0.1700
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0278	0.0400	0.0545	0.0662	0.0711	0.0801	0.0898	0.1000	0.1108
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0278	0.0400	0.0544	0.0625	0.0711	0.0801	0.0899	0.1002	0.1109
	0.0003	0.0011	0.0025	0.0044	0.0100	0.0178	0.0277	0.0399	0.0543	0.0622	0.0708	0.0799	0.0894	0.0994	0.1100
	0.0133	0.0536	0.1206	0.2144	0.4822	0.8572	1.3394	1.9289	2.6256	3.0139	3.4292	3.8711	4.3400	4.8356	5.3581
	2.2483	2.2978	2.3597	2.4339	2.6189	2.8531	3.1364	3.4689	3.8506	4.0597	4.2811	4.5147	4.7608	5.0192	5.2897
	0.0139	0.0631	0.1675	0.3394	0.8567	1.5044	2.1833	2.8444	3.4764	3.7831	4.0856	4.3858	4.6856	4.9869	5.2914
$\nu=5, \lambda_1=3, \lambda_2=10$	0.0133	0.0550	0.1289	0.2406	0.5847	1.0708	1.6561	2.2994	2.9725	3.3144	3.6586	4.0047	4.3522	4.7017	5.0531
	0.0133	0.0536	0.1206	0.2139	0.4792	0.8442	1.3011	1.8406	2.4525	2.7825	3.1267	3.4844	3.8542	4.2356	4.6275
	0.0133	0.0536	0.1206	0.2142	0.4797	0.8467	1.3081	1.8553	2.4792	2.8164	3.1694	3.5367	3.9172	4.3100	4.7142
	0.0133	0.0536	0.1203	0.2136	0.4758	0.8325	1.2728	1.7856	2.3600	2.6678	2.9878	3.3192	3.6614	4.0139	4.3761
	0.0019	0.0075	0.0169	0.0298	0.0673	0.1197	0.1869	0.2692	0.3664	0.4206	0.4786	0.5402	0.6056	0.6748	0.7477
	1.3356	1.3127	1.2917	1.2727	1.2406	1.2166	1.2003	1.1922	1.1919	1.1947	1.1995	1.2063	1.2150	1.2258	1.2386
	0.0019	0.0081	0.0209	0.0442	0.1334	0.2781	0.4545	0.6339	0.7970	0.8695	0.9358	0.9958	1.0502	1.0995	1.1447
$\nu=10, \lambda_1=3, \lambda_2=10$	0.0019	0.0075	0.0173	0.0320	0.0794	0.1555	0.2600	0.3867	0.5263	0.5980	0.6697	0.7406	0.8102	0.8777	0.9431
	0.0019	0.0075	0.0169	0.0298	0.0672	0.1191	0.1852	0.2645	0.3564	0.4067	0.4595	0.5150	0.5727	0.6325	0.6942
	0.0019	0.0075	0.0169	0.0298	0.0672	0.1192	0.1855	0.2655	0.3584	0.4094	0.4631	0.5197	0.5786	0.6400	0.7036
	0.0019	0.0075	0.0169	0.0298	0.0670	0.1184	0.1831	0.2600	0.3475	0.3948	0.4442	0.4955	0.5484	0.6030	0.6589
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0615	0.0886	0.1206	0.1384	0.1575	0.1778	0.1993	0.2221	0.2460
	1.1176	1.0964	1.0760	1.0562	1.0186	0.9837	0.9514	0.9217	0.8946	0.8820	0.8702	0.8589	0.8483	0.8384	0.8291
	0.0006	0.0025	0.0060	0.0124	0.0428	0.1102	0.2108	0.3269	0.4393	0.4898	0.5355	0.5760	0.6112	0.6414	0.6668
$\nu=100, \lambda_1=3, \lambda_2=10$	0.0006	0.0025	0.0055	0.0100	0.0244	0.0495	0.0895	0.1456	0.2147	0.2524	0.2911	0.3302	0.3688	0.4065	0.4427
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0614	0.0884	0.1201	0.1375	0.1562	0.1759	0.1966	0.2182	0.2409
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0614	0.0884	0.1202	0.1379	0.1566	0.1765	0.1974	0.2194	0.2424
	0.0006	0.0025	0.0055	0.0099	0.0222	0.0394	0.0613	0.0879	0.1188	0.1358	0.1536	0.1722	0.1917	0.2118	0.2325
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0356	0.0556	0.0800	0.1089	0.1250	0.1422	0.1606	0.1800	0.2006	0.2222
	1.1013	1.0808	1.0608	1.0414	1.0044	0.9698	0.9375	0.9076	0.8799	0.8670	0.8547	0.8429	0.8318	0.8212	0.8112
	0.0006	0.0022	0.0053	0.0109	0.0384	0.1006	0.1965	0.3089	0.4186	0.4683	0.5133	0.5532	0.5880	0.6175	0.6425
$\nu=\infty, \lambda_1=3, \lambda_2=10$	0.0006	0.0022	0.0050	0.0091	0.0217	0.0442	0.0806	0.1326	0.1976	0.2334	0.2704	0.3077	0.3448	0.3812	0.4161
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0355	0.0555	0.0798	0.1085	0.1244	0.1413	0.1592	0.1781	0.1978	0.2184
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0356	0.0555	0.0799	0.1087	0.1246	0.1415	0.1596	0.1787	0.1987	0.2195
	0.0006	0.0022	0.0050	0.0089	0.0200	0.0355	0.0554	0.0796	0.1075	0.1230	0.1392	0.1562	0.1741	0.1923	0.2113
	0.0161	0.0647	0.1458	0.2592	0.5833	1.0369	1.6203	2.3333	3.1758	3.6458	4.1481	4.6828	5.2500	5.8494	6.4814
	2.3075	2.4156	2.5353	2.6667	2.9642	3.3083	3.6989	4.1361	4.6197	4.8789	5.1497	5.4322	5.7261	6.0319	6.3492
	0.0167	0.0778	0.2125	0.4406	1.1394	1.9897	2.8222	3.5800	4.2744	4.6078	4.9367	5.2647	5.5944	5.9283	6.2675
$\nu=5, \lambda_1=10, \lambda_2=10$	0.0164	0.0667	0.1572	0.2958	0.7317	1.3533	2.0925	2.8811	3.6750	4.0678	4.4567	4.8425	5.2261	5.6089	5.9919
	0.0161	0.0647	0.1458	0.2589	0.5792	1.0194	1.5694	2.2153	2.9431	3.3333	3.7386	4.1575	4.5886	5.0308	5.4822
	0.0161	0.0647	0.1458	0.2589	0.5800	1.0231	1.5786	2.2356	2.9797	3.3803	3.7978	4.2306	4.6769	5.1353	5.6047
	0.0161	0.0647	0.1456	0.2581	0.5750	1.0039	1.5306	2.1386	2.8133	3.1714	3.5417	3.9228	4.3139	4.7142	5.1231
	0.0047	0.0188	0.0420	0.0748	0.1684	0.2994	0.4678	0.6736	0.9169	1.0525	1.1975	1.3519	1.5156	1.6888	1.8711
	1.3892	1.4200	1.4528	1.4877	1.5636	1.6480	1.7405	1.8414	1.9506	2.0083	2.0680	2.1298	2.1938	2.2597	2.3278
	0.0047	0.0208	0.0569	0.1273	0.3958	0.7747	1.1528	1.4711	1.7255	1.8341	1.9336	2.0261	2.1139	2.1981	2.2803
$\nu=10, \lambda_1=10, \lambda_2=10$	0.0047	0.0189	0.0439	0.0823	0.2133	0.4270	0.7070	1.0167	1.3231	1.4686	1.6072	1.7389	1.8639	1.9827	2.0958
	0.0047	0.0188	0.0420	0.0748	0.1680	0.2975	0.4617	0.6578	0.8823	1.0038	1.1306	1.2622	1.3978	1.5369	1.6789
	0.0047	0.0188	0.0420	0.0748	0.1681	0.2980	0.4631	0.6614	0.8895	1.0138	1.1441	1.2797	1.4202	1.5648	1.7133
	0.0047	0.0188	0.0420	0.0747	0.1675	0.2952	0.4545	0.6411	0.8494	0.9598	1.0738	1.1905	1.3092	1.4297	1.5513
	0.0034	0.0137	0.0308	0.0548	0.1233	0.2191	0.3424	0.4930	0.6711	0.7704	0.8765	0.9895	1.1093	1.2360	1.3695
	1.1681	1.1980	1.2292	1.2613	1.3293	1.4020	1.4793	1.5611	1.6476	1.6926	1.7388	1.7860	1.8344	1.8841	1.9348
	0.0034	0.0143	0.0374	0.0847	0.2902	0.6092	0.9404	1.2213	1.4426	1.5350	1.6180	1.6939	1.7643	1.8306	1.8942
$\nu=100, \lambda_1=10, \lambda_2=10$	0.0034	0.0137	0.0313	0.0578	0.1496	0.3113	0.5376	0.7983	1.0609	1.1860	1.3050	1.4174	1.5231	1.6224	1.7160
	0.0034	0.0137	0.0308	0.0548	0.1232	0.2189	0.3412	0.4892	0.6610	0.7551	0.8540	0.9574	1.0648	1.1754	1.2892
	0.0034	0.0137	0.0308	0.0548	0.1233	0.2190	0.3415	0.4903	0.6639	0.7594	0.8602	0.9661	1.0763	1.1909	1.3089
	0.0034	0.0137	0.0308	0.0548	0.1231	0.2181	0.3385	0.4817	0.6438	0.7306	0.8206	0.9132	1.0076	1.1036	1.2006
	0.0034	0.0135	0.0303	0.0538	0.1211	0.2153	0.3364	0.4844	0.6594	0.7569	0.8612	0.9723	1.0900	1.2145	1.3457
	1.1516	1.1818	1.2132	1.2457	1.3141	1.3869	1.4643	1.5461	1.6324	1.6773	1.7233	1.7704	1.8186	1.8679	1.9184
	0.0034	0.0140	0.0363	0.0822	0.2838	0.5996	0.9286	1.2082	1.4287	1.5207	1.6034	1.6790	1.7491	1.8151	1.8782
$\nu=\infty, \lambda_1=10, \lambda_2=10$	0.0034	0.0135	0.0307	0.0566	0.1463	0.3050	0.5285	0.7870	1.0481	1.1724	1.2910	1.4028	1.5081	1.6069	1.7000
	0.0034	0.0135	0.0303	0.0538	0.1211	0.2151	0.3355	0.4812	0.6506	0.7435	0.8411	0.9434	1.0496	1.1593	1.2719
	0.0034	0.0135	0.0303	0.0538	0.1211	0.2151	0.3358	0.4822	0.6532	0.7474	0.8469	0.9514	1.0605	1.1736	1.2905
	0.0034	0.0135	0.0303	0.0538	0.1210	0.2145	0.3331	0.4741	0.6344	0.7203	0.8094	0.9008	0.9945	1.0897	1.1858

TABLE A6.2.4: Relative risks of s^2_{NL}, s^2_{AL}, and s^2_{PL}.

$\nu_1 = 16$, $\nu_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

	λ_1														
Estimator	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	6.5972	6.6272	6.6650	6.7106	6.7639	6.8250	6.8942	6.9708	7.0556	7.2483	7.4722	7.7275	8.0139	8.3317	8.6806
	6.2500	6.2633	6.2800	6.3003	6.3242	6.3514	6.3819	6.4161	6.4536	6.5394	6.6389	6.7522	6.8797	7.0208	7.1758
	6.2086	6.2211	6.2372	6.2572	6.2811	6.3086	6.3394	6.3739	6.4119	6.4986	6.5989	6.7133	6.8417	6.9836	7.1397
$\nu=5,\phi=1,\lambda_2=0$	6.0661	6.0764	6.0914	6.1108	6.1344	6.1622	6.1936	6.2289	6.2678	6.3561	6.4589	6.5758	6.7064	6.8511	7.0094
	5.6242	5.6306	5.6428	5.6603	5.6828	5.7103	5.7425	5.7792	5.8203	5.9147	6.0250	6.1503	6.2906	6.4450	6.6136
	5.7144	5.7222	5.7361	5.7556	5.7803	5.8100	5.8444	5.8836	5.9272	6.0269	6.1425	6.2736	6.4192	6.5789	6.7525
	5.3086	5.3081	5.3131	5.3231	5.3381	5.3578	5.3822	5.4111	5.4444	5.5233	5.6183	5.7292	5.8553	5.9964	6.1522
	0.7813	0.8047	0.8359	0.8750	0.9219	0.9766	1.0391	1.1094	1.1875	1.3672	1.5781	1.8203	2.0938	2.3984	2.7344
	0.6944	0.7048	0.7188	0.7361	0.7569	0.7813	0.8091	0.8403	0.8750	0.9548	1.0486	1.1563	1.2778	1.4131	1.5625
	0.6938	0.7033	0.7167	0.7338	0.7545	0.7788	0.8064	0.8378	0.8725	0.9527	1.0466	1.1544	1.2761	1.4117	1.5613
$\nu=10,\phi=1,\lambda_2=0$	0.6863	0.6942	0.7064	0.7228	0.7431	0.7672	0.7948	0.8263	0.8613	0.9420	1.0366	1.1453	1.2677	1.4041	1.5541
	0.6528	0.6580	0.6683	0.6834	0.7031	0.7273	0.7558	0.7883	0.8250	0.9098	1.0098	1.1241	1.2523	1.3945	1.5502
	0.6639	0.6703	0.6820	0.6986	0.7198	0.7458	0.7759	0.8103	0.8488	0.9375	1.0411	1.1589	1.2908	1.4361	1.5948
	0.5970	0.5970	0.6020	0.6117	0.6258	0.6444	0.6673	0.6944	0.7255	0.7998	0.8895	0.9945	1.1142	1.2484	1.3972
	0.1250	0.1445	0.1719	0.2070	0.2500	0.3008	0.3594	0.4258	0.5000	0.6719	0.8750	1.1094	1.3750	1.6719	2.0000
	0.0833	0.0920	0.1042	0.1198	0.1389	0.1615	0.1875	0.2170	0.2500	0.3264	0.4167	0.4208	0.6389	0.7708	0.9167
	0.0857	0.0937	0.1054	0.1207	0.1396	0.1619	0.1878	0.2172	0.2501	0.3264	0.4167	0.5208	0.6389	0.7709	0.9167
$\nu=\infty,\phi=1,\lambda_2=0$	0.0902	0.0969	0.1076	0.1221	0.1403	0.1624	0.1879	0.2172	0.2500	0.3261	0.4164	0.5206	0.6387	0.7707	0.9165
	0.0965	0.1009	0.1101	0.1237	0.1417	0.1639	0.1901	0.2203	0.2542	0.3333	0.4268	0.5341	0.6553	0.7897	0.9376
	0.0985	0.1039	0.1142	0.1291	0.1483	0.1719	0.1994	0.2309	0.2663	0.3481	0.4443	0.5541	0.6773	0.8136	0.9629
	0.0931	0.0959	0.1031	0.1150	0.1310	0.1513	0.1757	0.2039	0.2361	0.3118	0.4023	0.5073	0.6265	0.7598	0.9069
	6.5972	6.6272	6.6650	6.7106	6.7639	6.8250	6.8942	6.9708	7.0556	7.2483	7.4722	7.7275	8.0139	8.3317	8.6806
	6.3819	6.4161	6.4536	6.4947	6.5394	6.5875	6.6389	6.6939	6.7522	6.8797	7.0208	7.1758	7.3450	7.5278	7.7244
	6.3839	6.4008	6.4258	6.4575	6.4956	6.5389	6.5875	6.6406	6.6981	6.8247	6.9667	7.1231	7.2936	7.4781	7.6767
$\nu=5,\phi=1,\lambda_2=3$	6.2383	6.2486	6.2667	6.2919	6.3242	6.3628	6.4075	6.4575	6.5131	6.6383	6.7808	6.9397	7.1133	7.3019	7.5044
	5.6947	5.7031	5.7181	5.7394	5.7672	5.8011	5.8411	5.8869	5.9386	6.0586	6.2000	6.3617	6.5422	6.7406	6.9561
	5.7844	5.7950	5.8117	5.8353	5.8650	5.9014	5.9436	5.9919	6.0464	6.1722	6.3200	6.4886	6.6767	6.8833	7.1072
	5.3975	5.3997	5.4086	5.4233	5.4444	5.4711	5.5039	5.5422	5.5858	5.6894	5.8133	5.9567	6.1183	6.2975	6.4936
	0.7813	0.8047	0.8359	0.8750	0.9219	0.9766	1.0391	1.1094	1.1875	1.3672	1.5781	1.8203	2.0938	2.3984	2.7344
	0.8091	0.8403	0.8750	0.9131	0.9548	1.0000	1.0486	1.1006	1.1563	1.2778	1.4131	1.5625	1.7256	1.9028	2.0938
	0.8205	0.8442	0.8736	0.9081	0.9473	0.9909	1.0388	1.0905	1.1459	1.2681	1.4044	1.5547	1.7189	1.8969	2.0888
$\nu=10,\phi=1,\lambda_2=3$	0.7959	0.8150	0.8403	0.8716	0.9083	0.9502	0.9969	1.0481	1.1036	1.2270	1.3658	1.5191	1.6864	1.8675	2.0622
	0.6898	0.7017	0.7202	0.7447	0.7753	0.8119	0.8542	0.9020	0.9553	1.0777	1.2195	1.3802	1.5583	1.7528	1.9630
	0.7006	0.7138	0.7336	0.7597	0.7919	0.8303	0.8745	0.9247	0.9803	1.1077	1.2555	1.4225	1.6077	1.8097	2.0278
	0.6477	0.6555	0.6694	0.6891	0.7144	0.7453	0.7817	0.8234	0.8702	0.9786	1.1058	1.2509	1.4133	1.5919	1.7863
	0.1250	0.1445	0.1719	0.2070	0.2500	0.3008	0.3594	0.4258	0.5000	0.6719	0.8750	1.1094	1.3750	1.6719	2.0000
	0.1875	0.2170	0.2500	0.2865	0.3264	0.3698	0.4167	0.4670	0.5208	0.6389	0.7708	0.9167	1.0764	1.2500	1.4375
	0.1829	0.2110	0.2432	0.2794	0.3195	0.3631	0.4105	0.4614	0.5158	0.6349	0.7679	0.9146	1.0748	1.2489	1.4368
$\nu=\infty,\phi=1,\lambda_2=3$	0.1618	0.1865	0.2165	0.2514	0.2909	0.3346	0.3827	0.4245	0.4900	0.6123	0.7487	0.8985	1.0619	1.2386	1.4286
	0.1140	0.1279	0.1482	0.1744	0.2068	0.2448	0.2884	0.3375	0.3918	0.5153	0.6576	0.8177	0.9944	1.1867	1.3940
	0.1158	0.1305	0.1517	0.1792	0.2129	0.2525	0.2979	0.3488	0.4052	0.5335	0.6814	0.8477	1.0312	1.2306	1.4452
	0.1096	0.1215	0.1395	0.1632	0.1925	0.2273	0.2674	0.3125	0.3627	0.4771	0.6098	0.7594	0.9257	1.1073	1.3041
	1.6494	1.6567	1.6661	1.6775	1.6911	1.7064	1.7236	1.7428	1.7639	1.8119	1.8681	1.9319	2.0036	2.0828	2.1703
	2.8936	2.9083	2.9242	2.9408	2.9583	2.9767	2.9958	3.0161	3.0369	3.0817	3.1297	3.1811	3.2361	3.2944	3.3564
	2.4906	2.5058	2.5228	2.5414	2.5614	2.5825	2.6047	2.6283	2.6528	2.7044	2.7597	2.8183	2.8803	2.9456	3.0142
$\nu=5,\phi=0.5,\lambda_2=0$	2.0650	2.0772	2.0917	2.1078	2.1258	2.1453	2.1667	2.1892	2.2131	2.2644	2.3206	2.3808	2.4450	2.5131	2.5844
	1.5911	1.5969	1.6042	1.6136	1.6244	1.6369	1.6514	1.6672	1.6847	1.7242	1.7694	1.8203	1.8764	1.9375	2.0036
	1.6003	1.6064	1.6142	1.6236	1.6350	1.6478	1.6625	1.6789	1.6969	1.7375	1.7839	1.8364	1.8939	1.9569	2.0247
	1.5667	1.5717	1.5783	1.5867	1.5969	1.6086	1.6219	1.6367	1.6531	1.6900	1.7325	1.7803	1.8331	1.8906	1.9528
	0.1953	0.2013	0.2091	0.2188	0.2305	0.2442	0.2598	0.2773	0.2969	0.3419	0.3945	0.4552	0.5234	0.5997	0.6836
	0.3617	0.3730	0.3852	0.3981	0.4120	0.4267	0.4423	0.4589	0.4763	0.5136	0.5544	0.5986	0.6464	0.6977	0.7523
	0.3095	0.3209	0.3338	0.3478	0.3630	0.3792	0.3964	0.4145	0.4336	0.4744	0.5184	0.5658	0.6164	0.6703	0.7273
$\nu=10,\phi=0.5,\lambda_2=0$	0.2522	0.2616	0.2725	0.2852	0.2992	0.3147	0.3316	0.3495	0.3686	0.4102	0.4558	0.5050	0.5578	0.6141	0.6738
	0.1869	0.1913	0.1975	0.2052	0.2147	0.2256	0.2381	0.2522	0.2678	0.3031	0.3438	0.3897	0.4405	0.4959	0.5559
	0.1883	0.1928	0.1992	0.2073	0.2170	0.2283	0.2413	0.2558	0.2717	0.3081	0.3502	0.3975	0.4498	0.5070	0.5688
	0.1833	0.1873	0.1928	0.2000	0.2088	0.2189	0.2306	0.2438	0.2581	0.2913	0.3294	0.3725	0.4203	0.4727	0.5292
	0.0313	0.0361	0.0430	0.0518	0.0625	0.0752	0.0898	0.1064	0.1250	0.1680	0.2188	0.2773	0.3438	0.4180	0.5000
	0.0694	0.0786	0.0885	0.0994	0.1111	0.1237	0.1372	0.1515	0.1667	0.1997	0.2361	0.2760	0.3194	0.3663	0.4167
	0.0584	0.0675	0.0778	0.0891	0.1014	0.1147	0.1289	0.1440	0.1599	0.1942	0.2318	0.2727	0.3169	0.3644	0.4153
$\nu=\infty,\phi=0.5,\lambda_2=0$	0.0451	0.0526	0.0616	0.0720	0.0838	0.0968	0.1109	0.1261	0.1424	0.1778	0.2169	0.2595	0.3054	0.3545	0.4069
	0.0290	0.0327	0.0380	0.0449	0.0534	0.0634	0.0749	0.0879	0.1022	0.1350	0.1729	0.2158	0.2632	0.3151	0.3710
	0.0293	0.0332	0.0387	0.0459	0.0547	0.0650	0.0769	0.0902	0.1051	0.1389	0.1781	0.2223	0.2713	0.3249	0.3827
	0.0280	0.0313	0.0362	0.0425	0.0504	0.0597	0.0704	0.0824	0.0959	0.1265	0.1620	0.2023	0.2469	0.2959	0.3488
	1.6494	1.6567	1.6661	1.6775	1.6911	1.7064	1.7236	1.7428	1.7639	1.8119	1.8681	1.9319	2.0036	2.0828	2.1703
	3.1644	3.1897	3.2158	3.2428	3.2708	3.2997	3.3292	3.3597	3.3911	3.4567	3.5256	3.5978	3.6736	3.7528	3.8356
	2.6231	2.6428	2.6650	2.6900	2.7169	2.7464	2.7778	2.8111	2.8464	2.9214	3.0022	3.0875	3.1769	3.2700	3.3661
$\nu=5,\phi=0.5,\lambda_2=3$	2.1100	2.1250	2.1422	2.1614	2.1831	2.2069	2.2331	2.2614	2.2917	2.3586	2.4328	2.5139	2.6014	2.6939	2.7917
	1.5989	1.6050	1.6131	1.6231	1.6347	1.6483	1.6636	1.6808	1.6997	1.7433	1.7936	1.8511	1.9153	1.9861	2.0633
	1.6072	1.6136	1.6217	1.6319	1.6439	1.6578	1.6733	1.6908	1.7103	1.7544	1.8056	1.8642	1.9294	2.0014	2.0803
	1.5764	1.5833	1.5894	1.5986	1.6097	1.6225	1.6372	1.6536	1.6717	1.7133	1.7614	1.8164	1.8778	1.9453	2.0192
	0.1953	0.2013	0.2091	0.2188	0.2305	0.2442	0.2598	0.2773	0.2969	0.3419	0.3945	0.4552	0.5234	0.5997	0.6836
	0.5805	0.6022	0.6247	0.6481	0.6725	0.6977	0.7236	0.7506	0.7783	0.8366	0.8981	0.9633	1.0319	1.1039	1.1794
	0.3914	0.4120	0.4350	0.4600	0.4870	0.5159	0.5466	0.5786	0.6119	0.6822	0.7566	0.8342	0.9150	0.9984	1.0844
$\nu=10,\phi=0.5,\lambda_2=3$	0.2759	0.2894	0.3050	0.3231	0.3436	0.3663	0.3909	0.4180	0.4467	0.5102	0.5803	0.6564	0.7378	0.8241	0.9144
	0.1909	0.1961	0.2033	0.2122	0.2231	0.2358	0.2503	0.2666	0.2847	0.3263	0.3747	0.4300	0.4920	0.5606	0.6355
	0.1917	0.1970	0.2042	0.2134	0.2244	0.2372	0.2519	0.2684	0.2869	0.3289	0.3781	0.4342	0.4973	0.5670	0.6431
	0.1886	0.1934	0.2003	0.2089	0.2194	0.2316	0.2455	0.2613	0.2786	0.3186	0.3653	0.4184	0.4780	0.5434	0.6150
	0.0313	0.0361	0.0430	0.0518	0.0625	0.0752	0.0898	0.1064	0.1250	0.1680	0.2188	0.2773	0.3438	0.4180	0.5000
	0.2569	0.2765	0.2969	0.3181	0.3403	0.3633	0.3872	0.4119	0.4375	0.4913	0.5486	0.6094	0.6736	0.7413	0.8125
	0.1099	0.1311	0.1544	0.1797	0.2068	0.2354	0.2653	0.2965	0.3288	0.3961	0.4666	0.5397	0.6151	0.6928	0.7726
$\nu=\infty,\phi=0.5,\lambda_2=3$	0.0566	0.0687	0.0834	0.1007	0.1202	0.1421	0.1661	0.1921	0.2202	0.2815	0.3492	0.4224	0.5005	0.5828	0.6687
	0.0309	0.0355	0.0421	0.0505	0.0608	0.0729	0.0869	0.1026	0.1202	0.1606	0.2080	0.2621	0.3229	0.3900	0.4634
	0.0309	0.0357	0.0423	0.0507	0.0611	0.0734	0.0874	0.1034	0.1211	0.1620	0.2100	0.2649	0.3266	0.3948	0.4695
	0.0306	0.0351	0.0414	0.0496	0.0596	0.0714	0.0850	0.1003	0.1173	0.1565	0.2022	0.2544	0.3127	0.3770	0.4471

TABLE A6.2.4 (continued)

Estimator						λ_1									
	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0661	0.0664	0.0667	0.0672	0.0678	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0803	0.0833	0.0869
	1.3750	1.3792	1.3836	1.3881	1.3925	1.3969	1.4014	1.4058	1.4103	1.4194	1.4289	1.4383	1.4481	1.4578	1.4675
	0.0847	0.0858	0.0869	0.0881	0.0897	0.0911	0.0931	0.0950	0.0969	0.1017	0.1069	0.1131	0.1200	0.1272	0.1353
$\nu=5, \phi=0.1, \lambda_2=0$	0.0689	0.0694	0.0700	0.0706	0.0711	0.0719	0.0728	0.0739	0.0750	0.0775	0.0803	0.0839	0.0875	0.0919	0.0954
	0.0658	0.0664	0.0667	0.0672	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0867
	0.0658	0.0664	0.0667	0.0672	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0867
	0.0658	0.0661	0.0667	0.0669	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0857
	0.0078	0.0081	0.0084	0.0088	0.0092	0.0098	0.0105	0.0111	0.0119	0.0138	0.0158	0.0183	0.0209	0.0241	0.0273
	0.2830	0.2863	0.2895	0.2928	0.2961	0.2995	0.3028	0.3063	0.3098	0.3169	0.3241	0.3314	0.3388	0.3464	0.3542
	0.0113	0.0120	0.0130	0.0141	0.0153	0.0167	0.0183	0.0200	0.0220	0.0264	0.0314	0.0373	0.0438	0.0509	0.0588
$\nu=10, \phi=0.1, \lambda_2=0$	0.0083	0.0088	0.0091	0.0097	0.0102	0.0109	0.0117	0.0127	0.0138	0.0161	0.0189	0.0222	0.0259	0.0300	0.0347
	0.0078	0.0080	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0273
	0.0078	0.0080	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0273
	0.0078	0.0080	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0272
	0.0013	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0088	0.0111	0.0138	0.0167	0.0200
	0.1183	0.1209	0.1235	0.1262	0.1289	0.1316	0.1344	0.1372	0.1400	0.1458	0.1517	0.1577	0.1639	0.1702	0.1767
	0.0025	0.0032	0.0040	0.0049	0.0061	0.0074	0.0089	0.0105	0.0124	0.0166	0.0215	0.0272	0.0336	0.0406	0.0483
$\nu=\infty, \phi=0.1, \lambda_2=0$	0.0014	0.0017	0.0021	0.0026	0.0031	0.0038	0.0046	0.0054	0.0064	0.0087	0.0115	0.0147	0.0184	0.0226	0.0272
	0.0012	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0199
	0.0012	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0042	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0199
	0.0012	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0042	0.0050	0.0067	0.0087	0.0110	0.0137	0.0166	0.0199
	0.0661	0.0664	0.0667	0.0672	0.0678	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0803	0.0833	0.0869
	1.7569	1.7633	1.7697	1.7761	1.7828	1.7892	1.7958	1.8022	1.8089	1.8222	1.8358	1.8494	1.8633	1.8772	1.8911
	0.0819	0.0825	0.0836	0.0844	0.0856	0.0869	0.0881	0.0897	0.0911	0.0947	0.0989	0.1036	0.1092	0.1150	0.1217
$\nu=5, \phi=0.1, \lambda_2=3$	0.0683	0.0689	0.0692	0.0697	0.0706	0.0711	0.0719	0.0728	0.0739	0.0761	0.0786	0.0814	0.0847	0.0883	0.0925
	0.0658	0.0664	0.0667	0.0672	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0867
	0.0658	0.0664	0.0667	0.0672	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0867
	0.0658	0.0664	0.0667	0.0672	0.0675	0.0683	0.0689	0.0697	0.0706	0.0725	0.0747	0.0772	0.0800	0.0833	0.0867
	0.0078	0.0081	0.0084	0.0088	0.0092	0.0098	0.0105	0.0111	0.0119	0.0138	0.0158	0.0183	0.0209	0.0241	0.0273
	0.5850	0.5903	0.5958	0.6011	0.6066	0.6120	0.6175	0.6230	0.6286	0.6397	0.6511	0.6627	0.6742	0.6859	0.6980
	0.0095	0.0100	0.0106	0.0113	0.0122	0.0131	0.0141	0.0153	0.0166	0.0197	0.0233	0.0275	0.0323	0.0380	0.0442
$\nu=10, \phi=0.1, \lambda_2=3$	0.0081	0.0083	0.0088	0.0091	0.0097	0.0102	0.0109	0.0117	0.0125	0.0145	0.0169	0.0195	0.0227	0.0261	0.0300
	0.0078	0.0080	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0273
	0.0078	0.0081	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0273
	0.0078	0.0080	0.0083	0.0088	0.0092	0.0097	0.0103	0.0111	0.0119	0.0136	0.0158	0.0181	0.0209	0.0239	0.0273
	0.0013	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0088	0.0111	0.0138	0.0167	0.0200
	0.3725	0.3772	0.3819	0.3866	0.3914	0.3962	0.4010	0.4059	0.4108	0.4208	0.4308	0.4410	0.4514	0.4619	0.4725
	0.0015	0.0018	0.0022	0.0028	0.0034	0.0042	0.0051	0.0061	0.0072	0.0099	0.0132	0.0172	0.0218	0.0271	0.0331
$\nu=\infty, \phi=0.1, \lambda_2=3$	0.0013	0.0015	0.0018	0.0021	0.0026	0.0032	0.0038	0.0045	0.0053	0.0071	0.0093	0.0119	0.0149	0.0183	0.0220
	0.0013	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0200
	0.0013	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0200
	0.0012	0.0014	0.0017	0.0021	0.0025	0.0030	0.0036	0.0043	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0200

TABLE A6.2.5: Relative risks of s^2_{NML}, s^2_{AML}, and s^2_{PML}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	λ_1 0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	4.7475	4.7411	4.7403	4.7447	4.7550	4.7708	4.7919	4.8186	4.8511	4.9322	5.0358	5.1614	5.3089	5.4789	5.6711
	4.1111	4.0975	4.0858	4.0767	4.0697	4.0647	4.0622	4.0619	4.0636	4.0742	4.0933	4.1214	4.1586	4.2044	4.2592
	4.0897	4.0750	4.0625	4.0528	4.0456	4.0406	4.0378	4.0375	4.0394	4.0503	4.0697	4.0983	4.1358	4.1822	4.2375
$\nu=5, \phi=1.0, \lambda_2=0$	4.0164	3.9989	3.9847	3.9739	3.9656	3.9600	3.9572	3.9567	3.9586	3.9697	3.9903	4.0197	4.0583	4.1058	4.1622
	3.9894	3.9703	3.9553	3.9444	3.9369	3.9331	3.9325	3.9350	3.9403	3.9592	3.9883	4.0272	4.0756	4.1328	4.1989
	4.0681	4.0500	4.0361	4.0267	4.0208	4.0186	4.0200	4.0244	4.0317	4.0553	4.0892	4.1331	4.1864	4.2486	4.3194
	3.7678	3.7456	3.7275	3.7128	3.7014	3.6931	3.6878	3.6856	3.6858	3.6950	3.7142	3.7436	3.7825	3.8308	3.8883
	0.5930	0.5888	0.5902	0.5972	0.6095	0.6277	0.6511	0.6802	0.7148	0.8008	0.9088	1.0389	1.1913	1.3658	1.5625
	0.5069	0.4969	0.4892	0.4836	0.4803	0.4792	0.4803	0.4836	0.4892	0.5069	0.5336	0.5692	0.6136	0.6669	0.7292
	0.5091	0.4980	0.4895	0.4836	0.4798	0.4786	0.4795	0.4828	0.4883	0.5059	0.5327	0.5683	0.6128	0.6663	0.7284
$\nu=10, \phi=1.0, \lambda_2=0$	0.5114	0.4984	0.4886	0.4814	0.4770	0.4750	0.4756	0.4786	0.4839	0.5014	0.5283	0.5641	0.6088	0.6623	0.7248
	0.5220	0.5080	0.4973	0.4903	0.4864	0.4855	0.4875	0.4922	0.4997	0.5222	0.5542	0.5955	0.6455	0.7042	0.7714
	0.5284	0.5152	0.5055	0.4995	0.4967	0.4972	0.5006	0.5069	0.5158	0.5414	0.5769	0.6214	0.6747	0.7364	0.8066
	0.5061	0.4898	0.4770	0.4672	0.4605	0.4566	0.4555	0.4570	0.4613	0.4775	0.5034	0.5391	0.5839	0.6381	0.7013
	0.1136	0.1108	0.1136	0.1219	0.1357	0.1551	0.1801	0.2105	0.2465	0.3352	0.4460	0.5789	0.7341	0.9114	1.1108
	0.0933	0.0856	0.0800	0.0767	0.0756	0.0767	0.0800	0.0856	0.0933	0.1156	0.1467	0.1867	0.2356	0.2933	0.3600
	0.0964	0.0878	0.0817	0.0779	0.0765	0.0773	0.0805	0.0859	0.0936	0.1157	0.1467	0.1867	0.2356	0.2934	0.3600
$\nu=\infty, \phi=1.0, \lambda_2=0$	0.1035	0.0935	0.0861	0.0813	0.0791	0.0794	0.0820	0.0871	0.0944	0.1161	0.1470	0.1868	0.2356	0.2934	0.3600
	0.1137	0.1026	0.0947	0.0898	0.0880	0.0889	0.0925	0.0987	0.1074	0.1319	0.1657	0.2082	0.2594	0.3189	0.3870
	0.1129	0.1025	0.0954	0.0914	0.0904	0.0925	0.0971	0.1044	0.1143	0.1411	0.1772	0.2220	0.2752	0.3365	0.4060
	0.1174	0.1047	0.0948	0.0881	0.0841	0.0828	0.0843	0.0883	0.0947	0.1153	0.1454	0.1849	0.2336	0.2914	0.3582
	4.7475	4.7411	4.7403	4.7447	4.7550	4.7708	4.7919	4.8186	4.8511	4.9322	5.0358	5.1614	5.3089	5.4789	5.6711
	4.0622	4.0619	4.0636	4.0678	4.0742	4.0825	4.0933	4.1064	4.1214	4.1586	4.2044	4.2592	4.3231	4.3956	4.4769
	4.0942	4.0775	4.0669	4.0617	4.0611	4.0644	4.0717	4.0819	4.0953	4.1300	4.1753	4.2300	4.2939	4.3669	4.4492
$\nu=5, \phi=1.0, \lambda_2=3$	4.0494	4.0261	4.0086	3.9964	3.9892	3.9869	3.9889	3.9953	4.0050	4.0353	4.0781	4.1319	4.1961	4.2700	4.3533
	4.0147	3.9931	3.9761	3.9639	3.9561	3.9528	3.9539	3.9592	3.9686	3.9994	4.0456	4.1056	4.1789	4.2644	4.3611
	4.0958	4.0756	4.0600	4.0492	4.0431	4.0414	4.0442	4.0514	4.0628	4.0981	4.1489	4.2144	4.2936	4.3858	4.4897
	3.7775	3.7525	3.7322	3.7161	3.7047	3.6975	3.6942	3.6950	3.6994	3.7197	3.7539	3.8008	3.8600	3.9308	4.0125
	0.5930	0.5888	0.5902	0.5972	0.6095	0.6277	0.6511	0.6802	0.7148	0.8008	0.9088	1.0389	1.1913	1.3658	1.5625
	0.4803	0.4836	0.4892	0.4969	0.5069	0.5192	0.5336	0.5503	0.5692	0.6136	0.6669	0.7292	0.8003	0.8803	0.9692
	0.5094	0.5041	0.5030	0.5059	0.5122	0.5217	0.5342	0.5495	0.5675	0.6109	0.6639	0.7261	0.7973	0.8777	0.9667
$\nu=10, \phi=1.0, \lambda_2=3$	0.5250	0.5145	0.5088	0.5073	0.5098	0.5163	0.5261	0.5394	0.5556	0.5970	0.6492	0.7114	0.7833	0.8642	0.9544
	0.5289	0.5170	0.5097	0.5069	0.5081	0.5138	0.5233	0.5367	0.5539	0.5992	0.6580	0.7294	0.8123	0.9061	1.0100
	0.5370	0.5259	0.5197	0.5178	0.5205	0.5273	0.5384	0.5534	0.5725	0.6219	0.6853	0.7620	0.8508	0.9508	1.0613
	0.5039	0.4894	0.4792	0.4731	0.4711	0.4728	0.4781	0.4872	0.4995	0.5344	0.5816	0.6405	0.7106	0.7911	0.8819
	0.1136	0.1108	0.1136	0.1219	0.1357	0.1551	0.1801	0.2105	0.2465	0.3352	0.4460	0.5789	0.7341	0.9114	1.1108
	0.0800	0.0856	0.0933	0.1033	0.1156	0.1300	0.1467	0.1656	0.1867	0.2356	0.2933	0.3600	0.4356	0.5200	0.6133
	0.0940	0.0957	0.1005	0.1083	0.1189	0.1321	0.1479	0.1662	0.1869	0.2352	0.2929	0.3596	0.4351	0.5197	0.6132
$\nu=\infty, \phi=1.0, \lambda_2=3$	0.1049	0.1026	0.1041	0.1091	0.1175	0.1291	0.1438	0.1611	0.1812	0.2292	0.2871	0.3542	0.4306	0.5158	0.6099
	0.1101	0.1039	0.1023	0.1048	0.1117	0.1224	0.1369	0.1553	0.1772	0.2311	0.2975	0.3756	0.4645	0.5633	0.6715
	0.1107	0.1050	0.1040	0.1074	0.1152	0.1272	0.1432	0.1630	0.1865	0.2443	0.3153	0.3987	0.4933	0.5982	0.7128
	0.1081	0.1000	0.0959	0.0957	0.0992	0.1063	0.1168	0.1307	0.1478	0.1913	0.2463	0.3124	0.3890	0.4755	0.5716
	1.1869	1.1853	1.1850	1.1861	1.1889	1.1928	1.1981	1.2047	1.2128	1.2331	1.2589	1.2903	1.3272	1.3697	1.4178
	1.8056	1.8094	1.8142	1.8192	1.8247	1.8311	1.8378	1.8450	1.8531	1.8703	1.8900	1.9119	1.9358	1.9622	1.9908
	1.5761	1.5783	1.5819	1.5867	1.5928	1.5994	1.6072	1.6156	1.6247	1.6453	1.6683	1.6936	1.7214	1.7511	1.7833
$\nu=5, \phi=0.5, \lambda_2=0$	1.3406	1.3406	1.3419	1.3444	1.3486	1.3536	1.3600	1.3672	1.3756	1.3947	1.4172	1.4425	1.4706	1.5014	1.5344
	1.1400	1.1375	1.1361	1.1361	1.1372	1.1397	1.1433	1.1481	1.1539	1.1689	1.1881	1.2114	1.2381	1.2681	1.3014
	1.1483	1.1458	1.1447	1.1447	1.1461	1.1489	1.1528	1.1578	1.1642	1.1800	1.2003	1.2244	1.2525	1.2842	1.3469
	1.1050	1.1017	1.0997	1.0989	1.0992	1.1006	1.1031	1.1067	1.1111	1.1231	1.1389	1.1581	1.1808	1.2064	1.2350
	0.1483	0.1472	0.1475	0.1492	0.1523	0.1569	0.1628	0.1700	0.1788	0.2002	0.2272	0.2597	0.2978	0.3414	0.3906
	0.2055	0.2084	0.2120	0.2163	0.2209	0.2263	0.2320	0.2384	0.2455	0.2609	0.2788	0.2988	0.3209	0.3455	0.3720
	0.1855	0.1872	0.1898	0.1936	0.1981	0.2034	0.2095	0.2164	0.2239	0.2406	0.2597	0.2811	0.3045	0.3303	0.3581
$\nu=10, \phi=0.5, \lambda_2=0$	0.1630	0.1630	0.1641	0.1664	0.1697	0.1741	0.1794	0.1856	0.1927	0.2089	0.2281	0.2498	0.2742	0.3008	0.3297
	0.1430	0.1411	0.1405	0.1409	0.1427	0.1455	0.1494	0.1544	0.1603	0.1753	0.1939	0.2161	0.2416	0.2702	0.3017
	0.1439	0.1422	0.1417	0.1423	0.1442	0.1473	0.1516	0.1567	0.1631	0.1789	0.1986	0.2220	0.2488	0.2789	0.3119
	0.1392	0.1367	0.1355	0.1353	0.1363	0.1381	0.1409	0.1448	0.1497	0.1619	0.1775	0.1963	0.2181	0.2428	0.2702
	0.0284	0.0277	0.0284	0.0305	0.0339	0.0388	0.0450	0.0526	0.0616	0.0838	0.1115	0.1447	0.1835	0.2278	0.2777
	0.0278	0.0303	0.0333	0.0369	0.0411	0.0458	0.0511	0.0569	0.0633	0.0778	0.0944	0.1133	0.1344	0.1578	0.1833
	0.0289	0.0303	0.0325	0.0356	0.0393	0.0439	0.0490	0.0548	0.0613	0.0759	0.0929	0.1120	0.1334	0.1570	0.1827
$\nu=\infty, \phi=0.5, \lambda_2=0$	0.0286	0.0286	0.0297	0.0318	0.0348	0.0387	0.0434	0.0489	0.0551	0.0697	0.0868	0.1064	0.1284	0.1525	0.1789
	0.0277	0.0263	0.0261	0.0269	0.0289	0.0320	0.0360	0.0411	0.0471	0.0619	0.0802	0.1018	0.1264	0.1539	0.1840
	0.0278	0.0266	0.0264	0.0275	0.0297	0.0329	0.0373	0.0427	0.0491	0.0648	0.0841	0.1069	0.1328	0.1618	0.1935
	0.0273	0.0255	0.0247	0.0249	0.0262	0.0284	0.0315	0.0356	0.0404	0.0527	0.0682	0.0866	0.1079	0.1318	0.1583
	1.1869	1.1853	1.1850	1.1861	1.1889	1.1928	1.1981	1.2047	1.2128	1.2331	1.2589	1.2903	1.3272	1.3697	1.4178
	1.9122	1.9228	1.9342	1.9458	1.9581	1.9711	1.9844	1.9983	2.0131	2.0436	2.0767	2.1119	2.1492	2.1889	2.2308
	1.6461	1.6483	1.6522	1.6578	1.6650	1.6742	1.6850	1.6972	1.7108	1.7422	1.7778	1.8172	1.8600	1.9056	1.9536
$\nu=5, \phi=0.5, \lambda_2=3$	1.3683	1.3683	1.3700	1.3731	1.3778	1.3836	1.3914	1.4003	1.4108	1.4358	1.4661	1.5014	1.5408	1.5844	1.6317
	1.1447	1.1419	1.1408	1.1406	1.1419	1.1447	1.1486	1.1536	1.1603	1.1769	1.1986	1.2253	1.2567	1.2928	1.3333
	1.1522	1.1497	1.1486	1.1489	1.1503	1.1531	1.1572	1.1628	1.1694	1.1869	1.2092	1.2367	1.2689	1.3061	1.3481
	1.1114	1.1081	1.1058	1.1050	1.1053	1.1072	1.1100	1.1142	1.1194	1.1339	1.1528	1.1761	1.2042	1.2361	1.2722
	0.1483	0.1472	0.1475	0.1492	0.1523	0.1569	0.1628	0.1700	0.1788	0.2002	0.2272	0.2597	0.2978	0.3414	0.3906
	0.2955	0.3052	0.3155	0.3263	0.3377	0.3495	0.3620	0.3752	0.3888	0.4177	0.4488	0.4820	0.5177	0.5555	0.5955
	0.2277	0.2327	0.2394	0.2477	0.2575	0.2688	0.2811	0.2947	0.3094	0.3416	0.3772	0.4155	0.4561	0.4991	0.5441
$\nu=10, \phi=0.5, \lambda_2=3$	0.1777	0.1791	0.1822	0.1867	0.1930	0.2005	0.2095	0.2198	0.2317	0.2589	0.2908	0.3269	0.3667	0.4098	0.4561
	0.1448	0.1434	0.1433	0.1444	0.1469	0.1506	0.1558	0.1620	0.1697	0.1886	0.2125	0.2411	0.2744	0.3120	0.3542
	0.1456	0.1442	0.1441	0.1453	0.1480	0.1519	0.1570	0.1636	0.1714	0.1908	0.2152	0.2445	0.2786	0.3173	0.3606
	0.1417	0.1398	0.1394	0.1400	0.1419	0.1450	0.1494	0.1550	0.1617	0.1786	0.2002	0.2258	0.2558	0.2897	0.3273
	0.0284	0.0277	0.0284	0.0305	0.0339	0.0388	0.0450	0.0526	0.0616	0.0838	0.1115	0.1447	0.1835	0.2278	0.2777
	0.1078	0.1169	0.1267	0.1369	0.1478	0.1592	0.1711	0.1836	0.1967	0.2244	0.2544	0.2867	0.3211	0.3578	0.3967
	0.0543	0.0613	0.0697	0.0797	0.0909	0.1033	0.1168	0.1312	0.1464	0.1792	0.2146	0.2522	0.2917	0.3331	0.3761
$\nu=\infty, \phi=0.5, \lambda_2=3$	0.0359	0.0380	0.0419	0.0474	0.0543	0.0628	0.0727	0.0839	0.0964	0.1250	0.1579	0.1945	0.2345	0.2775	0.3231
	0.0282	0.0274	0.0279	0.0297	0.0329	0.0373	0.0430	0.0501	0.0583	0.0786	0.1037	0.1336	0.1680	0.2068	0.2499
	0.0282	0.0275	0.0280	0.0299	0.0331	0.0377	0.0435	0.0506	0.0590	0.0797	0.1053	0.1358	0.1710	0.2107	0.2550
	0.0280	0.0270	0.0272	0.0288	0.0315	0.0356	0.0408	0.0472	0.0547	0.0733	0.0962	0.1234	0.1546	0.1896	0.2283

TABLE A6.2.5 (continued)

Estimator	0.0	0.5	1.0	1.5	2.0	2.5 λ_1	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0481	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0550	0.0567
	0.8544	0.8569	0.8594	0.8622	0.8647	0.8672	0.8700	0.8725	0.8753	0.8808	0.8864	0.8919	0.8975	0.9033	0.9092
	0.0581	0.0583	0.0589	0.0592	0.0597	0.0606	0.0611	0.0622	0.0631	0.0653	0.0681	0.0711	0.0747	0.0786	0.0831
$\nu=5, \phi=0.1, \lambda_2=0$	0.0492	0.0492	0.0492	0.0492	0.0494	0.0497	0.0500	0.0503	0.0508	0.0517	0.0531	0.0547	0.0567	0.0589	0.0614
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0483	0.0492	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.1667	0.1686	0.1706	0.1725	0.1745	0.1764	0.1784	0.1805	0.1825	0.1867	0.1909	0.1953	0.1998	0.2044	0.2089
	0.0077	0.0080	0.0083	0.0086	0.0092	0.0098	0.0106	0.0114	0.0123	0.0145	0.0172	0.0203	0.0239	0.0278	0.0322
$\nu=10, \phi=0.1, \lambda_2=0$	0.0063	0.0063	0.0063	0.0064	0.0066	0.0069	0.0072	0.0075	0.0080	0.0091	0.0105	0.0122	0.0142	0.0166	0.0191
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0067	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0155
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0016	0.0018	0.0021	0.0025	0.0034	0.0045	0.0058	0.0073	0.0091	0.0111
	0.0665	0.0681	0.0696	0.0712	0.0728	0.0744	0.0760	0.0777	0.0793	0.0828	0.0863	0.0899	0.0936	0.0973	0.1012
	0.0017	0.0019	0.0022	0.0026	0.0031	0.0037	0.0045	0.0053	0.0062	0.0085	0.0111	0.0143	0.0178	0.0218	0.0262
$\nu=\infty, \phi=0.1, \lambda_2=0$	0.0012	0.0012	0.0013	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0043	0.0058	0.0075	0.0096	0.0119	0.0146
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0015	0.0018	0.0021	0.0025	0.0033	0.0044	0.0058	0.0073	0.0091	0.0110
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0016	0.0018	0.0021	0.0025	0.0033	0.0044	0.0058	0.0073	0.0091	0.0111
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0015	0.0018	0.0021	0.0024	0.0033	0.0044	0.0057	0.0073	0.0090	0.0109
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0481	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0550	0.0567
	1.0856	1.0894	1.0933	1.0972	1.1011	1.1050	1.1092	1.1131	1.1169	1.1253	1.1333	1.1417	1.1500	1.1586	1.1669
	0.0567	0.0569	0.0572	0.0575	0.0581	0.0583	0.0589	0.0597	0.0603	0.0622	0.0642	0.0667	0.0694	0.0728	0.0764
$\nu=5, \phi=0.1, \lambda_2=3$	0.0489	0.0489	0.0489	0.0489	0.0492	0.0492	0.0494	0.0497	0.0503	0.0511	0.0522	0.0539	0.0556	0.0575	0.0597
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0486	0.0494	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0478	0.0478	0.0481	0.0486	0.0492	0.0503	0.0517	0.0531	0.0547	0.0567
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.3500	0.3533	0.3566	0.3598	0.3631	0.3664	0.3698	0.3731	0.3766	0.3834	0.3903	0.3973	0.4045	0.4117	0.4189
	0.0069	0.0070	0.0072	0.0073	0.0077	0.0081	0.0086	0.0091	0.0097	0.0113	0.0131	0.0155	0.0181	0.0214	0.0250
$\nu=10, \phi=0.1, \lambda_2=3$	0.0061	0.0061	0.0061	0.0061	0.0063	0.0066	0.0067	0.0070	0.0075	0.0084	0.0097	0.0111	0.0128	0.0147	0.0170
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.0059	0.0059	0.0059	0.0059	0.0061	0.0063	0.0066	0.0069	0.0072	0.0080	0.0091	0.0103	0.0119	0.0136	0.0156
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0016	0.0018	0.0021	0.0025	0.0034	0.0045	0.0058	0.0073	0.0091	0.0111
	0.2212	0.2241	0.2269	0.2298	0.2328	0.2357	0.2387	0.2417	0.2447	0.2508	0.2569	0.2632	0.2696	0.2760	0.2825
	0.0013	0.0013	0.0014	0.0016	0.0018	0.0022	0.0026	0.0031	0.0036	0.0051	0.0069	0.0091	0.0117	0.0148	0.0184
$\nu=\infty, \phi=0.1, \lambda_2=3$	0.0011	0.0011	0.0012	0.0013	0.0014	0.0016	0.0019	0.0022	0.0026	0.0036	0.0048	0.0062	0.0079	0.0099	0.0121
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0016	0.0018	0.0021	0.0025	0.0034	0.0045	0.0058	0.0073	0.0091	0.0111
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0016	0.0018	0.0021	0.0025	0.0034	0.0045	0.0058	0.0073	0.0091	0.0111
	0.0011	0.0011	0.0011	0.0012	0.0014	0.0015	0.0018	0.0021	0.0025	0.0033	0.0045	0.0058	0.0073	0.0091	0.0111

TABLE A6.2.6: Relative risks of s^2_{NM}, s^2_{AM}, and s^2_{PM}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

	λ_1														
Estimator	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	5.2469	5.2500	5.2592	5.2747	5.2964	5.3242	5.3581	5.3981	5.4444	5.5556	5.6914	5.8519	6.0369	6.2469	6.4814
	4.6486	4.6414	4.6367	4.6344	4.6350	4.6378	4.6433	4.6514	4.6622	4.6911	4.7303	4.7794	4.8389	4.9086	4.9886
	4.6222	4.6136	4.6083	4.6058	4.6058	4.6089	4.6144	4.6225	4.6333	4.6628	4.7025	4.7522	4.8125	4.8828	4.9631
$\nu=5,\phi=1.0,\lambda_2=0$	4.5314	4.5206	4.5136	4.5097	4.5092	4.5117	4.5172	4.5256	4.5367	4.5667	4.6075	4.6586	4.7203	4.7919	4.8739
	4.4172	4.4044	4.3958	4.3919	4.3919	4.3958	4.4033	4.4142	4.4283	4.4661	4.5156	4.5761	4.6472	4.7289	4.8206
	4.5011	4.4894	4.4825	4.4800	4.4819	4.4875	4.4972	4.5103	4.5267	4.5692	4.6236	4.6894	4.7658	4.8528	4.9494
	4.2008	4.1850	4.1733	4.1656	4.1614	4.1608	4.1639	4.1700	4.1794	4.2078	4.2475	4.2986	4.3611	4.4342	4.5178
	0.6366	0.6397	0.6489	0.6644	0.6859	0.7138	0.7477	0.7878	0.8341	0.9452	1.0811	1.2416	1.4267	1.6366	1.8711
	0.5420	0.5370	0.5344	0.5344	0.5370	0.5420	0.5497	0.5600	0.5727	0.6059	0.6492	0.7028	0.7666	0.8406	0.9247
	0.5436	0.5375	0.5342	0.5339	0.5361	0.5409	0.5486	0.5588	0.5714	0.6045	0.6480	0.7017	0.7656	0.8397	0.9239
$\nu=10,\phi=1.0,\lambda_2=0$	0.5436	0.5356	0.5311	0.5297	0.5313	0.5356	0.5428	0.5528	0.5653	0.5986	0.6422	0.6963	0.7605	0.8347	0.9195
	0.5478	0.5383	0.5328	0.5313	0.5331	0.5384	0.5469	0.5586	0.5733	0.6113	0.6603	0.7197	0.7894	0.8691	0.9586
	0.5558	0.5473	0.5430	0.5425	0.5458	0.5525	0.5627	0.5759	0.5922	0.6336	0.6861	0.7491	0.8222	0.9050	0.9973
	0.5286	0.5170	0.5091	0.5047	0.5036	0.5061	0.5114	0.5200	0.5316	0.5634	0.6066	0.6606	0.7255	0.8006	0.8864
	0.1111	0.1142	0.1235	0.1389	0.1605	0.1883	0.2222	0.2623	0.3086	0.4198	0.5556	0.7160	0.9012	1.1111	1.3457
	0.0816	0.0778	0.0765	0.0778	0.0816	0.0880	0.0969	0.1084	0.1224	0.1582	0.2041	0.2602	0.3265	0.4031	0.4898
	0.0846	0.0800	0.0782	0.0790	0.0826	0.0886	0.0974	0.1088	0.1226	0.1582	0.2041	0.2602	0.3266	0.4031	0.4898
$\nu=\infty,\phi=1.0,\lambda_2=0$	0.0912	0.0851	0.0820	0.0820	0.0847	0.0904	0.0986	0.1097	0.1233	0.1586	0.2043	0.2603	0.3265	0.4031	0.4898
	0.1027	0.0953	0.0916	0.0913	0.0943	0.1005	0.1096	0.1218	0.1368	0.1749	0.2235	0.2823	0.3510	0.4294	0.5176
	0.1028	0.0962	0.0934	0.0941	0.0981	0.1056	0.1159	0.1293	0.1455	0.1861	0.2373	0.2984	0.3691	0.4494	0.5390
	0.1036	0.0944	0.0888	0.0864	0.0873	0.0913	0.0983	0.1083	0.1211	0.1553	0.2004	0.2563	0.3227	0.3995	0.4866
	5.2469	5.2500	5.2592	5.2747	5.2964	5.3242	5.3581	5.3981	5.4444	5.5556	5.6914	5.8519	6.0369	6.2469	6.4814
	4.6433	4.6514	4.6622	4.6753	4.6911	4.7094	4.7303	4.7536	4.7794	4.8389	4.9086	4.9886	5.0789	5.1792	5.2897
	4.6686	4.6600	4.6581	4.6617	4.6706	4.6839	4.7011	4.7219	4.7461	4.8039	4.8731	4.9533	5.0442	5.1453	5.2567
$\nu=5,\phi=1.0,\lambda_2=3$	4.5994	4.5842	4.5750	4.5719	4.5742	4.5819	4.5944	4.6114	4.6325	4.6861	4.7539	4.8339	4.9253	5.0281	5.1414
	4.4561	4.4419	4.4331	4.4292	4.4303	4.4364	4.4475	4.4631	4.4833	4.5369	4.6072	4.6931	4.7939	4.9083	5.0353
	4.5417	4.5289	4.5217	4.5194	4.5225	4.5306	4.5433	4.5611	4.5836	4.6422	4.7178	4.8100	4.9172	5.0389	5.1736
	4.2272	4.2094	4.1969	4.1892	4.1864	4.1881	4.1944	4.2053	4.2206	4.2631	4.3211	4.3939	4.4806	4.5800	4.6922
	0.6366	0.6397	0.6489	0.6644	0.6859	0.7138	0.7477	0.7878	0.8341	0.9452	1.0811	1.2416	1.4267	1.6366	1.8711
	0.5497	0.5600	0.5727	0.5880	0.6059	0.6263	0.6492	0.6747	0.7028	0.7666	0.8406	0.9247	1.0191	1.1238	1.2386
	0.5750	0.5767	0.5831	0.5938	0.6081	0.6261	0.6473	0.6717	0.6991	0.7622	0.8361	0.9205	1.0153	1.1203	1.2355
$\nu=10,\phi=1.0,\lambda_2=3$	0.5814	0.5780	0.5797	0.5863	0.5972	0.6122	0.6313	0.6539	0.6798	0.7417	0.8156	0.9008	0.9966	1.1030	1.2195
	0.5636	0.5578	0.5570	0.5611	0.5698	0.5834	0.6014	0.6236	0.6502	0.7152	0.7955	0.8897	0.9972	1.1169	1.2480
	0.5727	0.5678	0.5683	0.5736	0.5838	0.5988	0.6184	0.6427	0.6711	0.7408	0.8261	0.9261	1.0398	1.1664	1.3047
	0.5373	0.5288	0.5250	0.5258	0.5309	0.5405	0.5542	0.5719	0.5934	0.6478	0.7164	0.7983	0.8927	0.9992	1.1173
	0.1111	0.1142	0.1235	0.1389	0.1605	0.1883	0.2222	0.2623	0.3086	0.4198	0.5556	0.7160	0.9012	1.1111	1.3457
	0.0969	0.1084	0.1224	0.1390	0.1582	0.1798	0.2041	0.2309	0.2602	0.3265	0.4031	0.4898	0.5867	0.6939	0.8112
	0.1067	0.1148	0.1264	0.1412	0.1591	0.1799	0.2036	0.2300	0.2592	0.3253	0.4020	0.4890	0.5860	0.6934	0.8110
$\nu=\infty,\phi=1.0,\lambda_2=3$	0.1102	0.1144	0.1229	0.1325	0.1513	0.1708	0.1938	0.2198	0.2487	0.3153	0.3930	0.4810	0.5794	0.6879	0.8065
	0.1053	0.1043	0.1082	0.1169	0.1304	0.1481	0.1702	0.1965	0.2268	0.2987	0.3848	0.4842	0.5957	0.7186	0.8524
	0.1063	0.1059	0.1106	0.1203	0.1348	0.1540	0.1776	0.2055	0.2377	0.3139	0.4050	0.5099	0.6276	0.7571	0.8977
	0.1020	0.0989	0.1003	0.1061	0.1161	0.1302	0.1483	0.1700	0.1954	0.2570	0.3317	0.4190	0.5183	0.6289	0.7508
	1.3117	1.3125	1.3147	1.3186	1.3242	1.3311	1.3394	1.3494	1.3611	1.3889	1.4228	1.4631	1.5092	1.5617	1.6203
	2.0833	2.0900	2.0975	2.1053	2.1139	2.1231	2.1331	2.1436	2.1547	2.1789	2.2058	2.2350	2.2669	2.3014	2.3383
	1.8094	1.8147	1.8217	1.8300	1.8394	1.8497	1.8611	1.8733	1.8864	1.9147	1.9458	1.9797	2.0158	2.0547	2.0961
$\nu=5,\phi=0.5,\lambda_2=0$	1.5258	1.5286	1.5333	1.5392	1.5467	1.5556	1.5653	1.5764	1.5886	1.6158	1.6467	1.6808	1.7181	1.7583	1.8011
	1.2608	1.2606	1.2614	1.2636	1.2675	1.2725	1.2789	1.2867	1.2956	1.3169	1.3428	1.3733	1.4075	1.4458	1.4878
	1.2697	1.2694	1.2706	1.2731	1.2769	1.2825	1.2892	1.2972	1.3064	1.3289	1.3558	1.3875	1.4233	1.4633	1.5069
	1.2281	1.2269	1.2269	1.2283	1.2311	1.2353	1.2406	1.2469	1.2544	1.2731	1.2956	1.3222	1.3525	1.3864	1.4236
	0.1592	0.1598	0.1622	0.1661	0.1716	0.1784	0.1869	0.1969	0.2086	0.2363	0.2703	0.3103	0.3567	0.4092	0.4678
	0.2419	0.2469	0.2527	0.2591	0.2661	0.2738	0.2820	0.2909	0.3005	0.3216	0.3452	0.3713	0.4000	0.4313	0.4650
	0.2138	0.2180	0.2231	0.2294	0.2366	0.2447	0.2536	0.2631	0.2736	0.2964	0.3217	0.3497	0.3802	0.4130	0.4483
$\nu=10,\phi=0.5,\lambda_2=0$	0.1830	0.1852	0.1886	0.1934	0.1995	0.2067	0.2148	0.2239	0.2339	0.2566	0.2825	0.3111	0.3427	0.3767	0.4134
	0.1528	0.1527	0.1538	0.1561	0.1598	0.1647	0.1709	0.1783	0.1867	0.2069	0.2314	0.2598	0.2919	0.3275	0.3664
	0.1539	0.1538	0.1552	0.1577	0.1617	0.1669	0.1734	0.1811	0.1900	0.2111	0.2367	0.2663	0.2998	0.3370	0.3777
	0.1489	0.1481	0.1486	0.1502	0.1531	0.1570	0.1622	0.1684	0.1758	0.1933	0.2147	0.2397	0.2683	0.3000	0.3350
	0.0278	0.0285	0.0309	0.0347	0.0401	0.0471	0.0556	0.0656	0.0772	0.1049	0.1389	0.1790	0.2253	0.2778	0.3364
	0.0357	0.0399	0.0446	0.0501	0.0561	0.0628	0.0702	0.0781	0.0867	0.1059	0.1276	0.1518	0.1786	0.2079	0.2398
	0.0339	0.0372	0.0414	0.0465	0.0524	0.0591	0.0665	0.0747	0.0835	0.1031	0.1253	0.1500	0.1772	0.2068	0.2390
$\nu=\infty,\phi=0.5,\lambda_2=0$	0.0306	0.0324	0.0354	0.0395	0.0447	0.0509	0.0579	0.0658	0.0746	0.0944	0.1171	0.1425	0.1705	0.2010	0.2340
	0.0266	0.0265	0.0277	0.0301	0.0338	0.0387	0.0447	0.0518	0.0600	0.0795	0.1029	0.1301	0.1606	0.1944	0.2313
	0.0268	0.0269	0.0282	0.0308	0.0347	0.0399	0.0462	0.0537	0.0623	0.0827	0.1073	0.1357	0.1677	0.2031	0.2416
	0.0259	0.0254	0.0260	0.0278	0.0308	0.0348	0.0398	0.0459	0.0529	0.0699	0.0906	0.1146	0.1418	0.1722	0.2054
	1.3117	1.3125	1.3147	1.3186	1.3242	1.3311	1.3394	1.3494	1.3611	1.3889	1.4228	1.4631	1.5092	1.5617	1.6203
	2.2314	2.2456	2.2606	2.2764	2.2925	2.3094	2.3269	2.3450	2.3639	2.4036	2.4456	2.4903	2.5375	2.5872	2.6394
	1.8958	1.9022	1.9106	1.9208	1.9333	1.9472	1.9633	1.9808	1.9997	2.0422	2.0892	2.1403	2.1947	2.2525	2.3128
$\nu=5,\phi=0.5,\lambda_2=3$	1.5586	1.5625	1.5678	1.5750	1.5836	1.5942	1.6064	1.6200	1.6356	1.6711	1.7125	1.7592	1.8108	1.8667	1.9269
	1.2667	1.2661	1.2672	1.2700	1.2739	1.2794	1.2864	1.2947	1.3044	1.3283	1.3575	1.3925	1.4325	1.4778	1.5283
	1.2744	1.2744	1.2756	1.2783	1.2828	1.2883	1.2956	1.3042	1.3144	1.3389	1.3689	1.4044	1.4456	1.4922	1.5439
	1.2356	1.2344	1.2350	1.2367	1.2397	1.2444	1.2503	1.2578	1.2664	1.2878	1.3144	1.3461	1.3828	1.4242	1.4703
	0.1592	0.1598	0.1622	0.1661	0.1716	0.1784	0.1869	0.1969	0.2086	0.2363	0.2703	0.3103	0.3567	0.4092	0.4678
	0.3642	0.3770	0.3905	0.4044	0.4191	0.4344	0.4503	0.4669	0.4842	0.5205	0.5594	0.6009	0.6448	0.6914	0.7405
	0.3914	0.4120	0.4350	0.4600	0.4870	0.5159	0.5466	0.5786	0.6119	0.6822	0.7566	0.8342	0.9150	0.9984	1.0844
$\nu=10,\phi=0.5,\lambda_2=3$	0.2003	0.2047	0.2109	0.2189	0.2286	0.2398	0.2530	0.2675	0.2834	0.3198	0.3614	0.4077	0.4580	0.5120	0.5695
	0.1555	0.1558	0.1575	0.1606	0.1653	0.1714	0.1791	0.1880	0.1984	0.2234	0.2538	0.2895	0.3305	0.3766	0.4273
	0.1561	0.1566	0.1584	0.1617	0.1666	0.1728	0.1805	0.1897	0.2003	0.2258	0.2567	0.2933	0.3352	0.3822	0.4344
	0.1523	0.1523	0.1536	0.1564	0.1605	0.1661	0.1730	0.1811	0.1908	0.2138	0.2419	0.2748	0.3125	0.3548	0.4014
	0.0278	0.0285	0.0309	0.0347	0.0401	0.0471	0.0556	0.0656	0.0772	0.1049	0.1389	0.1790	0.2253	0.2778	0.3364
	0.1429	0.1547	0.1671	0.1802	0.1939	0.2082	0.2232	0.2388	0.2551	0.2895	0.3265	0.3661	0.4082	0.4528	0.5000
	0.0662	0.0767	0.0889	0.1027	0.1179	0.1344	0.1520	0.1706	0.1902	0.2317	0.2761	0.3227	0.3714	0.4221	0.4745
$\nu=\infty,\phi=0.5,\lambda_2=3$	0.0393	0.0439	0.0505	0.0589	0.0690	0.0806	0.0943	0.1093	0.1257	0.1625	0.2043	0.2502	0.2999	0.3529	0.4088
	0.0276	0.0281	0.0302	0.0338	0.0389	0.0454	0.0533	0.0626	0.0733	0.0990	0.1301	0.1665	0.2080	0.2544	0.3056
	0.0276	0.0283	0.0304	0.0340	0.0391	0.0457	0.0537	0.0632	0.0742	0.1002	0.1318	0.1689	0.2112	0.2587	0.3112
	0.0272	0.0277	0.0295	0.0328	0.0375	0.0436	0.0510	0.0598	0.0698	0.0938	0.1228	0.1566	0.1950	0.2378	0.2848

TABLE A6.2.6 (continued)

Estimator	0.0	0.5	1.0	1.5	2.0	2.5 λ_1	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.9903	0.9933	0.9964	0.9994	1.0025	1.0056	1.0086	1.0119	1.0150	1.0214	1.0278	1.0344	1.0411	1.0481	1.0550
	0.0653	0.0656	0.0661	0.0669	0.0678	0.0686	0.0697	0.0708	0.0719	0.0750	0.0783	0.0822	0.0864	0.0914	0.0967
$\nu=5, \phi=0.1, \lambda_2=0$	0.0544	0.0544	0.0547	0.0550	0.0553	0.0556	0.0561	0.0567	0.0572	0.0586	0.0603	0.0625	0.0650	0.0675	0.0706
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0583	0.0603	0.0625	0.0647
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0188
	0.1967	0.1991	0.2013	0.2036	0.2059	0.2083	0.2106	0.2130	0.2155	0.2203	0.2253	0.2305	0.2358	0.2411	0.2466
	0.0086	0.0089	0.0094	0.0100	0.0106	0.0116	0.0125	0.0136	0.0148	0.0175	0.0208	0.0247	0.0289	0.0338	0.0391
$\nu=10, \phi=0.1, \lambda_2=0$	0.0067	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0088	0.0094	0.0109	0.0127	0.0148	0.0172	0.0200	0.0231
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0163	0.0186
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0186
	0.0064	0.0064	0.0064	0.0066	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0163	0.0186
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0056	0.0072	0.0090	0.0111	0.0135
	0.0798	0.0816	0.0834	0.0853	0.0871	0.0890	0.0910	0.0929	0.0949	0.0989	0.1031	0.1073	0.1116	0.1161	0.1206
	0.0019	0.0022	0.0026	0.0031	0.0038	0.0046	0.0055	0.0066	0.0077	0.0105	0.0138	0.0175	0.0218	0.0266	0.0319
$\nu=\infty, \phi=0.1, \lambda_2=0$	0.0012	0.0013	0.0014	0.0017	0.0020	0.0023	0.0028	0.0033	0.0039	0.0054	0.0072	0.0093	0.0118	0.0146	0.0178
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0055	0.0071	0.0090	0.0111	0.0134
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0055	0.0071	0.0090	0.0111	0.0134
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0055	0.0071	0.0089	0.0110	0.0133
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	1.2608	1.2653	1.2697	1.2744	1.2789	1.2836	1.2883	1.2931	1.2978	1.3072	1.3167	1.3264	1.3361	1.3461	1.3561
	0.0633	0.0639	0.0642	0.0647	0.0653	0.0658	0.0667	0.0675	0.0686	0.0708	0.0733	0.0764	0.0800	0.0839	0.0883
$\nu=5, \phi=0.1, \lambda_2=3$	0.0542	0.0542	0.0544	0.0544	0.0547	0.0550	0.0556	0.0561	0.0567	0.0578	0.0594	0.0611	0.0633	0.0658	0.0683
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.0525	0.0525	0.0525	0.0528	0.0531	0.0533	0.0536	0.0539	0.0544	0.0556	0.0569	0.0586	0.0603	0.0625	0.0647
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0188
	0.4111	0.4148	0.4186	0.4225	0.4263	0.4302	0.4341	0.4380	0.4419	0.4500	0.4580	0.4663	0.4745	0.4830	0.4914
	0.0075	0.0077	0.0080	0.0083	0.0088	0.0094	0.0100	0.0106	0.0114	0.0134	0.0158	0.0186	0.0219	0.0256	0.0300
$\nu=10, \phi=0.1, \lambda_2=3$	0.0066	0.0066	0.0067	0.0069	0.0065	0.0075	0.0078	0.0083	0.0088	0.0100	0.0116	0.0133	0.0153	0.0177	0.0203
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0188
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0188
	0.0064	0.0064	0.0066	0.0067	0.0069	0.0072	0.0075	0.0078	0.0083	0.0094	0.0108	0.0123	0.0142	0.0164	0.0188
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0056	0.0072	0.0090	0.0111	0.0135
	0.2604	0.2637	0.2671	0.2705	0.2739	0.2773	0.2808	0.2842	0.2878	0.2948	0.3020	0.3093	0.3167	0.3242	0.3318
	0.0013	0.0014	0.0016	0.0018	0.0022	0.0026	0.0032	0.0038	0.0045	0.0063	0.0085	0.0112	0.0143	0.0180	0.0222
$\nu=\infty, \phi=0.1, \lambda_2=3$	0.0011	0.0012	0.0013	0.0014	0.0017	0.0020	0.0023	0.0028	0.0033	0.0045	0.0059	0.0077	0.0097	0.0121	0.0147
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0056	0.0072	0.0090	0.0111	0.0135
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0056	0.0072	0.0090	0.0111	0.0135
	0.0011	0.0011	0.0012	0.0014	0.0016	0.0019	0.0022	0.0026	0.0031	0.0042	0.0055	0.0071	0.0090	0.0111	0.0134

TABLE A6.2.7: Relative risks of s^2_{NL}, s^2_{AL}, and s^2_{PL}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	λ_2 0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972
	6.2500	6.2633	6.2800	6.3003	6.3242	6.3514	6.3819	6.4161	6.4536	6.5394	6.6389	6.7522	6.8797	7.0208	7.1758
	6.2086	6.2244	6.2453	6.2719	6.3042	6.3414	6.3839	6.4311	6.4822	6.5947	6.7175	6.8469	6.9800	7.1142	7.2472
$\nu=5, \phi=1.0, \lambda_1=0$	6.0661	6.0825	6.1047	6.1322	6.1642	6.1997	6.2383	6.2792	6.3211	6.4075	6.4942	6.5783	6.6589	6.7347	6.8058
	5.6242	5.6319	5.6422	5.6542	5.6672	5.6808	5.6947	5.7086	5.7222	5.7489	5.7739	5.7975	5.8197	5.8403	5.8594
	5.7144	5.7222	5.7328	5.7447	5.7578	5.7711	5.7844	5.7981	5.8111	5.8364	5.8600	5.8822	5.9028	5.9217	5.9394
	5.3086	5.3197	5.3331	5.3481	5.3642	5.3806	5.3975	5.4142	5.4308	5.4633	5.4942	5.5233	5.5511	5.5769	5.6011
	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813	0.7813
	0.6944	0.7048	0.7188	0.7361	0.7569	0.7813	0.8091	0.8403	0.8750	0.9548	1.0486	1.1563	1.2778	1.4131	1.5625
	0.6938	0.7055	0.7213	0.7408	0.7641	0.7908	0.8205	0.8530	0.8878	0.9630	1.0425	1.1236	1.2039	1.2813	1.3542
$\nu=10, \phi=1.0, \lambda_1=0$	0.6863	0.6978	0.7130	0.7308	0.7511	0.7730	0.7959	0.8197	0.8436	0.8905	0.9344	0.9739	1.0084	1.0378	1.0619
	0.6528	0.6578	0.6638	0.6703	0.6769	0.6834	0.6898	0.6958	0.7016	0.7120	0.7209	0.7288	0.7353	0.7408	0.7456
	0.6639	0.6692	0.6753	0.6817	0.6881	0.6945	0.7006	0.7063	0.7116	0.7213	0.7294	0.7363	0.7422	0.7470	0.7513
	0.5970	0.6047	0.6131	0.6219	0.6306	0.6394	0.6477	0.6558	0.6634	0.6773	0.6897	0.7005	0.7097	0.7178	0.7247
	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250	0.1250
	0.0833	0.0920	0.1042	0.1198	0.1389	0.1615	0.1875	0.2170	0.2500	0.3264	0.4167	0.5208	0.6389	0.7708	0.9167
	0.0857	0.0952	0.1077	0.1231	0.1410	0.1611	0.1829	0.2063	0.2307	0.2814	0.3321	0.3803	0.4238	0.4615	0.4917
$\nu=\infty, \phi=1.0, \lambda_1=0$	0.0902	0.0992	0.1100	0.1220	0.1350	0.1484	0.1618	0.1748	0.1873	0.2093	0.2268	0.2390	0.2463	0.2488	0.2471
	0.0965	0.1000	0.1033	0.1063	0.1092	0.1118	0.1140	0.1160	0.1176	0.1201	0.1219	0.1230	0.1237	0.1242	0.1246
	0.0985	0.1021	0.1056	0.1087	0.1114	0.1138	0.1158	0.1175	0.1189	0.1212	0.1226	0.1235	0.1241	0.1246	0.1246
	0.0931	0.0958	0.0987	0.1016	0.1044	0.1071	0.1096	0.1118	0.1137	0.1170	0.1194	0.1212	0.1224	0.1234	0.1239
	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972	6.5972
	6.3819	6.4161	6.4536	6.4947	6.5394	6.5875	6.6389	6.6939	6.7522	6.8797	7.0208	7.1758	7.3450	7.5278	7.7244
	6.3394	6.3719	6.4078	6.4472	6.4903	6.5372	6.5875	6.6417	6.6992	6.8247	6.9631	7.1122	7.2706	7.4350	7.6025
$\nu=5, \phi=1.0, \lambda_1=3$	6.1936	6.2206	6.2511	6.2853	6.3228	6.3636	6.4075	6.4542	6.5031	6.6069	6.7156	6.8256	6.9339	7.0386	7.1381
	5.7425	5.7536	5.7678	5.7844	5.8025	5.8214	5.8411	5.8608	5.8806	5.9186	5.9544	5.9878	6.0186	6.0469	6.0731
	5.8444	5.8561	5.8706	5.8872	5.9053	5.9244	5.9436	5.9631	5.9819	6.0186	6.0525	6.0839	6.1125	6.1389	6.1631
	5.3822	5.3983	5.4169	5.4369	5.4586	5.4811	5.5039	5.5269	5.5500	5.5950	5.6381	5.6783	5.7164	5.7517	5.7844
	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391	1.0391
	0.8091	0.8403	0.8750	0.9131	0.9548	1.0000	1.0486	1.1006	1.1563	1.2778	1.4131	1.5625	1.7256	1.9028	2.0938
	0.8064	0.8372	0.8711	0.9083	0.9488	0.9922	1.0388	1.0880	1.1398	1.2506	1.3689	1.4925	1.6188	1.7453	1.8698
$\nu=10, \phi=1.0, \lambda_1=3$	0.7948	0.8228	0.8534	0.8864	0.9216	0.9584	0.9969	1.0363	1.0763	1.1567	1.2352	1.3094	1.3770	1.4372	1.4892
	0.7558	0.7725	0.7895	0.8066	0.8231	0.8391	0.8542	0.8684	0.8816	0.9052	0.9252	0.9419	0.9558	0.9673	0.9772
	0.7759	0.7933	0.8106	0.8278	0.8442	0.8600	0.8745	0.8881	0.9008	0.9227	0.9409	0.9561	0.9684	0.9788	0.9872
	0.6673	0.6867	0.7064	0.7259	0.7452	0.7639	0.7817	0.7989	0.8150	0.8444	0.8700	0.8920	0.9109	0.9272	0.9409
	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594	0.3594
	0.1875	0.2170	0.2500	0.2865	0.3264	0.3698	0.4167	0.4670	0.5208	0.6389	0.7708	0.9167	1.0764	1.2500	1.4375
	0.1878	0.2173	0.2501	0.2859	0.3248	0.3664	0.4105	0.4570	0.5055	0.6073	0.7135	0.8212	0.9278	1.0309	1.1276
$\nu=\infty, \phi=1.0, \lambda_1=3$	0.1879	0.2162	0.2468	0.2792	0.3129	0.3475	0.3827	0.4177	0.4523	0.5188	0.5793	0.6316	0.6744	0.7070	0.7294
	0.1901	0.2101	0.2289	0.2463	0.2620	0.2760	0.2884	0.2993	0.3089	0.3240	0.3352	0.3430	0.3484	0.3522	0.3546
	0.1994	0.2204	0.2396	0.2569	0.2724	0.2859	0.2979	0.3081	0.3168	0.3304	0.3402	0.3467	0.3512	0.3542	0.3560
	0.1757	0.1930	0.2098	0.2258	0.2407	0.2546	0.2674	0.2789	0.2894	0.3070	0.3208	0.3313	0.3394	0.3452	0.3494
	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494	1.6494
	2.8936	2.9300	2.9700	3.0133	3.0742	3.1106	3.1644	3.2217	3.2825	3.4144	3.5603	3.7200	3.8936	4.0811	4.2825
	2.4906	2.5106	2.5322	2.5550	2.5781	2.6008	2.6231	2.6442	2.6644	2.7014	2.7333	2.7603	2.7831	2.8017	2.8167
$\nu=5, \phi=0.5, \lambda_1=0$	2.0650	2.0731	2.0814	2.0894	2.0969	2.1039	2.1100	2.1153	2.1197	2.1269	2.1314	2.1342	2.1353	2.1350	2.1339
	1.5911	1.5922	1.5936	1.5950	1.5964	1.5978	1.5989	1.6003	1.6014	1.6033	1.6053	1.6069	1.6083	1.6097	1.6108
	1.6003	1.6014	1.6025	1.6039	1.6050	1.6061	1.6072	1.6083	1.6092	1.6111	1.6125	1.6139	1.6153	1.6164	1.6175
	1.5667	1.5681	1.5697	1.5714	1.5731	1.5750	1.5764	1.5781	1.5797	1.5825	1.5850	1.5872	1.5894	1.5911	1.5931
	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953	0.1953
	0.3617	0.3895	0.4208	0.4555	0.4936	0.5353	0.5805	0.6291	0.6811	0.7958	0.9242	1.0666	1.2228	1.3930	1.5770
	0.3095	0.3239	0.3386	0.3530	0.3667	0.3797	0.3914	0.4020	0.4114	0.4261	0.4361	0.4417	0.4438	0.4430	0.4400
$\nu=10, \phi=0.5, \lambda_1=0$	0.2522	0.2578	0.2630	0.2673	0.2709	0.2739	0.2759	0.2775	0.2783	0.2784	0.2769	0.2744	0.2711	0.2673	0.2634
	0.1869	0.1877	0.1884	0.1892	0.1898	0.1903	0.1909	0.1913	0.1917	0.1923	0.1928	0.1933	0.1936	0.1939	0.1941
	0.1883	0.1889	0.1895	0.1902	0.1908	0.1913	0.1917	0.1920	0.1923	0.1930	0.1933	0.1938	0.1939	0.1942	0.1944
	0.1833	0.1844	0.1853	0.1863	0.1870	0.1878	0.1886	0.1892	0.1897	0.1906	0.1914	0.1920	0.1925	0.1930	0.1933
	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313	0.0313
	0.0694	0.0920	0.1181	0.1476	0.1806	0.2170	0.2569	0.3003	0.3472	0.4514	0.5694	0.7014	0.8472	1.0069	1.1806
	0.0584	0.0694	0.0798	0.0893	0.0975	0.1044	0.1099	0.1141	0.1170	0.1191	0.1171	0.1122	0.1051	0.0970	0.0886
$\nu=\infty, \phi=0.5, \lambda_1=0$	0.0451	0.0492	0.0423	0.0545	0.0559	0.0566	0.0566	0.0561	0.0553	0.0527	0.0497	0.0465	0.0435	0.0407	0.0386
	0.0290	0.0295	0.0300	0.0303	0.0305	0.0307	0.0309	0.0310	0.0310	0.0312	0.0312	0.0312	0.0312	0.0313	0.0313
	0.0293	0.0298	0.0302	0.0305	0.0307	0.0308	0.0309	0.0310	0.0311	0.0312	0.0312	0.0312	0.0312	0.0312	0.0313
	0.0280	0.0287	0.0293	0.0297	0.0301	0.0303	0.0306	0.0307	0.0309	0.0311	0.0311	0.0312	0.0312	0.0312	0.0312
	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236	1.7236
	2.9958	3.0428	3.0931	3.1469	3.2042	3.2650	3.3292	3.3969	3.4681	3.6208	3.7875	3.9681	4.1625	4.3708	4.5931
	2.6047	2.6311	2.6589	2.6878	2.7175	2.7475	2.7778	2.8075	2.8367	2.8917	2.9408	2.9836	3.0197	3.0494	3.0733
$\nu=5, \phi=0.5, \lambda_1=3$	2.1667	2.1772	2.1889	2.2006	2.2119	2.2231	2.2331	2.2425	2.2506	2.2639	2.2731	2.2786	2.2814	2.2819	2.2808
	1.6514	1.6531	1.6550	1.6572	1.6594	1.6617	1.6636	1.6656	1.6672	1.6706	1.6731	1.6756	1.6778	1.6794	1.6811
	1.6625	1.6642	1.6658	1.6678	1.6697	1.6717	1.6733	1.6750	1.6764	1.6792	1.6817	1.6836	1.6853	1.6869	1.6883
	1.6219	1.6239	1.6267	1.6292	1.6319	1.6347	1.6372	1.6397	1.6419	1.6461	1.6500	1.6533	1.6561	1.6589	1.6611
	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598	0.2598
	0.4423	0.4806	0.5223	0.5673	0.6161	0.6681	0.7236	0.7827	0.8452	0.9806	1.1298	1.2931	1.4702	1.6611	1.8661
	0.3964	0.4216	0.4473	0.4731	0.4984	0.5230	0.5466	0.5684	0.5889	0.6239	0.6511	0.6702	0.6820	0.6872	0.6869
$\nu=10, \phi=0.5, \lambda_1=3$	0.3316	0.3442	0.3561	0.3669	0.3763	0.3844	0.3909	0.3963	0.4002	0.4042	0.4045	0.4016	0.3967	0.3905	0.3834
	0.2381	0.2409	0.2434	0.2455	0.2473	0.2489	0.2503	0.2514	0.2523	0.2541	0.2552	0.2561	0.2567	0.2573	0.2577
	0.2413	0.2438	0.2459	0.2478	0.2494	0.2508	0.2519	0.2530	0.2538	0.2552	0.2561	0.2569	0.2573	0.2578	0.2581
	0.2306	0.2338	0.2367	0.2394	0.2416	0.2438	0.2455	0.2470	0.2484	0.2506	0.2525	0.2538	0.2397	0.2556	0.2563
	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898	0.0898
	0.1372	0.1701	0.2066	0.2465	0.2899	0.3368	0.3872	0.4410	0.4983	0.6233	0.7622	0.9149	1.0816	1.2622	1.4566
	0.1289	0.1535	0.1780	0.2019	0.2246	0.2459	0.2653	0.2827	0.2979	0.3212	0.3351	0.3401	0.3375	0.3286	0.3149
$\nu=\infty, \phi=0.5, \lambda_1=3$	0.1109	0.1248	0.1369	0.1471	0.1553	0.1616	0.1661	0.1689	0.1702	0.1692	0.1644	0.1573	0.1489	0.1403	0.1320
	0.0749	0.0783	0.0810	0.0831	0.0847	0.0859	0.0869	0.0876	0.0882	0.0889	0.0893	0.0895	0.0897	0.0897	0.0898
	0.0769	0.0800	0.0823	0.0842	0.0856	0.0867	0.0874	0.0881	0.0885	0.0891	0.0894	0.0897	0.0897	0.0898	0.0898
	0.0704	0.0742	0.0774	0.0800	0.0820	0.0837	0.0850	0.0860	0.0869	0.0881	0.0888	0.0892	0.0895	0.0896	0.0898

TABLE A6.2.7 (continued)

Estimator	0.0	0.5	1.0	1.5	2.0	2.5 λ_2	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661	0.0661
	1.3750	1.4300	1.4883	1.5503	1.6158	1.6847	1.7569	1.8328	1.9119	2.0811	2.2639	2.4606	2.6714	2.8958	3.1342
	0.0847	0.0842	0.0836	0.0831	0.0828	0.0822	0.0819	0.0814	0.0811	0.0806	0.0800	0.0794	0.0789	0.0783	0.0781
$\nu=5, \tau=0.1, \lambda_1=0$	0.0689	0.0689	0.0689	0.0686	0.0686	0.0686	0.0683	0.0683	0.0683	0.0683	0.0681	0.0681	0.0681	0.0678	0.0678
	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0833	0.0867
	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658
	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658	0.0658
	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
	0.2830	0.3247	0.3698	0.4184	0.4705	0.5261	0.5850	0.6475	0.7136	0.8559	1.0122	1.1823	1.3663	1.5642	1.7761
	0.0113	0.0109	0.0106	0.0103	0.0100	0.0098	0.0095	0.0094	0.0092	0.0089	0.0088	0.0086	0.0084	0.0084	0.0083
$\nu=10, \tau=0.1, \lambda_1=0$	0.0083	0.0083	0.0083	0.0081	0.0081	0.0081	0.0081	0.0080	0.0080	0.0080	0.0080	0.0080	0.0080	0.0078	0.0078
	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078	0.0078
	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
	0.1183	0.1520	0.1892	0.2298	0.2739	0.3215	0.3725	0.4270	0.4850	0.6114	0.7517	0.9058	1.0739	1.2558	1.4517
	0.0025	0.0023	0.0020	0.0019	0.0017	0.0016	0.0015	0.0014	0.0014	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
$\nu=\infty, \tau=0.1, \lambda_1=0$	0.0014	0.0014	0.0014	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
	0.0012	0.0012	0.0012	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
	0.0012	0.0012	0.0012	0.0013	0.0025	0.0030	0.0036	0.0042	0.0050	0.0067	0.0087	0.0111	0.0137	0.0167	0.0199
	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013	0.0013
	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689
	1.4014	1.4583	1.5189	1.5831	1.6503	1.7214	1.7958	1.8736	1.9550	2.1281	2.3153	2.5161	2.7308	2.9597	3.2022
	0.0931	0.0919	0.0911	0.0903	0.0894	0.0889	0.0881	0.0875	0.0869	0.0858	0.0850	0.0842	0.0836	0.0831	0.0825
$\nu=5, \tau=0.1, \lambda_1=3$	0.0728	0.0728	0.0725	0.0725	0.0722	0.0722	0.0719	0.0719	0.0717	0.0717	0.0714	0.0714	0.0711	0.0711	0.0711
	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689
	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689
	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689	0.0689
	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105
	0.3028	0.3466	0.3939	0.4445	0.4988	0.5564	0.6175	0.6820	0.7502	0.8966	1.0570	1.2314	1.4195	1.6216	1.8377
	0.0183	0.0173	0.0166	0.0158	0.0152	0.0147	0.0141	0.0138	0.0133	0.0127	0.0122	0.0119	0.0116	0.0114	0.0113
$\nu=10, \tau=0.1, \lambda_1=3$	0.0117	0.0116	0.0114	0.0113	0.0111	0.0109	0.0109	0.0108	0.0108	0.0106	0.0106	0.0106	0.0105	0.0105	0.0105
	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0105	0.0105	0.0105	0.0105	0.0105
	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0105	0.0105	0.0105	0.0105	0.0105	0.0105
	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0103	0.0105	0.0105
	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036
	0.1344	0.1701	0.2094	0.2521	0.2983	0.3479	0.4010	0.4576	0.5177	0.6483	0.7927	0.9510	1.1233	1.3094	1.5094
	0.0089	0.0080	0.0072	0.0065	0.0059	0.0054	0.0051	0.0047	0.0045	0.0041	0.0039	0.0038	0.0037	0.0036	0.0036
$\nu=\infty, \tau=0.1, \lambda_1=3$	0.0046	0.0043	0.0042	0.0040	0.0039	0.0038	0.0038	0.0038	0.0037	0.0037	0.0037	0.0036	0.0036	0.0036	0.0036
	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036
	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036
	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036	0.0036

TABLE A6.2.8: Relative risks of s^2_{NML}, s^2_{AML}, and s^2_{PML}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	λ_2 0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475	4.7475
	4.1111	4.0975	4.0858	4.0767	4.0697	4.0647	4.0622	4.0619	4.0636	4.0742	4.0933	4.1214	4.1586	4.2044	4.2592
	4.0897	4.0797	4.0736	4.0722	4.0750	4.0825	4.0942	4.1097	4.1292	4.1775	4.2364	4.3031	4.3750	4.4503	4.5272
$\nu=5,\phi=1.0,\lambda_1=0$	4.0164	4.0106	4.0097	4.0139	4.0222	4.0344	4.0494	4.0672	4.0869	4.1303	4.1769	4.2247	4.2722	4.3186	4.3631
	3.9894	3.9875	3.9892	3.9933	3.9994	4.0067	4.0147	4.0233	4.0322	4.0503	4.0681	4.0853	4.1017	4.1172	4.1319
	4.0681	4.0667	4.0692	4.0739	4.0803	4.0878	4.0958	4.1042	4.1128	4.1300	4.1469	4.1631	4.1783	4.1928	4.2061
	3.7678	3.7628	3.7614	3.7628	3.7661	3.7711	3.7775	3.7844	3.7922	3.8092	3.8267	3.8444	3.8622	3.8792	3.8961
	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930	0.5930
	0.5069	0.4969	0.4892	0.4836	0.4803	0.4792	0.4803	0.4836	0.4892	0.5069	0.5336	0.5692	0.6136	0.6669	0.7292
	0.5091	0.5014	0.4969	0.4955	0.4972	0.5019	0.5094	0.5194	0.5319	0.5627	0.5997	0.6409	0.6842	0.7281	0.7714
$\nu=10,\phi=1.0,\lambda_1=0$	0.5114	0.5066	0.5052	0.5067	0.5108	0.5170	0.5250	0.5344	0.5448	0.5675	0.5911	0.6139	0.6353	0.6547	0.6717
	0.5220	0.5200	0.5200	0.5213	0.5233	0.5259	0.5289	0.5320	0.5353	0.5416	0.5473	0.5525	0.5572	0.5611	0.5647
	0.5284	0.5270	0.5273	0.5289	0.5313	0.5341	0.5370	0.5400	0.5431	0.5489	0.5542	0.5589	0.5630	0.5666	0.5695
	0.5061	0.5022	0.5000	0.4995	0.5002	0.5017	0.5039	0.5064	0.5094	0.5155	0.5219	0.5281	0.5339	0.5392	0.5441
	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136	0.1136
	0.0933	0.0856	0.0800	0.0767	0.0756	0.0767	0.0800	0.0856	0.0933	0.1156	0.1467	0.1867	0.2356	0.2933	0.3600
	0.0964	0.0904	0.0868	0.0855	0.0864	0.0893	0.0940	0.1004	0.1082	0.1272	0.1494	0.1728	0.1959	0.2180	0.2375
$\nu=\infty,\phi=1.0,\lambda_1=0$	0.1035	0.0995	0.0976	0.0975	0.0989	0.1014	0.1049	0.1091	0.1137	0.1233	0.1326	0.1405	0.1468	0.1512	0.1536
	0.1137	0.1117	0.1105	0.1099	0.1097	0.1098	0.1101	0.1104	0.1108	0.1115	0.1121	0.1126	0.1129	0.1131	0.1133
	0.1129	0.1114	0.1106	0.1102	0.1102	0.1104	0.1107	0.1110	0.1113	0.1120	0.1125	0.1129	0.1131	0.1134	0.1134
	0.1174	0.1139	0.1115	0.1098	0.1089	0.1083	0.1081	0.1082	0.1084	0.1092	0.1101	0.1108	0.1115	0.1120	0.1125
	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919	4.7919
	4.0622	4.0619	4.0636	4.0678	4.0742	4.0825	4.0933	4.1064	4.1214	4.1586	4.2044	4.2592	4.3231	4.3956	4.4769
	4.0378	4.0369	4.0386	4.0431	4.0497	4.0592	4.0717	4.0867	4.1047	4.1492	4.2050	4.2711	4.3461	4.4292	4.5172
$\nu=5,\phi=1.0,\lambda_1=3$	3.9572	3.9550	3.9556	3.9594	3.9661	3.9761	3.9889	4.0047	4.0231	4.0669	4.1181	4.1742	4.2331	4.2928	4.3514
	3.9325	3.9281	3.9275	3.9306	3.9364	3.9442	3.9539	3.9644	3.9758	3.9997	4.0236	4.0469	4.0692	4.0903	4.1100
	4.0200	4.0156	4.0158	4.0194	4.0258	4.0342	4.0442	4.0550	4.0661	4.0894	4.1125	4.1344	4.1553	4.1747	4.1928
	3.6878	3.6822	3.6797	3.6800	3.6828	3.6875	3.6942	3.7022	3.7114	3.7319	3.7544	3.7778	3.8008	3.8236	3.8456
	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511	0.6511
	0.4803	0.4836	0.4892	0.4969	0.5069	0.5192	0.5336	0.5503	0.5692	0.6136	0.6669	0.7292	0.8003	0.8803	0.9692
	0.4795	0.4830	0.4886	0.4966	0.5069	0.5195	0.5342	0.5513	0.5705	0.6147	0.6663	0.7238	0.7856	0.8508	0.9173
$\nu=10,\phi=1.0,\lambda_1=3$	0.4756	0.4788	0.4841	0.4916	0.5013	0.5128	0.5261	0.5409	0.5572	0.5925	0.6303	0.6686	0.7061	0.7413	0.7734
	0.4875	0.4905	0.4952	0.5013	0.5081	0.5156	0.5233	0.5309	0.5386	0.5528	0.5658	0.5770	0.5867	0.5952	0.6022
	0.5006	0.5042	0.5095	0.5159	0.5231	0.5308	0.5384	0.5459	0.5531	0.5667	0.5786	0.5888	0.5973	0.6047	0.6108
	0.4555	0.4561	0.4583	0.4619	0.4666	0.4720	0.4781	0.4848	0.4917	0.5058	0.5195	0.5327	0.5447	0.5556	0.5653
	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801	0.1801
	0.0800	0.0856	0.0933	0.1033	0.1156	0.1300	0.1467	0.1656	0.1867	0.2356	0.2933	0.3600	0.4356	0.5200	0.6133
	0.0805	0.0864	0.0945	0.1047	0.1171	0.1316	0.1479	0.1661	0.1860	0.2301	0.2791	0.3311	0.3849	0.4389	0.4913
$\nu=\infty,\phi=1.0,\lambda_1=3$	0.0820	0.0881	0.0963	0.1061	0.1175	0.1300	0.1438	0.1581	0.1730	0.2034	0.2331	0.2607	0.2849	0.3050	0.3208
	0.0925	0.1001	0.1079	0.1157	0.1233	0.1304	0.1369	0.1429	0.1483	0.1572	0.1640	0.1689	0.1725	0.1750	0.1767
	0.0971	0.1056	0.1140	0.1221	0.1298	0.1367	0.1432	0.1488	0.1537	0.1617	0.1676	0.1716	0.1745	0.1765	0.1777
	0.0843	0.0885	0.0935	0.0991	0.1049	0.1109	0.1168	0.1226	0.1282	0.1384	0.1473	0.1549	0.1608	0.1656	0.1693
	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869	1.1869
	1.8056	1.8178	1.8322	1.8489	1.8678	1.8889	1.9122	1.9378	1.9656	2.0278	2.0989	2.1789	2.2678	2.3656	2.4722
	1.5761	1.5844	1.5947	1.6067	1.6194	1.6328	1.6461	1.6594	1.6725	1.6969	1.7189	1.7383	1.7550	1.7694	1.7817
$\nu=5,\phi=0.5,\lambda_1=0$	1.3406	1.3444	1.3489	1.3539	1.3589	1.3639	1.3683	1.3725	1.3764	1.3828	1.3878	1.3919	1.3947	1.3967	1.3981
	1.1400	1.1406	1.1411	1.1419	1.1428	1.1439	1.1447	1.1456	1.1464	1.1481	1.1494	1.1508	1.1519	1.1531	1.1542
	1.1483	1.1486	1.1492	1.1500	1.1508	1.1514	1.1522	1.1531	1.1536	1.1550	1.1564	1.1575	1.1583	1.1592	1.1600
	1.1050	1.1056	1.1064	1.1075	1.1086	1.1100	1.1114	1.1125	1.1139	1.1164	1.1186	1.1208	1.1228	1.1247	1.1264
	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483	0.1483
	0.2055	0.2148	0.2266	0.2405	0.2566	0.2748	0.2955	0.3183	0.3433	0.3998	0.4655	0.5398	0.6233	0.7155	0.8166
	0.1855	0.1913	0.1981	0.2055	0.2130	0.2205	0.2277	0.2344	0.2406	0.2514	0.2597	0.2656	0.2694	0.2714	0.2719
$\nu=10,\phi=0.5,\lambda_1=0$	0.1630	0.1655	0.1683	0.1709	0.1734	0.1756	0.1777	0.1792	0.1805	0.1822	0.1828	0.1827	0.1822	0.1811	0.1798
	0.1430	0.1431	0.1436	0.1439	0.1442	0.1445	0.1448	0.1452	0.1455	0.1459	0.1463	0.1466	0.1469	0.1470	0.1472
	0.1439	0.1441	0.1444	0.1447	0.1450	0.1453	0.1456	0.1458	0.1461	0.1464	0.1467	0.1470	0.1472	0.1473	0.1475
	0.1392	0.1394	0.1397	0.1403	0.1408	0.1413	0.1417	0.1422	0.1427	0.1434	0.1442	0.1447	0.1453	0.1456	0.1459
	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284
	0.0278	0.0356	0.0456	0.0578	0.0722	0.0889	0.1078	0.1289	0.1522	0.2056	0.2678	0.3389	0.4189	0.5078	0.6056
	0.0289	0.0336	0.0380	0.0425	0.0469	0.0508	0.0543	0.0573	0.0597	0.0627	0.0637	0.0630	0.0610	0.0582	0.0549
$\nu=\infty,\phi=0.5,\lambda_1=0$	0.0286	0.0304	0.0321	0.0334	0.0345	0.0353	0.0359	0.0361	0.0363	0.0359	0.0352	0.0343	0.0333	0.0323	0.0315
	0.0277	0.0278	0.0279	0.0280	0.0281	0.0282	0.0282	0.0283	0.0283	0.0284	0.0284	0.0284	0.0284	0.0284	0.0284
	0.0278	0.0279	0.0280	0.0281	0.0282	0.0282	0.0282	0.0283	0.0284	0.0283	0.0283	0.0284	0.0284	0.0284	0.0284
	0.0273	0.0274	0.0275	0.0276	0.0278	0.0279	0.0280	0.0280	0.0281	0.0282	0.0283	0.0283	0.0284	0.0284	0.0284
	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981	1.1981
	1.8378	1.8567	1.8778	1.9011	1.9267	1.9544	1.9844	2.0167	2.0511	2.1267	2.2111	2.3044	2.4067	2.5178	2.6378
	1.6072	1.6164	1.6275	1.6403	1.6542	1.6692	1.6850	1.7014	1.7178	1.7508	1.7819	1.8106	1.8361	1.8581	1.8764
$\nu=5,\phi=0.5,\lambda_1=3$	1.3600	1.3631	1.3672	1.3728	1.3786	1.3850	1.3914	1.3975	1.4033	1.4136	1.4222	1.4289	1.4336	1.4372	1.4394
	1.1433	1.1431	1.1436	1.1447	1.1458	1.1472	1.1486	1.1497	1.1511	1.1533	1.1556	1.1572	1.1589	1.1606	1.1619
	1.1528	1.1525	1.1531	1.1539	1.1550	1.1561	1.1572	1.1583	1.1594	1.1614	1.1633	1.1647	1.1661	1.1675	1.1686
	1.1031	1.1031	1.1036	1.1047	1.1064	1.1081	1.1100	1.1119	1.1136	1.1172	1.1206	1.1236	1.1264	1.1289	1.1311
	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628	0.1628
	0.2320	0.2483	0.2666	0.2870	0.3098	0.3348	0.3620	0.3916	0.4233	0.4933	0.5720	0.6598	0.7566	0.8620	0.9766
	0.2095	0.2202	0.2316	0.2436	0.2559	0.2686	0.2811	0.2933	0.3050	0.3263	0.3441	0.3580	0.3680	0.3745	0.3778
$\nu=10,\phi=0.5,\lambda_1=3$	0.1794	0.1847	0.1900	0.1953	0.2005	0.2053	0.2095	0.2133	0.2164	0.2209	0.2234	0.2242	0.2236	0.2220	0.2198
	0.1494	0.1505	0.1517	0.1528	0.1539	0.1548	0.1558	0.1566	0.1572	0.1583	0.1592	0.1598	0.1603	0.1608	0.1611
	0.1516	0.1525	0.1536	0.1545	0.1555	0.1564	0.1570	0.1577	0.1583	0.1592	0.1600	0.1605	0.1609	0.1613	0.1614
	0.1409	0.1423	0.1438	0.1453	0.1467	0.1481	0.1494	0.1506	0.1517	0.1536	0.1552	0.1564	0.1575	0.1583	0.1591
	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450	0.0450
	0.0511	0.0656	0.0822	0.1011	0.1222	0.1456	0.1711	0.1989	0.2289	0.2956	0.3711	0.4556	0.5489	0.6511	0.7622
	0.0490	0.0602	0.0717	0.0835	0.0950	0.1062	0.1168	0.1265	0.1353	0.1496	0.1594	0.1646	0.1657	0.1633	0.1581
$\nu=\infty,\phi=0.5,\lambda_1=3$	0.0434	0.0499	0.0559	0.0613	0.0659	0.0697	0.0727	0.0750	0.0765	0.0777	0.0769	0.0747	0.0716	0.0681	0.0646
	0.0360	0.0379	0.0395	0.0407	0.0417	0.0424	0.0430	0.0435	0.0439	0.0443	0.0446	0.0448	0.0449	0.0449	0.0450
	0.0373	0.0390	0.0404	0.0415	0.0423	0.0430	0.0435	0.0439	0.0442	0.0445	0.0447	0.0449	0.0449	0.0450	0.0450
	0.0315	0.0337	0.0356	0.0372	0.0386	0.0398	0.0408	0.0416	0.0423	0.0432	0.0439	0.0443	0.0446	0.0448	0.0448

TABLE A6.2.8 (continued)

Estimator	λ_2														
	0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
	0.8544	0.8875	0.9225	0.9600	0.9997	1.0414	1.0856	1.1319	1.1803	1.2842	1.3967	1.5181	1.6486	1.7878	1.9358
	0.0581	0.0578	0.0578	0.0575	0.0572	0.0569	0.0567	0.0567	0.0564	0.0561	0.0556	0.0553	0.0550	0.0547	0.0547
$\nu=5, \phi=0.1, \lambda_1=0$	0.0492	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0489	0.0486	0.0486	0.0486	0.0486	0.0486	0.0486	0.0486
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475	0.0475
	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059
	0.1667	0.1917	0.2189	0.2484	0.2800	0.3139	0.3500	0.3884	0.4289	0.5167	0.6134	0.7189	0.8334	0.9567	1.0889
	0.0077	0.0075	0.0073	0.0072	0.0070	0.0070	0.0069	0.0069	0.0067	0.0066	0.0064	0.0064	0.0063	0.0063	0.0063
$\nu=10, \phi=0.1, \lambda_1=0$	0.0063	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0061	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059
	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059
	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059
	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059	0.0059
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0665	0.0868	0.1092	0.1339	0.1608	0.1899	0.2212	0.2548	0.2905	0.3688	0.4559	0.5519	0.6568	0.7705	0.8932
	0.0017	0.0016	0.0015	0.0014	0.0014	0.0013	0.0013	0.0012	0.0012	0.0012	0.0012	0.0011	0.0011	0.0011	0.0011
$\nu=\infty, \phi=0.1, \lambda_1=0$	0.0012	0.0012	0.0012	0.0012	0.0012	0.0012	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481	0.0481
	0.8700	0.9042	0.9408	0.9794	1.0206	1.0636	1.1092	1.1567	1.2067	1.3131	1.4281	1.5522	1.6853	1.8272	1.9781
	0.0611	0.0608	0.0606	0.0600	0.0597	0.0594	0.0589	0.0586	0.0583	0.0578	0.0572	0.0569	0.0564	0.0561	0.0558
$\nu=5, \phi=0.1, \lambda_1=3$	0.0500	0.0497	0.0497	0.0497	0.0497	0.0494	0.0494	0.0494	0.0494	0.0494	0.0492	0.0492	0.0492	0.0492	0.0492
	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478
	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478
	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478	0.0478
	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066
	0.1784	0.2048	0.2333	0.2641	0.2972	0.3323	0.3698	0.4095	0.4514	0.5419	0.6411	0.7494	0.8664	0.9925	1.1273
	0.0106	0.0102	0.0097	0.0094	0.0091	0.0088	0.0086	0.0083	0.0081	0.0078	0.0075	0.0073	0.0072	0.0070	0.0070
$\nu=10, \phi=0.1, \lambda_1=3$	0.0072	0.0070	0.0070	0.0069	0.0069	0.0069	0.0067	0.0067	0.0067	0.0067	0.0066	0.0066	0.0066	0.0066	0.0066
	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066
	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066
	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066
	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
	0.0760	0.0976	0.1213	0.1473	0.1756	0.2060	0.2387	0.2736	0.3107	0.3916	0.4813	0.5800	0.6876	0.8040	0.9293
	0.0045	0.0040	0.0036	0.0033	0.0030	0.0028	0.0026	0.0024	0.0023	0.0021	0.0020	0.0019	0.0019	0.0018	0.0018
$\nu=\infty, \phi=0.1, \lambda_1=3$	0.0022	0.0021	0.0021	0.0020	0.0019	0.0019	0.0019	0.0019	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018
	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018	0.0018

TABLE A6.2.9: Relative risks of s^2_{NM}, s^2_{AM}, and s^2_{PM}.

$v_1 = 16$, $v_2 = 8$, $T_1 = 19$, $T_2 = 11$, $k = 3$.

Estimator	λ_2 0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469	5.2469
	4.6486	4.6414	4.6367	4.6344	4.6350	4.6378	4.6433	4.6514	4.6622	4.6911	4.7303	4.7794	4.8389	4.9086	4.9886
	4.6222	4.6183	4.6189	4.6242	4.6344	4.6492	4.6686	4.6922	4.7194	4.7842	4.8594	4.9422	5.0300	5.1206	5.2122
$\nu=5, \ddagger=1.0, \lambda_1=0$	4.5314	4.5308	4.5358	4.5458	4.5603	4.5783	4.5994	4.6233	4.6486	4.7033	4.7606	4.8183	4.8747	4.9289	4.9806
	4.4172	4.4181	4.4222	4.4289	4.4372	4.4464	4.4561	4.4664	4.4767	4.4975	4.5175	4.5367	4.5550	4.5722	4.5886
	4.5011	4.5025	4.5072	4.5142	4.5225	4.5319	4.5417	4.5514	4.5614	4.5814	4.6003	4.6183	4.6353	4.6511	4.6661
	4.2008	4.1992	4.2008	4.2050	4.2111	4.2186	4.2272	4.2364	4.2461	4.2664	4.2869	4.3075	4.3272	4.3464	4.3650
	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366	0.6366
	0.5420	0.5370	0.5344	0.5344	0.5370	0.5420	0.5497	0.5600	0.5727	0.6059	0.6492	0.7028	0.7666	0.8406	0.9247
	0.5436	0.5406	0.5409	0.5447	0.5517	0.5619	0.5750	0.5906	0.6088	0.6508	0.6988	0.7503	0.8033	0.8559	0.9072
$\nu=10, :=1.0, \lambda_1=0$	0.5436	0.5428	0.5455	0.5511	0.5592	0.5695	0.5814	0.5945	0.6084	0.6375	0.6666	0.6941	0.7192	0.7416	0.7608
	0.5478	0.5478	0.5495	0.5523	0.5558	0.5595	0.5636	0.5677	0.5716	0.5792	0.5859	0.5920	0.5972	0.6019	0.6058
	0.5558	0.5564	0.5584	0.5614	0.5650	0.5688	0.5727	0.5766	0.5803	0.5873	0.5936	0.5989	0.6036	0.6075	0.6109
	0.5288	0.5269	0.5270	0.5284	0.5308	0.5338	0.5373	0.5411	0.5452	0.5531	0.5608	0.5681	0.5747	0.5806	0.5861
	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
	0.0816	0.0778	0.0765	0.0778	0.0816	0.0880	0.0969	0.1084	0.1224	0.1582	0.2041	0.2602	0.3265	0.4031	0.4898
	0.0846	0.0823	0.0826	0.0854	0.0905	0.0977	0.1067	0.1173	0.1293	0.1564	0.1858	0.2157	0.2442	0.2705	0.2930
$\nu=\infty, :=1.0, \lambda_1=0$	0.0912	0.0904	0.0916	0.0945	0.0988	0.1041	0.1102	0.1167	0.1233	0.1364	0.1480	0.1573	0.1642	0.1684	0.1700
	0.1027	0.1023	0.1024	0.1028	0.1036	0.1045	0.1053	0.1061	0.1069	0.1082	0.1091	0.1098	0.1102	0.1105	0.1100
	0.1028	0.1027	0.1032	0.1038	0.1046	0.1055	0.1063	0.1070	0.1077	0.1088	0.1096	0.1102	0.1105	0.1109	0.1100
	0.1036	0.1018	0.1010	0.1007	0.1009	0.1013	0.1020	0.1028	0.1036	0.1052	0.1067	0.1078	0.1086	0.1094	0.1100
	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581	5.3581
	4.6433	4.6514	4.6622	4.6753	4.6911	4.7094	4.7303	4.7536	4.7794	4.8389	4.9086	4.9886	5.0789	5.1792	5.2897
	4.6144	4.6217	4.6319	4.6447	4.6606	4.6792	4.7011	4.7258	4.7536	4.8186	4.8953	4.9825	5.0789	5.1828	5.2914
$\nu=5, :=1.0, \lambda_1=3$	4.5172	4.5222	4.5303	4.5417	4.5561	4.5736	4.5944	4.6178	4.6442	4.7033	4.7694	4.8397	4.9117	4.9833	5.0531
	4.4033	4.4033	4.4069	4.4142	4.4236	4.4347	4.4475	4.4608	4.4747	4.5031	4.5308	4.5572	4.5822	4.6056	4.6275
	4.4972	4.4975	4.5017	4.5092	4.5192	4.5308	4.5433	4.5567	4.5706	4.5978	4.6242	4.6492	4.6725	4.6942	4.7142
	4.1639	4.1625	4.1644	4.1689	4.1756	4.1842	4.1944	4.2058	4.2183	4.2444	4.2719	4.2992	4.3258	4.3517	4.3761
	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477	0.7477
	0.5497	0.5600	0.5727	0.5880	0.6059	0.6263	0.6492	0.6747	0.7028	0.7666	0.8406	0.9247	1.0191	1.1238	1.2386
	0.5486	0.5586	0.5713	0.5866	0.6044	0.6247	0.6473	0.6725	0.6998	0.7609	0.8294	0.9038	0.9822	1.0631	1.1447
$\nu=10, :=1.0, \lambda_1=3$	0.5428	0.5520	0.5636	0.5775	0.5936	0.6116	0.6313	0.6523	0.6745	0.7216	0.7700	0.8178	0.8633	0.9053	0.9431
	0.5469	0.5539	0.5623	0.5716	0.5814	0.5914	0.6014	0.6111	0.6203	0.6377	0.6527	0.6658	0.6769	0.6863	0.6942
	0.5627	0.5702	0.5791	0.5888	0.5988	0.6088	0.6184	0.6278	0.6367	0.6530	0.6667	0.6784	0.6883	0.6966	0.7036
	0.5114	0.5159	0.5219	0.5289	0.5369	0.5453	0.5542	0.5633	0.5723	0.5900	0.6067	0.6220	0.6358	0.6481	0.6589
	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222	0.2222
	0.0969	0.1084	0.1224	0.1390	0.1582	0.1798	0.2041	0.2309	0.2602	0.3265	0.4031	0.4898	0.5867	0.6939	0.8112
	0.0974	0.1091	0.1233	0.1400	0.1590	0.1802	0.2036	0.2289	0.2560	0.3147	0.3783	0.4446	0.5119	0.5787	0.6425
$\nu=\infty, :=1.0, \lambda_1=3$	0.0986	0.1102	0.1240	0.1395	0.1565	0.1746	0.1938	0.2134	0.2334	0.2731	0.3109	0.3451	0.3745	0.3982	0.4161
	0.1096	0.1209	0.1320	0.1426	0.1527	0.1619	0.1702	0.1777	0.1844	0.1953	0.2035	0.2093	0.2135	0.2165	0.2184
	0.1159	0.1281	0.1397	0.1505	0.1606	0.1695	0.1776	0.1847	0.1908	0.2005	0.2076	0.2124	0.2158	0.2181	0.2195
	0.0983	0.1062	0.1146	0.1232	0.1318	0.1401	0.1483	0.1559	0.1630	0.1758	0.1865	0.1951	0.2020	0.2072	0.2113
	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117	1.3117
	2.0833	2.1017	2.1225	2.1458	2.1717	2.2003	2.2314	2.2650	2.3011	2.3808	2.4711	2.5714	2.6819	2.8028	2.9336
	1.8094	1.8208	1.8342	1.8486	1.8642	1.8800	1.8958	1.9111	1.9261	1.9539	1.9789	2.0003	2.0189	2.0344	2.0478
$\nu=5, :=0.5, \lambda_1=0$	1.5258	1.5308	1.5364	1.5422	1.5481	1.5536	1.5586	1.5631	1.5672	1.5742	1.5794	1.5831	1.5858	1.5872	1.5883
	1.2608	1.2617	1.2625	1.2636	1.2644	1.2656	1.2667	1.2675	1.2686	1.2703	1.2719	1.2733	1.2744	1.2758	1.2769
	1.2697	1.2703	1.2711	1.2719	1.2728	1.2736	1.2744	1.2753	1.2761	1.2775	1.2789	1.2803	1.2814	1.2822	1.2831
	1.2281	1.2289	1.2300	1.2314	1.2328	1.2342	1.2356	1.2369	1.2383	1.2411	1.2436	1.2458	1.2478	1.2497	1.2514
	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592	0.1592
	0.2419	0.2558	0.2725	0.2916	0.3133	0.3375	0.3642	0.3936	0.4255	0.4969	0.5786	0.6703	0.7725	0.8847	1.0072
	0.2138	0.2219	0.2308	0.2398	0.2491	0.2580	0.2664	0.2742	0.2814	0.2933	0.3022	0.3083	0.3117	0.3131	0.3127
$\nu=10, :=0.5, \lambda_1=0$	0.1830	0.1863	0.1897	0.1928	0.1958	0.1983	0.2003	0.2019	0.2031	0.2045	0.2048	0.2042	0.2030	0.2014	0.1995
	0.1528	0.1533	0.1538	0.1542	0.1547	0.1550	0.1555	0.1558	0.1561	0.1566	0.1570	0.1573	0.1577	0.1578	0.1581
	0.1539	0.1542	0.1547	0.1552	0.1555	0.1558	0.1561	0.1564	0.1567	0.1572	0.1575	0.1578	0.1580	0.1581	0.1583
	0.1489	0.1494	0.1500	0.1506	0.1513	0.1519	0.1523	0.1530	0.1534	0.1542	0.1550	0.1556	0.1561	0.1566	0.1569
	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278	0.0278
	0.0357	0.0472	0.0612	0.0778	0.0969	0.1186	0.1429	0.1696	0.1990	0.2653	0.3418	0.4286	0.5255	0.6327	0.7500
	0.0339	0.0401	0.0462	0.0520	0.0574	0.0621	0.0662	0.0695	0.0721	0.0751	0.0754	0.0737	0.0706	0.0664	0.0619
$\nu=\infty, :=0.5, \lambda_1=0$	0.0306	0.0330	0.0351	0.0367	0.0380	0.0388	0.0393	0.0394	0.0394	0.0385	0.0373	0.0359	0.0344	0.0330	0.0319
	0.0266	0.0268	0.0271	0.0272	0.0273	0.0275	0.0275	0.0276	0.0276	0.0277	0.0277	0.0278	0.0278	0.0278	0.0278
	0.0268	0.0270	0.0272	0.0273	0.0274	0.0275	0.0276	0.0276	0.0277	0.0277	0.0277	0.0277	0.0278	0.0278	0.0278
	0.0259	0.0262	0.0265	0.0267	0.0269	0.0271	0.0272	0.0274	0.0275	0.0276	0.0277	0.0277	0.0278	0.0278	0.0284
	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394	1.3394
	2.1331	2.1589	2.1875	2.2186	2.2522	2.2883	2.3269	2.3683	2.4119	2.5072	2.6128	2.7283	2.8542	2.9903	3.1364
	1.8611	1.8747	1.8900	1.9067	1.9247	1.9436	1.9633	1.9831	2.0031	2.0417	2.0778	2.1103	2.1389	2.1631	2.1833
$\nu=5, :=0.5, \lambda_1=3$	1.5653	1.5706	1.5767	1.5839	1.5914	1.5989	1.6064	1.6133	1.6200	1.6314	1.6406	1.6472	1.6517	1.6547	1.6561
	1.2789	1.2792	1.2803	1.2817	1.2833	1.2847	1.2864	1.2878	1.2892	1.2919	1.2942	1.2961	1.2981	1.2997	1.3011
	1.2892	1.2894	1.2903	1.2917	1.2931	1.2942	1.2956	1.2969	1.2981	1.3003	1.3022	1.3039	1.3056	1.3069	1.3081
	1.2406	1.2411	1.2425	1.2442	1.2461	1.2483	1.2503	1.2525	1.2544	1.2583	1.2619	1.2650	1.2681	1.2706	1.2728
	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869	0.1869
	0.2820	0.3038	0.3280	0.3547	0.3841	0.4159	0.4503	0.4873	0.5269	0.6136	0.7106	0.8177	0.9350	1.0627	1.2003
	0.2536	0.2678	0.2828	0.2984	0.3142	0.3298	0.3453	0.3600	0.3741	0.3991	0.4194	0.4348	0.4455	0.4519	0.4544
$\nu=10, :=0.5, \lambda_1=3$	0.2148	0.2220	0.2291	0.2359	0.2422	0.2480	0.2530	0.2572	0.2606	0.2653	0.2673	0.2675	0.2659	0.2633	0.2600
	0.1709	0.1725	0.1741	0.1755	0.1769	0.1780	0.1791	0.1800	0.1808	0.1820	0.1830	0.1838	0.1842	0.1847	0.1852
	0.1736	0.1748	0.1763	0.1775	0.1786	0.1797	0.1805	0.1813	0.1819	0.1830	0.1838	0.1844	0.1848	0.1852	0.1855
	0.1622	0.1642	0.1661	0.1680	0.1698	0.1714	0.1730	0.1742	0.1755	0.1775	0.1792	0.1805	0.1816	0.1823	0.1831
	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556	0.0556
	0.0702	0.0893	0.1110	0.1352	0.1620	0.1913	0.2232	0.2577	0.2946	0.3763	0.4681	0.5702	0.6824	0.8048	0.9375
	0.0665	0.0811	0.0960	0.1109	0.1253	0.1391	0.1520	0.1637	0.1742	0.1910	0.2020	0.2073	0.2076	0.2037	0.1965
$\nu=\infty, :=0.5, \lambda_1=3$	0.0579	0.0664	0.0740	0.0807	0.0863	0.0908	0.0943	0.0968	0.0984	0.0992	0.0975	0.0942	0.0899	0.0853	0.0806
	0.0447	0.0470	0.0489	0.0504	0.0516	0.0525	0.0533	0.0538	0.0542	0.0548	0.0551	0.0553	0.0554	0.0555	0.0555
	0.0462	0.0483	0.0500	0.0514	0.0524	0.0532	0.0537	0.0542	0.0546	0.0550	0.0552	0.0554	0.0554	0.0555	0.0555
	0.0398	0.0426	0.0449	0.0469	0.0485	0.0499	0.0510	0.0519	0.0526	0.0537	0.0545	0.0549	0.0551	0.0553	0.0554

TABLE A6.2.9 (continued)

Estimator	λ_2 0.0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	5.0	6.0	7.0	8.0	9.0	10.0
	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525
	0.9903	1.0292	1.0703	1.1142	1.1606	1.2094	1.2608	1.3147	1.3714	1.4919	1.6231	1.7642	1.9156	2.0772	2.2489
	0.0653	0.0650	0.0644	0.0642	0.0639	0.0636	0.0633	0.0631	0.0631	0.0625	0.0622	0.0617	0.0614	0.0611	0.0608
$\nu=5, \phi=0.1, \lambda_1=0$	0.0544	0.0544	0.0542	0.0542	0.0542	0.0542	0.0542	0.0542	0.0539	0.0539	0.0539	0.0539	0.0539	0.0536	0.0536
	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525
	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525
	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525	0.0525
	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
	0.1967	0.2261	0.2580	0.2923	0.3294	0.3689	0.4111	0.4556	0.5028	0.6048	0.7172	0.8395	0.9722	1.1152	1.2681
	0.0086	0.0083	0.0081	0.0080	0.0078	0.0077	0.0075	0.0073	0.0073	0.0072	0.0070	0.0069	0.0069	0.0067	0.0067
$\nu=10, \phi=0.1, \lambda_1=0$	0.0067	0.0067	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0066	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064	0.0064
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0798	0.1035	0.1298	0.1586	0.1900	0.2239	0.2604	0.2994	0.3410	0.4318	0.5329	0.6441	0.7655	0.8971	1.0390
	0.0019	0.0017	0.0016	0.0015	0.0014	0.0013	0.0013	0.0012	0.0012	0.0012	0.0012	0.0011	0.0011	0.0011	0.0011
$\nu=\infty, \phi=0.1, \lambda_1=0$	0.0012	0.0012	0.0012	0.0012	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011	0.0011
	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536
	1.0086	1.0489	1.0917	1.1369	1.1850	1.2353	1.2883	1.3439	1.4019	1.5256	1.6597	1.8039	1.9583	2.1231	2.2978
	0.0697	0.0692	0.0686	0.0681	0.0675	0.0672	0.0667	0.0664	0.0658	0.0653	0.0647	0.0642	0.0636	0.0633	0.0631
$\nu=5, \phi=0.1, \lambda_1=3$	0.0561	0.0558	0.0558	0.0558	0.0556	0.0556	0.0556	0.0556	0.0553	0.0553	0.0553	0.0550	0.0550	0.0550	0.0550
	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536
	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536
	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536	0.0536
	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
	0.2106	0.2414	0.2748	0.3108	0.3494	0.3905	0.4341	0.4803	0.5289	0.6341	0.7494	0.8748	1.0106	1.1566	1.3127
	0.0125	0.0120	0.0114	0.0109	0.0106	0.0103	0.0100	0.0097	0.0094	0.0091	0.0088	0.0084	0.0083	0.0081	0.0081
$\nu=10, \phi=0.1, \lambda_1=3$	0.0083	0.0081	0.0081	0.0080	0.0080	0.0078	0.0078	0.0078	0.0077	0.0077	0.0077	0.0077	0.0075	0.0075	0.0075
	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075	0.0075
	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022
	0.0910	0.1162	0.1440	0.1744	0.2073	0.2428	0.2808	0.3213	0.3644	0.4583	0.5624	0.6767	0.8012	0.9359	1.0808
	0.0055	0.0050	0.0045	0.0041	0.0037	0.0034	0.0032	0.0030	0.0028	0.0026	0.0024	0.0024	0.0023	0.0022	0.0022
$\nu=\infty, \phi=0.1, \lambda_1=3$	0.0028	0.0027	0.0026	0.0025	0.0024	0.0024	0.0023	0.0023	0.0023	0.0023	0.0022	0.0022	0.0022	0.0022	0.0022
	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022
	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022
	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022	0.0022

CHAPTER SEVEN

FINAL REMARKS

The principal aim of this thesis is to investigate the consequences of pre-testing in a mis-specified linear regression model. We consider two forms of (possibly simultaneous) mis-specification. First, that the usual normality assumption should be a broader one of spherical symmetry, and secondly, that variables are possibly omitted from the design matrix. We investigate first, the estimation of the prediction vector and of the error variance after a pre-test for exact linear restrictions and secondly, the estimation of the error variance after a pre-test for homogeneity in the two-sample linear regression model.

The exact bias functions and the exact risk functions of the pre-test estimators and their components are derived , and we then compare these finite sample properties of the estimators for a given level of mis-specification and as the degree of mis-specification varies. We consider the bias as well as the risk of the estimators, given that there is a pre-occupation in classical econometrics with the use of unbiased estimators, and also to determine whether the mis-specification affects the bias functions in the same way as it does the risk functions.

We investigate the estimators of the conditional forecast of y (Xb, Xb^*, $X\hat{b}$) in the linear regression model after a pre-test for exact linear restrictions in Chapter Four. We find that the bias functions of Xb and Xb^* do not depend on the specific variance mixing distribution, and so are the same for all members of the SSD_N family. However, $f(\tau)$ affects the bias($X\hat{b}$). This is not surprising given the dependence of the pre-test estimator on the non-null distribution of the test statistic, u. In contrast, the risk functions of Xb, Xb*, and of $X\hat{b}$ are determined in part by $f(\tau)$, as each of these functions depends on the variance-covariance matrix of e.

When estimating the prediction vector we find that the features of the risk functions observed in the literature for normal errors qualitatively carry over to the broader assumption of SSD_N regression disturbances, though there are implications for the choice of estimator, assuming the hypothesis error is known, when we incorrectly assume normality. In particular, there is no guarantee that imposing valid restrictions will result in a reduction in risk once we allow for the possibility of omitted regressors.

When estimating the error variance, however, after either a pre-test for linear restrictions (Chapter Five) or after a pre-test for homogeneity (Chapter Six) we find that mis-specifying the error distribution affects the bias and the risk functions both quantitatively and qualitatively. If there are no excluded regressors, and e is Mt with small degrees of freedom ν, then the optimal strategy, with respect to risk, is to pre-test using $c=c^*$, even if the prior information is valid, and regardless of which pre-test we are undertaking. c^* is the critical value which minimises both the pre-test risk and the pre-test bias functions and, interestingly, it does not depend on the degree of mis-specification of the design matrix or on $f(\tau)$.

When testing for homogeneity of the error variances $c^*=\left(v_1(T_2+\mu)\right)/\left(v_2(T_1+\mu)\right)$, which implies that $c_L^*=1$, $c_{ML}^*=(v_1T_2)/(v_2T_1)$, and $c_M^*=\left(v_1(v_2+2)\right)/\left(v_2(v_1+2)\right)$; while when testing for exact linear restrictions $c^*=\left(v(h-g)\right)/\left(m(T+g)\right)$, so that $c_L^*=1$, $c_{ML}^*=0$, and $c_M^*=v/(v+2)$. c_L^* is unity regardless of which pre-test we are considering, and when pre-testing for linear restrictions using the ML components we should always ignore the prior information, even if it is valid. When pre-testing for homogeneity though and using the ML components, $c^*_{ML}>0$. We note that these critical values rarely imply a test size of 1 or 5% ; the optimal test size is more likely to be at least 30% .

Of the three component estimators we consider, the numerical results suggest, if ν is small and we are pre-testing using a minimax risk criterion, then it is preferable to use the ML components and, specifically, to pre-test using $c=c^*_{ML}$. This implies that it is better to simply ignore the prior information when testing for linear restrictions while we should pre-test using $c=(v_1T_2)/(v_2T_1)$ when pre-testing for homogeneity.

If, however, the researcher has not mis-specified the error distribution (that is, e is normal) or the design matrix then it is generally preferable, in terms of risk, to impose the valid linear restrictions or to always-pool the samples if the error variances are equal. Our evaluations suggest that the only exception to this is for the L components with one linear restriction. Then it seems preferable to employ $\hat{\sigma}^2_L$ with c=1.

In this situation of normal regression disturbances and a correctly specified design matrix, our results suggest that, if one pre-tests using a minimax risk criterion then, the M components are preferred if $\alpha=0.01$ but that the L components result in a smaller maximum risk if $\alpha\geq0.05$ when we are pre-testing for homogeneity. However, if we are pre-testing for linear restrictions then the M components have the smallest maximum risk.

These are not the conclusions we arrive at when we pre-test and aim to minimise the maximum absolute bias. Then, if the design matrix is correctly specified, and we are pre-testing the validity of a set of linear restrictions, our results suggest that we should use the L component estimators when $\alpha \geq 0.05$ and the M components when $\alpha = 0.01$ when the errors are normal. For small ν, in this situation, the L component estimators are preferred. Similarly, the L component estimators appear to have the smallest maximum absolute bias for all α when we are pre-testing for homogeneity of the error variances for $\nu > 2$.

Once we admit that we may have omitted regressors then we can no longer guarantee a reduction in risk when we impose valid prior information. The results suggest that the optimal strategy is to always use the ML components when we are pre-testing for linear restrictions, regardless of whether ν is relatively small or large. If the model is sufficiently mis-specified then $\tilde{\sigma}^2_{ML}$ strictly dominates all of the other considered estimators in terms of both bias and risk. We recall that $c^*_{ML} = 0$ for this problem.

We also prefer the ML pre-test estimator which uses c^*_{ML} when we are pre-testing for homogeneity if both samples are sufficiently mis-specified. This estimator results in the smallest (absolute) bias and the smallest risk. However, if only $\lambda_2 > 0$ and if ν is small, then our preferred estimators are those we discussed above for the case of no omitted regressors. While, for relatively large values of ν our results suggest we should use the ML components for $\alpha \leq 0.05$ and the M components otherwise. The conclusions we discussed for the bias with $\lambda_1 = \lambda_2 = 0$ hold for $\lambda_1 = 0$, $\lambda_2 \geq 0$.

It is difficult to ascertain how sensitive the results are to our choice of a squared error loss function. Lehmann (1983 p.55-56) argues that under such a loss function the performance of the estimators is strongly influenced by the tail behaviour of the assumed distribution of the random variable. (See also Andrews and Phillips (1987).) Further research is required on the consequences of pre-testing under alternative loss structures.

Moreover, we require the existence of the first two moments of the error distribution for risk under squared error loss to provide a meaningful comparison of the estimators. This means we have to preclude those spherically symmetric distributions with infinite first and second moments, such as the Cauchy distribution. Other criteria exist which do not require this restrictive assumption. For example, criteria such as the probability of concentration or the probability of nearness have appeal (see, for instance, Rao(1981)). Unfortunately, use of these criteria requires knowledge of the complete distribution function of the estimator. Our knowledge of the exact distributions of traditional pre-test estimators is limited, though this issue is receiving some attention (see, for instance, Giles(1989)). Interestingly, the use of such criteria, as compared to minimising risk under squared error loss, is known to produce conflicting results. For example, when estimating the error variance of a normal random population with (say) n observations, it is well known that s^2=SSR/(n+1) has smaller risk than s_*^2=SSR/(n-1), which is the usual unbiased estimator of the error variance, where SSR is the sum of squared residuals. However, under a probability of nearness criterion we prefer s_*^2 to s^2 (Rao(1981)).

The extent to which our results depend on the particular form of non-normality that we investigate also requires attention. The extension to a spherically symmetric family of distributions results in uncorrelated though dependent error terms. The error terms are only independent when they are normally distributed. Further research is required on the consequences of pre-testing when the regression disturbances are non-normal but are identically and independently distributed. Research by, for instance, Phillips and Hajivassiliou (1987) and Lye (1990) suggest that this distinction is an important one in non-normal models, though the Monte Carlo experiments of Miyazaki *et al.* (1986) suggest otherwise. It remains to be seen whether their qualitative results carry over to the problems that we investigate here.

The results that we obtain assume that the regression disturbances are specified as normal. It would be interesting to suppose that the researcher admits the possibilty of non-normal errors but mis-specifies the degree of non-normality. For instance, he may correctly assume Mt errors but he may mis-specify the value of ν. Miyazaki *et al.* (1986) examine this problem for some robust and Stein-like estimators using Monte Carlo techniques. A related issue is the question of the consequences of using an estimate of ν.

Our conclusions regarding the homogeneity pre-test problem depend crucially on the assumption of a one-sided alternative hypothesis. It would be relatively straightforward to extend the analysis to the two-sided case but the findings that we have discussed need not carry over. This remains for future research, as does the investigation of the pre-test estimator of the coefficient vector after a pre-test for homogeneity once we allow for mis-specification of the regression model[1].

We have paid scant attention to the choice of an optimal test critical value. Where we have shown that there exists a strictly dominating estimator this problem is not an issue. However, the choice of an optimal test critical value is still to be adequately investigated if there is no dominating estimator and the model is mis-specified.

Finally, we have only investigated traditional pre-test estimators,

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[1] Preliminary research into this problem suggests that the extension to SSD_N regression disturbances is feasible, provided that the design matrix is correctly specified. However, there appear to be some difficulties when there are omitted regressors.

and their components, after two common pre-tests. There are many other estimators and many other pre-tests that we could conceivably extend our analysis to.

However, despite these limitations to our study, and despite the research which remains to be done, one major conclusion that may be drawn from this thesis is that the assumption of a correctly specified regression model is precarious. We have shown that the properties of pre-test estimators may or may not be robust to mis-specification of the model. This will depend on the estimator of interest and the pre-test under investigation. This should be of interest to applied workers given that the use of pre-test estimators in applied econometric research is the norm rather than the exception, and that our models are invariably mis-specified to some degree.

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